On Inexact Solvers for the Coarse Problem of BDDC

Clark R. Dohrmann, Kendall H. Pierson, and Olof B. Widlund

1 Introduction

In this study, we present Balancing Domain Decomposition by Constraints (BDDC) preconditioners for three-dimensional scalar elliptic and linear elasticity problems in which the direct solution of the coarse problem is replaced by a preconditioner based on a smaller vertex-based coarse space. By doing so, the computational and memory requirements can be reduced significantly. Although the use of standard coarse spaces based on subdomain vertices (corners) alone has similar memory benefits, the associated rate of convergence is not attractive as the number of elements per subdomain grows [10]. This point is illustrated by a simple motivating example in the next section.

There exists a rich theory for Finite Element Tearing and Interconnecting Dual Primal (FETI-DP) and BDDC algorithms for scalar elliptic and linear elasticity problems in three dimensions (see, e.g., [10], [9] or §6.4.2 of [15]). In many cases, theoretical results for either FETI-DP or BDDC apply directly to the other because of the equivalence of eigenvalues of the preconditioned operators [13, 11, 1]. This equivalence does not hold in the present study because the basic FETI-DP algorithm [6] is not easily adapted to use a preconditioner instead of a direct solver for the symmetric and positive definite coarse problem. In contrast, such a change is accommodated easily by BDDC in both theory and practice [4]. Nevertheless, we expect that our approach could find use in the irFETI-DP algorithm described in [8].

The approach to preconditioning the BDDC coarse problem is motivated in part by more recent developments of small coarse spaces for domain decomposition

Clark R. Dohrmann
Sandia National Laboratories, Albuquerque, New Mexico, U.S.A., e-mail: crdohrm@sandia.gov

Kendall H. Pierson
Sandia National Laboratories, Albuquerque, New Mexico, U.S.A., e-mail: khpiers@sandia.gov

Olof B. Widlund
Courant Institute, New York, New York, U.S.A., e-mail: widlund@cims.nyu.edu

algorithms [5]. Although that study was focused on overlapping Schwarz methods, similar ideas can be used to construct coarse spaces for preconditioning the BDDC coarse problem. Compared with larger edge-based or face-based coarse spaces, we find that similar condition number bounds can be achieved at much lower cost under certain assumptions on material property jumps between adjacent subdomains.

We note that three-level and multi-level BDDC algorithms [17, 16, 14] can also be viewed as using an inexact solver for the coarse problem, but such approaches are fundamentally different from ours. Namely, these algorithms construct and apply (recursively for multi-level approaches) a BDDC preconditioner for the original two-level coarse problem. In contrast, we do not introduce additional coarse levels and make use of standard two-level additive Schwarz concepts for preconditioning the coarse problem. One important result of using smaller coarse spaces is that larger numbers of subdomains are feasible before needing to use a three- or multi-level approach. Consequently, the number of coarse levels can potentially be reduced and result in fewer synchronization points for parallel implementations. We also note that approximate solvers of the coarse problem were introduced in [8] as in the context of a saddle-point formulation for FETI-DP.

Reducing the size of the coarse problem while retaining favorable convergence rates was also the subject of Algorithm D in [10]. The basic idea there was to use a coarse space based on a subset of subdomain edges and corners (vertices) rather than all of them. The authors note that their recipe for selecting such edges and corners is relatively complicated, but it can effectively reduce the coarse problem dimension. In contrast to their approach, the present one uses all subdomain edges, but replaces the direct solver for the coarse problem with a preconditioner.

A motivating example is presented in the next section for the proposed approach which is summarized in §3. Theoretical results for scalar elliptic and linear elasticity problems are presented in §4. Complete proofs are provided in the article, [2], that has appeared since this paper was submitted; it also contains implementation details, extensions to face-based coarse spaces, and additional numerical examples. The final section of this paper contains numerical results, which confirm the theory and demonstrate the computational advantages of our approach.

2 Motivation

To help motivate the proposed approach, consider a unit cube domain partitioned into 27 smaller cubic subdomains. Each of these subdomains is discretized using H/h lowest order hexahedral elements in each coordinate direction for the Poisson equation with constant material properties. Homogeneous essential boundary conditions are applied to one side of the domain and a random load vector b is used for the right-hand side of the linear system $Ax = b$. We note that our algorithm iterates on the interface problem $Su = g$ after eliminating residuals in subdomain interiors (initial static condensation step). Here, S is the Schur complement matrix for the interface problem.

We first consider coarse spaces based on subdomain vertices alone or edges alone. Table 1 shows the condition number estimates for the preconditioned operator along with the number of iterations needed to achieve a relative residual tolerance of 10^{-8} using the conjugate gradient algorithm preconditioned using BDDC. The fast growth of condition numbers in the third column is consistent with a condition number bound proportional to $(H/h)(1+\log(H/h))^2$ as given in Remark 2 of [10]. The shortcomings of using coarse spaces based on vertices alone were recognized early in the history of FETI-DP [7]. Notice the results for the proposed approach show significant improvements in comparison to the standard vertex (corner) based coarse space.

Table 1: Poisson equation results. Number of iterations (iter) and condition number estimates (cond) are shown for a unit cube domain constrained on one side and decomposed into 27 smaller cubic subdomains. In this table and others, H/h denotes the number of elements in each coordinate direction for each subdomain. More generally, H/h refers to the maximum ratio of subdomain diameter H_i to smallest element diameter h_i for any subdomain Ω_i.

	standard approach				proposed approach	
	vertices		edges			
H/h	iter	cond	iter	cond	iter	cond
4	28	27.1	12	2.36	14	2.50
8	38	75.2	14	2.93	16	3.13
12	45	132	16	3.37	18	3.59
16	47	195	17	3.73	19	3.97

Results are shown in Table 2 for increasing numbers of subdomains N and fixed $H/h = 8$. Notice the dimensions n_c of the coarse space for edge-based coarse spaces are significantly larger than those for the proposed approach. Again, the advantages of the new approach are evident in the final three columns of the table where the number of iterations and condition numbers are much smaller than those for the standard vertex-based coarse space.

Table 2: Poisson equation results. Coarse space dimension n_c and convergence results are shown for increasing numbers of subdomains N and fixed $H/h = 8$.

	standard approach						proposed approach		
	vertices			edges					
N	n_c	iter	cond	n_c	iter	cond	n_c	iter	cond
64	27	55	74.5	108	15	2.98	27	17	3.25
216	125	70	73.7	450	15	2.94	125	17	3.26
512	343	74	73.6	1176	15	2.95	343	17	3.30
1000	729	75	73.6	2430	15	2.95	729	17	3.32

A primary goal of this study is to present an approach that combines the best of both worlds. That is, an approach that has the attractive convergence rates of edge-based coarse spaces and the more streamlined computational requirements of a smaller vertex-based coarse space.

3 Overview of BDDC and Our Inexact Approach

The domain Ω for the problem is assumed to be partitioned into nonoverlapping subdomains $\Omega_1, \ldots, \Omega_N$. The set of interface points that are common to two or more subdomain boundaries is denoted by Γ, and the set of interface points for Ω_i is denoted by $\Gamma_i := \Gamma \cap \partial\Omega_i$. Finite element nodes on Γ_i are partitioned into different equivalence classes such as those of subdomain vertices, edges, or faces depending on which subdomain boundaries contain them (see, e.g., [3] or [5] for more details).

A two-level BDDC preconditioner (see, e.g., [3] can be expressed concisely in additive form as

$$M^{-1} = M_{local}^{-1} + \Phi_D K_c^{-1} \Phi_D^T, \tag{1}$$

where K_c is the coarse matrix and Φ_D is a weighted interpolation matrix. We note that the application of the local component M_{local}^{-1} requires solutions of problems local to each subdomain, which can be done in parallel.

The coarse matrix is obtained from the assembly of coarse subdomain matrices and given by

$$K_c = \sum_{i=1}^{N} R_{ic}^T K_{ic} R_{ic},$$

where K_{ic} is the coarse matrix for Ω_i and $u_{ic} = R_{ic} u_c$ is the restriction of a coarse vector u_c to Ω_i. Let M_c^{-1} denote a preconditioner for K_c which satisfies the bounds

$$\beta_1 u_c^T K_c^{-1} u_c \le u_c^T M_c^{-1} u_c \le \beta_2 u_c^T K_c^{-1} u_c \quad \forall u_c, \tag{2}$$

where $0 < \beta_1 \le \beta_2$. Defining the approximate BDDC preconditioner M_a^{-1} as

$$M_a^{-1} := M_{local}^{-1} + \Phi_D M_c^{-1} \Phi_D^T,$$

we find from (1) and (2) that

$$\begin{aligned} p^T M_a^{-1} p &= p^T (M^{-1} + \Phi_D (M_c^{-1} - K_c^{-1}) \Phi_D^T) p \\ &\le p^T (M^{-1} + (\beta_2 - 1) \Phi_D K_c^{-1} \Phi_D^T) p \\ &\le \max(1, \beta_2) p^T M^{-1} p. \end{aligned} \tag{3}$$

Similarly,

$$p^T M_a^{-1} p \ge \min(1, \beta_1) p^T M^{-1} p. \tag{4}$$

Let κ denote the condition number of the original BDDC preconditioned operator. It then follows from (3) and (4) that

$$\kappa_a \leq \frac{\max(1, \beta_2)}{\min(1, \beta_1)} \kappa, \tag{5}$$

where κ_a is the condition number of the approximate BDDC preconditioned operator. Here we only consider preconditioners for the coarse matrix K_c, but approximations for other components of the BDDC preconditioner have also been studied [12, 4].

The construction of the preconditioner M_c^{-1} for K_c was inspired in part by our recent work on small coarse spaces [5]. What we have called vertices thus far here are generalized there and called coarse nodes. We recall that the coarse degrees of freedom for BDDC or FETI-DP are often associated with average values over the different equivalence classes. The basic idea of the coarse component of the preconditioner M_c^{-1} is to approximate these averages using adjacent vertex values.

Using the notation of [5], let $C_{\mathcal{N}}$ denote the set of ancestor vertices for a nodal equivalence class $\mathcal{N}$ (e.g. $\mathcal{N}$ may be the nodes of a subdomain edge or face). Let u_Ψ denote a vector of vertex values. We introduce the coarse interpolation $u_{c0} = \Psi u_\Psi$ between vertex values and nodal equivalence class averages such that each of these averages equals the average of its ancestor vertex values. Thus, a row of Ψ associated with an edge of the center subdomain in the motivating example has two entries of 1/2 (one entry for each vertex at its ends), while all other entries are 0. Notice that the number of rows in Ψ is the number of active coarse degrees of freedom for the original BDDC preconditioner. For instance, if only edges are used this number equals the total number of subdomain edges.

The reduced coarse matrix is defined as $K_{cr} := \Psi^T K_c \Psi$. The number of rows and columns in K_{cr} is the number of vertices for scalar problems. We consider the following preconditioner for K_c.

$$M_c^{-1} = \Psi K_{cr}^{-1} \Psi^T + \operatorname{diag}(K_c)^{-1}, \tag{6}$$

where diag denotes the diagonal of the matrix (for elasticity problems the second term on the right hand side of (6) is block diagonal). Notice M_c^{-1} is simply a Jacobi preconditioner with an additive coarse correction. Thus, since the number of subdomains incident to an edge is bounded, a uniform upper bound on β_2 for M_c^{-1} can be obtained using a standard coloring argument. Therefore, the analysis focuses on obtaining lower bound estimates for β_1. We comment that higher quality local preconditioning can be used (e.g., replacing Jacobi smoothing by symmetric Gauss-Seidel). Indeed, the numerical results in §5 were obtained using such an approach.

4 Main Results

We presently restrict our attention to edge-based BDDC coarse spaces for both scalar elliptic and linear elasticity problems (cf. §4 and §5 of [5] for problem specifications). For the scalar case, we assume quasi-monotone edge-connected paths as in Assumptions 4.5 of [5]. For elasticity problems, we must make the stronger assumption of quasi-monotone face-connected paths as in Assumption 4.4 of [5]. We also assume that material properties are constant within each subdomain and that the ratio $(H_j/h_j)/(H_k/h_k)$ is uniformly bounded for any two subdomains Ω_j and Ω_k sharing any subdomain vertex.

Theorem 1 *For edge-based BDDC coarse spaces and with quasi-monotone edge-connected paths, the condition number of the preconditioned operator that is obtained by replacing the direct solver for the coarse problem by the preconditioner* M_c^{-1} *defined in (6) is bounded by*

$$\kappa_a \leq C(1 + \log(H/h))^2$$

for scalar elliptic problems.

Theorem 2 *For edge-based BDDC coarse spaces and with quasi-monotone face-connected paths, the condition number of the preconditioned operator that is obtained by replacing the direct solver for the coarse problem by the preconditioner* M_c^{-1} *defined in (6) is bounded by*

$$\kappa_a \leq C(1 + \log(H/h))^2$$

for compressible linear elasticity problems.

The proofs of these theorems use classical additive Schwarz theory, an estimate in Lemma 4.2 of [17], and a variety of standard domain decomposition estimates. Further, the analysis for linear elasticity relies on Korn inequalities and on rigid body fits of subdomain face deformations (cf. [9] for a related approach).

5 Numerical Results

The results in Tables 1 and 2 are in good agreement with the theory for the scalar case, and demonstrate that comparable performance to the standard edge-based BDDC preconditioner can be obtained more efficiently. Notice in Table 2 that the coarse space dimension n_c is approximately 3 times smaller for the proposed approach than that of the standard edge-based approach for larger numbers of subdomains. Similar results were obtained for linear elasticity (not shown), but the reductions in coarse space dimension were more modest.

The next example deals with a cubic domain decomposed into 64 smaller cubic subdomains and constrained on its left side. Three different distributions of material properties are considered as shown in Figure 1. The leftmost one has quasi-monotone face-connected paths, the middle one has quasi-monotone edge-connected paths, and the rightmost one has a checkerboard arrangement which is not covered by our theory.

The material properties in the lighter colored regions are given by $\rho = 1$ for the scalar case and $E = 1$, $\nu = 0.3$ for elasticity. Likewise, the other regions have $\rho = 10^3$,

$E = 10^3$, and $\nu = 0.3$. Results for the scalar case and elasticity are shown in Table 3. Consistent with the theory, condition numbers for the scalar case grow sublinearly with respect to H/h for both face-connected and edge-connected paths. As expected, similar growth in condition numbers is observed for linear elasticity in the case of face-connected paths. Recall that the case of edge-connected paths is not covered by our theory for elasticity, and much larger condition numbers are apparent in the table. Remarkably, very good results are obtained for the checkerboard arrangement of material properties for both the scalar case and linear elasticity.

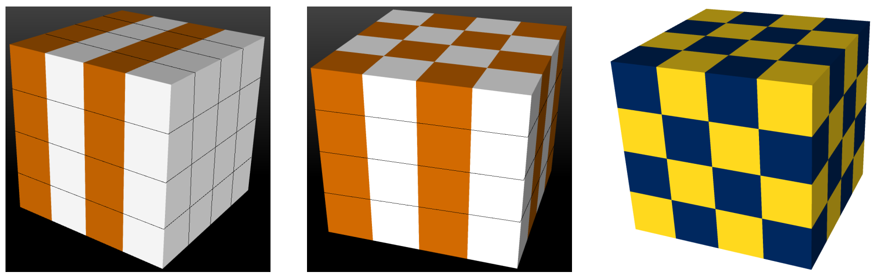

Fig. 1: Material property distributions for a cube decomposed into 64 smaller cubic subdomains. The leftmost figure has quasi-monotone face-connected paths while the middle one only has quasi-monotone edge-connected paths. The rightmost figure shows a checkerboard arrangement of material properties.

Table 3: Results for the models in Figure 1.

scalar case						
	face-connected		edge-connected		checkerboard	
H/h	iter	cond	iter	cond	iter	cond
4	14	2.41	16	3.58	9	1.45
8	16	2.95	20	4.81	11	1.71
12	18	3.40	22	5.65	12	1.99
16	19	3.75	24	6.32	13	2.19
linear elasticity						
H/h	face-connected		edge-connected		checkerboard	
	iter	cond	iter	cond	iter	cond
4	25	6.10	40	72.9	24	6.55
8	33	11.1	53	113	31	11.1
12	38	14.8	61	137	35	14.4
16	42	17.8	68	154	38	16.9

Additional numerical results have been generated for face-based rather than edge-based coarse spaces, for unstructured meshes, and performance tests are given which show reduced compute times. They are reported in the article, [2], which has appeared since this conference paper was submitted.

In closing, we expect that the approach presented here could be combined with an adaptive coarse space to handle problems where material properties vary greatly within a subdomain. The basic idea would be to use existing adaptive approaches for challenging subdomains, while using the present approach for less problematic ones.

References

1. Brenner, S.C., Sung, L.Y.: BDDC and FETI-DP without matrices or vectors. Comput. Methods Appl. Mech. Engrg. **196**(8), 1429–1435 (2007). DOI:10.1016/j.cma.2006.03.012
2. Dohrmann, C., Pierson, K., Widlund, O.: Vertex-based preconditioners for the coarse problem of BDDC. SIAM J. Sci. Comput. **41**(5), A3021–A3044 (2019). DOI:10.1137/19M1237557
3. Dohrmann, C.R.: A preconditioner for substructuring based on constrained energy minimization. SIAM J. Sci. Comput. **25**(1), 246–258 (2003). DOI:10.1137/S1064827502412887
4. Dohrmann, C.R.: An approximate BDDC preconditioner. Numer. Linear Algebra Appl. **14**(2), 149–168 (2007). DOI:10.1002/nla.514
5. Dohrmann, C.R., Widlund, O.B.: On the design of small coarse spaces for domain decomposition algorithms. SIAM J. Sci. Comput. **39**(4), A1466–A1488 (2017). DOI:10.1137/17M1114272
6. Farhat, C., Lesoinne, M., LeTallec, P., Pierson, K., Rixen, D.: FETI-DP: a dual-primal unified FETI method. I. A faster alternative to the two-level FETI method. Internat. J. Numer. Methods Engrg. **50**(7), 1523–1544 (2001). DOI:10.1002/nme.76
7. Farhat, C., Lesoinne, M., Pierson, K.: A scalable dual-primal domain decomposition method. Numer. Linear Algebra Appl. **7**(7-8), 687–714 (2000). DOI:10.1002/1099-1506(200010/12)7:7/8<687::AID-NLA219>3.0.CO;2-S. Preconditioning techniques for large sparse matrix problems in industrial applications (Minneapolis, MN, 1999)
8. Klawonn, A., Rheinbach, O.: Inexact FETI-DP methods. Internat. J. Numer. Methods Engrg. **69**(2), 284–307 (2007). DOI:10.1002/nme.1758
9. Klawonn, A., Widlund, O.B.: Dual-primal FETI methods for linear elasticity. Comm. Pure Appl. Math. **59**(11), 1523–1572 (2006). DOI:10.1002/cpa.20156
10. Klawonn, A., Widlund, O.B., Dryja, M.: Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. SIAM J. Numer. Anal. **40**(1), 159–179 (2002). DOI:10.1137/S0036142901388081
11. Li, J., Widlund, O.B.: FETI-DP, BDDC, and block Cholesky methods. Internat. J. Numer. Methods Engrg. **66**(2), 250–271 (2006). DOI:10.1002/nme.1553
12. Li, J., Widlund, O.B.: On the use of inexact subdomain solvers for BDDC algorithms. Comput. Methods Appl. Mech. Engrg. **196**(8), 1415–1428 (2007). DOI:10.1016/j.cma.2006.03.011
13. Mandel, J., Dohrmann, C.R., Tezaur, R.: An algebraic theory for primal and dual substructuring methods by constraints. Appl. Numer. Math. **54**(2), 167–193 (2005). DOI:10.1016/j.apnum.2004.09.022
14. Mandel, J., Sousedík, B., Dohrmann, C.R.: Multispace and multilevel BDDC. Computing **83**(2-3), 55–85 (2008). DOI:10.1007/s00607-008-0014-7
15. Toselli, A., Widlund, O.: Domain decomposition methods—algorithms and theory, *Springer Series in Computational Mathematics*, vol. 34. Springer-Verlag, Berlin (2005). DOI:10.1007/b137868
16. Tu, X.: Three-level BDDC in three dimensions. SIAM J. Sci. Comput. **29**(4), 1759–1780 (2007). DOI:10.1137/050629902
17. Tu, X.: Three-level BDDC in two dimensions. Internat. J. Numer. Methods Engrg. **69**(1), 33–59 (2007). DOI:10.1002/nme.1753