Adaptive BDDC Based on Local Eigenproblems

Clemens Pechstein

1 Introduction

FETI-DP (dual-primal finite element tearing and interconnecting) and BDDC (balancing domain decomposition by constraints) are among the leading non-overlapping domain decomposition preconditioners. For standard symmetric positive definite (SPD) problems and standard discretizations, the spectral condition number κ of the preconditioned systems of either FETI-DP or BDDC can be bounded from above by $C\,(1 + \log(H/h))^2$. Here, H is the subdomain diameter and h the discretization parameter, and C is a constant independent of H, h, and the number of subdomains; for more details see e.g. [19]. In the past decade, there has been significant effort in analyzing the dependence of the constant C on problem parameters, such as coefficient values. This research has also led to new parameter choices of the preconditioners themselves, such as more sophisticated scalings and primal constraints. In particular, *adaptive* choices of primal constraints have been studied, starting with the pioneering work by Mandel and Sousedík [13] and later with Šístek [15], continued from different angles by Spillane and Rixen [18] as well as Klawonn, Radtke, and Rheinbach [8], and meanwhile pursued by various researchers [2, 5, 6, 9, 10, 20].

Our own work started with talks and slides [4, 16] and led to the comprehensive paper [17], where we present a rigorous and quite general theoretical framework for adaptive BDDC preconditioners and show the connections and differences between various existing methods. The paper [17] appears to be rather long and technical. In the contribution at hand, we would like to summarize the big picture from a less detailed perspective in favor of simplicity.

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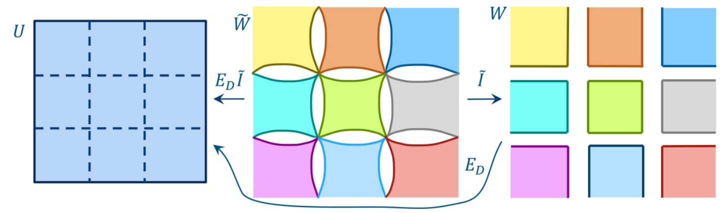

Fig. 1: Sketch of the spaces U, $\widetilde{W}$, and W for primal dofs on subdomain vertices in 2D.

2 BDDC Basics

The original problem to be solved reads $Au^* = f$, with the SPD matrix $A \in \mathbb{R}^{n\times n}$, originating from a PDE on the global domain Ω and projected to a finite-dimensional space via a finite element method (FEM), discontinuous Galerkin (DG) method, or isogeometric analysis (IGA). Using a decomposition of the domain Ω into non-overlapping subdomains $\{\Omega_i\}_{i=1}^N$, each degree of freedom (dof) is associated with one or several subdomains. After formal static condensation of the *inner* dofs (those owned only by one subdomain), we are left with the interface system

$$\widehat{S}\widehat{u} = \widehat{g}, \tag{1}$$

where $\widehat{S} : U \to U$ is again SPD. The global Schur complement $\widehat{S}$ can be assembled from subdomain contributions S_i in the following way,

$$\widehat{S} = R^T S R = \sum_{i=1}^{N} R_i^T S_i R_i \,, \tag{2}$$

where $R_i : U \to W_i$ is the restriction matrix that selects from all global dofs those of subdomain i, and $S = \operatorname{diag}(S_i)_{i=1}^N : W \to W := \prod_{i=1}^N W_i$. Throughout the paper we assume that each matrix S_i is symmetric positive semidefinite (SPSD).

The space U of interface dofs can be visualized as the global *continuous* space, whereas W can be visualized as a *discontinuous* space, see Fig. 1. The subspace of W containing *continuous* functions is $\widehat{W} := \operatorname{range}(R)$.

The balancing domain decomposition by constraints (BDDC) preconditioner [3, 12] can be seen as a fictitious space preconditioner [17, Appendix A]: Selecting a subspace $\widetilde{W} \subset W$ such that $\widetilde{W} \supset \widehat{W}$, the preconditioner reads

$$M_{\text{BDCC}}^{-1} := E_D \tilde{I} \widetilde{S}^{-1} \tilde{I}^T E_D^T \,, \tag{3}$$

where $\widetilde{S}$ is the restriction of S to $\widetilde{W}$, $\tilde{I} : \widetilde{W} \to W$ is the natural embedding operator, and $E_D : W \to U$ is a linear averaging operator mapping back to the original space U. In the following, we assume that $E_D R = I$, such that $R E_D$ becomes a projection.

In this case, $\lambda_{\min}(M_{\mathrm{BDDC}}^{-1}\widehat{S}) \geq 1$ and $\lambda_{\max}(M_{\mathrm{BDDC}}^{-1}\widehat{S}) \leq C$ where[1]

$$|(I - RE_D)w|_S^2 \leq C|w|_S^2 \qquad \forall w \in \widetilde{W}, \tag{4}$$

cf. [12]. For parallel computing, the subspace $\widetilde{W}$ should have small co-dimension with respect to W. In order to ensure invertibility of $\widetilde{S}$, the space $\widetilde{W}$ has to be made smaller than W and with that allow some coupling between individual subdomains. In case of highly varying coefficients, one typically needs even more coupling. For some motivating numerical results obtained by adaptively chosen spaces $\widetilde{W}$ see e.g. [6, Sect. 8] (using the pair-based approach).

We decompose the global set of interface dofs into equivalence classes called *globs* such that dofs within a glob are shared by the same set of subdomains. In the sequel, we refer to globs shared by two subdomains simply as *faces*.[2] Next, we define a *primal dof* as a linear combination of regular dofs within the same glob.

For the following investigation we make two assumptions:

1. The space $\widetilde{W}$ is based on *primal constraints*, i.e., it is the subspace of W where on each glob, the associated primal dofs are continuous across the subdomains.
2. The averaging operator is block-diagonal w.r.t. the glob partition, i.e., the global dofs of $E_D w$ associated with a glob only depend on the values of w_i on that glob.

To formulate these assumptions more precisely, we need some notation. Let R_{iG} extract the dofs of W_i that belong to glob G and let $\mathcal{N}_G$ denote the set of subdomains sharing glob G. Then Assumption 1 reads

$$\widetilde{W} = \{w \in W \colon Q_G^T(R_{iG}w_i - R_{jG}w_j) = 0 \quad \forall i, j \in \mathcal{N}_G\}, \tag{5}$$

where Q_G^T is the matrix evaluating all primal dofs on G. Assumption 2 reads $\widehat{R}_G E_D w = \sum_{i \in \mathcal{N}_G} D_{iG} R_{iG} w_i$, where $\widehat{R}_G$ extracts the dofs of U that belong to glob G and $\{D_{jG}\}_{j \in \mathcal{N}_G}$ are local weighing matrices, not necessarily diagonal. To ensure that RE_D is a projection, we assume the glob-wise partition of unity property

$$\sum\nolimits_{j \in \mathcal{N}_G} D_{jG} = I. \tag{6}$$

There are several ways to realize the application of $\widetilde{I}\widetilde{S}^{-1}\widetilde{I}^T$ in practice (see [3, 11] and [17, Appendix C]), but all essentially boil down to block factorization where a sparse matrix on the space of primal dofs forms the coarse problem, whereas independent subdomain problems with the primal dofs being fixed form the remainder.

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[1] assuming that RE_D is different from zero and identity

[2] In simple setups, one may visualize globs as open faces, open edges, and vertices, but this can change due to the geometry and/or the particular discretization.

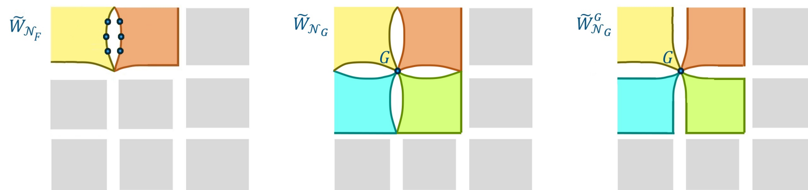

Fig. 2: Sketch of spaces and support (indicated by dots) of local operator $P_{D,G}$ for glob-based approach. *Left:* $\widetilde{W}_{\mathcal{N}_F}$ for face $G = F$. *Middle:* $\widetilde{W}_{\mathcal{N}_G}$ for vertex G. *Right:* $\widetilde{W}^G_{\mathcal{N}_G}$ for vertex G.

3 Localization, Eigenproblems, and Adaptivity

Under the assumptions from the previous section, the global estimate (4) can be localized. In the following, we consider two kinds of localizations and work out the associated generalized eigenproblem and adaptive coarse space enrichment.

3.1 Glob-based approach

Let $P_{D,G} \colon W_{\mathcal{N}_G} \to W_{\mathcal{N}_G} := \prod_{i\in\mathcal{N}_G} W_i$ be given by

$$(P_{D,G}w)_i = R_{iG}^T \sum\nolimits_{j\in\mathcal{N}_G} D_{jG}(R_{iG}w_i - R_{jG}w_j), \tag{7}$$

and let $\widetilde{W}_{\mathcal{N}_G} \subset W_{\mathcal{N}_G}$ denote the subspace of functions where on all *neighboring* globs of G, the primal constraints of the global problem are enforced (see Fig. 2).[3] Two globs G and G' are *neighbors* if they share at least two common subdomains.

Theorem 1 *If for each glob G the inequality*

$$\sum\nolimits_{i\in\mathcal{N}_G} |(P_{D,G}w)_i|^2_{S_i} \leq \omega_G \sum\nolimits_{i\in\mathcal{N}_G} |w_i|^2_{S_i} \qquad \forall w \in \widetilde{W}_{\mathcal{N}_G} \tag{8}$$

holds, then

$$\kappa(M^{-1}_{\mathrm{BDDC}}\widehat{S}) \leq \big(\max_{i=1,\dots,N} |\mathcal{G}_i|^2\big)\big(\max_G \omega_G\big), \tag{9}$$

where $|\mathcal{G}_i|$ is the number of globs associated with subdomain i.

Proof We only have to show estimate (4), i.e., $|P_D w|^2_S \leq C|w|^2_S$ for all $w \in \widetilde{W}$ where $P_D := (I - RE_D)$, cf. [12, Thm. 5] and [17, Sect. 2.3]. Under our assumptions, for any $w \in \widetilde{W}$,

[3] Precisely, $\widetilde{W}_{\mathcal{N}_G} = \{w \in W_{\mathcal{N}_G} : \forall i \neq j \in \mathcal{N}_G \forall G', \{i, j\} \subset \mathcal{N}_{G'} : Q^T_{G'}(R_{iG'}w_i - R_{jG'}w_j) = 0\}$.

$$(P_D w)_i = \sum_{G\in\mathcal{G}_i} R_{iG}^T \sum_{j\in\mathcal{N}_G} D_{jG}(R_{iG}w_i - R_{jG}w_j) = \sum_{G\in\mathcal{G}_i} (P_{D,G} \underbrace{[w_j]_{j\in\mathcal{N}_G}}_{\in \widetilde{W}_{\mathcal{N}_G}})_i\,. \tag{10}$$

I.e., the operators $P_{D,G}$ are localizations of P_D, and applying Cauchy's inequality and using (8) yields the desired result, see [17, Thm. 3.10]. □

Remark 1 If all dofs of glob G are primal dofs ($Q_G = I$) then $P_{D,G} = 0$. For such globs, estimate (8) holds with $\omega_G = 0$ and need not be accounted for in $|\mathcal{G}_i|$ in (9).

Generalized Eigenproblem. With $S_{\mathcal{N}_G} := \mathrm{diag}(S_j)_{j\in\mathcal{N}_G}$, estimate (8) is linked to the generalized eigenproblem, find $(w,\lambda) \in \widetilde{W}_{\mathcal{N}_G} \times \mathbb{R}$ such that

$$z^T S_{\mathcal{N}_G} w = \lambda\, z^T P_{D,G}^T S_{\mathcal{N}_G} P_{D,G} w \qquad \forall z \in \widetilde{W}_{\mathcal{N}_G}\,, \tag{11}$$

Since the local operator $P_{D,G}$ is a projection [17, Lemma 3.8], (11) is of the form $Ax = \lambda P^T A P x$ where A is SPSD and P a projection. Therefore all finite eigenvalues λ of (11) fulfill $\lambda \le 1$, see Lemma 1 in the Appendix. Moreover, if the smallest eigenvalue λ_1 is positive, then (8) holds with $\omega_G = \lambda_1^{-1}$. We further obtain the improved bound

$$|P_{D,G} w|_{S_{\mathcal{N}_G}}^2 \le \lambda_{k+1}^{-1} |w|_{S_{\mathcal{N}_G}}^2 \tag{12}$$

for all $w \in \widetilde{W}_{\mathcal{N}_G}$ such that

$$(y^{(\ell)})^T P_{D,G}^T S_{\mathcal{N}_G} P_{D,G} w = 0 \qquad \forall \ell = 1,\ldots,k, \tag{13}$$

where $y^{(1)},\ldots,y^{(k)}$ are the eigenvectors corresponding to the k smallest eigenvalues of (11), cf. [13]. As a viable alternative, we can replace the space $\widetilde{W}_{\mathcal{N}_G}$ in (11) by the space $\widetilde{W}_{\mathcal{N}_G}^G$ where just the primal constraints on G are enforced (but not on its neighbors),[4] see Fig. 2, cf. [17, Strategy 4]. This discards any (good) influence of primal constraints on neighboring globs but makes the underlying operator much more simple to implement.

Adaptive enrichment. We show now how to realize (13) by primal constraints.

If $G = F$ is a face shared by two subdomains i and j then (13) reads

$$\underbrace{(R_{iF} y_i^{(\ell)} - R_{jF} y_j^{(\ell)})^T \begin{bmatrix} D_{jF} \\ -D_{iF} \end{bmatrix}^T \begin{bmatrix} S_{iF} & 0 \\ 0 & S_{jF} \end{bmatrix} \begin{bmatrix} D_{jF} \\ -D_{iF} \end{bmatrix}}_{=:(Q_F^*)^T} (R_{iF} w_i - R_{jF} w_j) = 0, \tag{14}$$

where $S_{kF} := R_{kF}^T S_k R_{kF}$ is the principal minor of S_k associated with the dofs on F. Apparently, the columns of Q_F^* make up the new primal dofs.

For globs shared by more than just two subdomains, it has turned out to be a challenge to enforce (13) in terms of primal constraints. For simplicity we assume

[4] Precisely, $\widetilde{W}_{\mathcal{N}_G}^G := \{w \in W_{\mathcal{N}_G} : Q_G^T (R_{iG} w_i - R_{jG} w_j) = 0 \quad \forall i,j \in \mathcal{N}_G\}$.

that G is a glob shared by three subdomains i, j, and k (the general case follows the same idea, cf. [17, Sect. 5.4]). Then the constraints (13) take the form

$$\begin{bmatrix} c_i^{(\ell)} \\ c_j^{(\ell)} \\ c_k^{(\ell)} \end{bmatrix}^T \begin{bmatrix} (I - D_{iG})w_{iG} - D_{jG}w_{jG} - D_{kG}w_{kG} \\ -D_{iG}w_{iG} + (I - D_{jG})w_{jG} - D_{kG}w_{kG} \\ -D_{iG}w_{iG} - D_{jG}w_{jG} + (I - D_{kG})w_{kG} \end{bmatrix} = 0, \tag{15}$$

where $c^{(\ell)} = S_{\mathcal{N}_G} P_{D,G} y^{(\ell)}$ and w_{iG} is a short hand for $R_{iG}w_i$. We introduce

$$\widehat{w}_G := \tfrac{1}{3}(w_{iG} + w_{jG} + w_{kG}), \quad \check{w}_{2G} := w_{iG} - w_{jG}, \quad \check{w}_{3G} := w_{iG} - w_{kG}, \tag{16}$$

together with the corresponding inverse transformation

$$w_{iG} = \widehat{w}_G - \tfrac{1}{3}\check{w}_{2G} - \tfrac{1}{3}\check{w}_{3G}\,, \qquad w_{jG} = \widehat{w}_G + \tfrac{2}{3}\check{w}_{2G} - \tfrac{1}{3}\check{w}_{3G}\,, \qquad w_{kG} = \widehat{w}_G - \tfrac{1}{3}\check{w}_{2G} + \tfrac{2}{3}\check{w}_{3G}\,. \tag{17}$$

We apply this transformation to (15) and find out that due to the partition of unity property (6), the whole expression is independent of $\widehat{w}_G$, and so (15) is of form

$$(\check{c}_2^{(\ell)})^T\,\check{w}_{2G} + (\check{c}_3^{(\ell)})^T\,\check{w}_{3G} = 0. \tag{18}$$

In [17] we enforce this constraint by the two stronger constraints $(\check{c}_2^{(\ell)})^T\,\check{w}_{2G} = 0$ and $(\check{c}_3^{(\ell)})^T\,\check{w}_{3G} = 0$, rewritten in the original variables,

$$(\check{c}_2^{(\ell)})^T\,(R_{iG}w_i - R_{jG}w_j) = 0, \qquad (\check{c}_3^{(\ell)})^T\,(R_{iG}w_i - R_{kG}w_k) = 0. \tag{19}$$

With the enforcement of an even stronger set, namely,

$$\left.\begin{aligned} (\check{c}_2^{(\ell)})^T\,(R_{mG}w_m - R_{nG}w_n) = 0 \\ (\check{c}_3^{(\ell)})^T\,(R_{mG}w_m - R_{nG}w_n) = 0 \end{aligned}\right\} \qquad \forall m, n \in \{i, j, k\}, \tag{20}$$

we see that $\check{c}_2^{(\ell)}$, $\check{c}_3^{(\ell)}$ define the new primal dofs. In [17, Thm. 5.18], we show that it is more favorable to use the stronger primal constraints (20) than using so-called generalized primal constraints realizing exactly (15).

3.2 Pair-based approach

A different way of writing the P_D operator (compared to (10)) is

$$(P_D w)_i = \sum_{j \in \mathcal{N}_i} \; \sum_{G:\{i,j\} \subset \mathcal{N}_G} R_{iG}^T D_{jG}(R_{iG}w_i - R_{jG}w_j), \tag{21}$$

Fig. 3: Sketch of spaces and support of $P_{D,\Gamma_{ij}}$ for pair-based approach. Dots indicate support of the local operator $P_{D,\Gamma_{ij}}$. *Left:* Sketch of the space $\widetilde{W}_{ij}$. *Right:* Sketch of $\widetilde{W}_{ik}$.

where $\mathcal{N}_i$ denotes the set of subdomains that share a non-trivial set of globs with subdomain i. It was used in the early works [14, 15] and put on solid ground in [6].

Defining for $i \neq j$ the *generalized facet*

$$\Gamma_{ij} := \bigcup\nolimits_{G:\ \{i,j\}\subset\mathcal{N}_G} G, \tag{22}$$

and collecting only the non-trivial ones into the set Υ, we obtain

$$(P_D w)_i = \sum_{j:\ \Gamma_{ij}\in\Upsilon} \underbrace{R_{i\Gamma_{ij}}^T D_{j\Gamma_{ij}} (R_{i\Gamma_{ij}} w_i - R_{j\Gamma_{ij}} w_j)}_{=:(P_{D,\Gamma_{ij}} w_{ij})_i}, \tag{23}$$

where $R_{i\Gamma_{ij}}$ extracts the dofs on Γ_{ij}, the matrix $D_{j\Gamma_{ij}}$ is block-diagonal with blocks $\{D_{jG}\}_{G\subset\Gamma_{ij}}$, and $P_{D,\Gamma_{ij}} : W_{ij} \longrightarrow W_{ij} := W_i \times W_j$. Before we can formulate the counterpart of Theorem 1, we have to introduce the subspace $\widetilde{W}_{ij}$ of W_{ij} where all primal constraints between subdomain i and j are enforced,[5] see Fig. 3.

Theorem 2 *If for each generalized facet $\Gamma_{ij} \in \Upsilon$ the inequality*

$$|(P_{D,\Gamma_{ij}} w)_i|_{S_i}^2 + |(P_{D,\Gamma_{ij}} w)_j|_{S_j}^2 \leq \omega_{ij}\big(|w_i|_{S_i}^2 + |w_j|_{S_i}^2\big) \qquad \forall w \in \widetilde{W}_{ij} \tag{24}$$

holds, then

$$\kappa(M_{\mathrm{BDDC}}^{-1}\widehat{S}) \leq \big(\max_{i=1,\ldots,N} n_i^2\big) \max_{\Gamma_{ij}\in\Upsilon} \omega_{ij}\,,$$

with $n_i := |\{j : \Gamma_{ij} \in \Upsilon\}|$ the number of pairs associated with subdomain i.

Proof The proof is similar to that of Theorem 1, see also [17, Lemma 3.16]. □

Generalized eigenproblem. The generalized eigenproblem associated with estimate (24) is finding $(w, \lambda) \in \widetilde{W}_{ij} \times \mathbb{R}$ such that

$$z^T S_{ij} w = \lambda\, z^T P_{D,\Gamma_{ij}}^T S_{ij} P_{D,\Gamma_{ij}} w \qquad \forall z \in \widetilde{W}_{ij}\,, \tag{25}$$

where $S_{ij} := \operatorname{diag}(S_i, S_j)$.

[5] Precisely, $\widetilde{W}_{ij} := \{(w_i, w_j) \in W_i \times W_j : \forall G, \{i,j\} \subset \mathcal{N}_G : Q_G^T(R_{iG} w_i - R_{jG}) = 0\}$.

Remark 2 Unlike the operator $P_{D,G}$ from Sect. 3.1, the operator $P_{D,\Gamma_{ij}}$ in general *fails* to be a projection,[6] so Lemma 1 from the appendix (and some other tools from [17]) cannot be applied.

We obtain the improved bound

$$|P_{D,\Gamma_{ij}} w|^2_{S_{ij}} \leq \lambda^{-1}_{k+1} |w|^2_{S_{ij}} \tag{26}$$

for all $w \in \widetilde{W}_{ij}$ such that

$$(y^{(\ell)})^T P^T_{D,\Gamma_{ij}} S_{ij} P_{D,\Gamma_{ij}} w = 0 \qquad \forall \ell = 1, \ldots, k, \tag{27}$$

where $y^{(1)}, \ldots, y^{(k)}$ are the first k eigenvectors of (25).

Adaptive enrichment. We wish to enforce condition (27) by primal constraints and follow [15, 10, 6]. Because of its particular form, $P_{D,\Gamma_{ij}} w$ only depends on the *difference* of w_i and w_j on Γ_{ij}, and so for fixed ℓ, (27) can be written as

$$(c^{(\ell)})^T (R_{i\Gamma_{ij}} w_i - R_{j\Gamma_{ij}} w_j) = 0. \tag{28}$$

Splitting the dofs of Γ_{ij} into globs, we can express the latter as

$$\sum_{G:\ \{i,j\} \subset \mathcal{N}_G} (c_G^{(\ell)})^T (R_{iG} w_i - R_{jG} w_j) = 0, \tag{29}$$

where $c^{(\ell)} = [c_G^{(\ell)}]_{G:\ \{i,j\} \subset \mathcal{N}_G}$, up to possible renumbering. Apparently, (29) holds if we enforce the stronger conditions

$$(c_G^{(\ell)})^T (R_{kG} w_k - R_{\ell G} w_\ell) = 0 \qquad \forall k, \ell \in \mathcal{N}_G \tag{30}$$

for each glob G such that $\{i, j\} \subset \mathcal{N}_G$. Conditions (30) have exactly the form of primal constraints and will imply (27).

Remark 3 Apparently, for glob G shared by $m > 2$ subdomains, we have to collect the adaptive primal constraints originating from $(m-1)(m-2)$ pairs. E.g., for an edge (in three dimensions) shared by three subdomains, these are three pairs; if it is shared by four subdomains, six pairs. In order to avoid redundancy, an orthonormalization procedure should be applied, e.g., modified Gram-Schmidt. For the typical 3D mesh decompositions created by METIS, it is very unlikely that an edge will be shared by more than three subdomains [6]. Nevertheless, there can be a large number of short edges or thin faces, see also [6, Sect. 7].

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[6] A simple counterexample can be constructed by looking at an edge shared by two subdomains with its endpoints shared by four and using the multiplicity scaling. At an interior dofs x of the edge $P_{D,\Gamma_{ij}}$ evaluates as $\pm\frac{1}{2}(w_i(x) - w_j(x))$ whereas at an endpoint x as $\pm\frac{1}{4}(w_i(x) - w_j(x))$.

3.3 Three different approaches

We basically have three approaches:

1. The glob-based approach with the original space $\widetilde{W}_{\mathcal{N}_G}$ where primal constraints on neighboring globs are enforced ([17, Strategy 1–3]),
2. the glob-based approach with $\widetilde{W}^G_{\mathcal{N}_G}$ where no constraints are enforced on neighboring globs ([17, Strategy 4]),
3. the pair-based approach (where constraints on neighboring globs are sometimes enforced, sometimes not, see also [6]).

The difference between the glob- and pair-based approach is not only the space but also the localized P_D operator, see Fig. 2 and Fig. 3. A priori, there is no theoretical argument on which of the three approaches is better, and dedicated numerical studies will be necessary to find out more. For typical METIS partitions, the pair-based approach involves a smaller number of eigenproblems, while potentially creating some unnecessary constraints.[7] Each of the approaches can be hard to load balance, the glob-based approach likely more difficult (if one thinks of subdomain edges).

4 Simplification of the generalized eigenproblems

In this subsection, we pursue only the glob-based localization from Sect. 3.1 with the space $\widetilde{W}_{\mathcal{N}_G}$ in (11) being replaced by the space $\widetilde{W}^G_{\mathcal{N}_G}$ where just the primal constraints on G are enforced (but not on its neighbors), see Fig. 2.

In the following, suppose that F is a face shared by subdomains i and j. The eigenproblem (11) involves the dofs on the subdomain (boundary) *i and j*, so more than twice as many dofs as on F. However, the matrix $P_{D,F}$ has a large kernel with co-dimension equal to the number of dofs on F. Hence, there are many infinite eigenvalues that are irrelevant to our consideration (recall that we are after the first few smallest eigenvalues and their associated eigenvectors). It turns out that using Schur complement techniques, the eigenproblem (11) can be reduced to an equivalent one in the sense that the number of infinite eigenvalues is reduced, the rest of the spectrum is untouched, and the full eigenvectors can easily be reconstructed from the reduced ones, see [17, Principle 4.4]. For the face, (11) (on $\widetilde{W}^G_{\mathcal{N}_G}$) is equivalent (up to infinite eigenvalues) to

$$\check{z}_F^T (S^\star_{iF} : S^\star_{jF}) \check{w}_F = \lambda\, \check{z}_F^T M_F \check{w}_F \tag{31}$$

where $\check{w}_F = w_{iF} - w_{jF}$, and so the initially chosen primal dofs on F of $\check{w}_F$, $\check{z}_F$ vanish. Above, $S^\star_{kF}$ is the Schur complement of S_k eliminating all dofs except those

[7] Let us note that the sizes of the corresponding eigenproblems will not differ much, provided that one applies the same reduction technique. A comparison of approach 2. and 3. with *different* reduction techniques can be found in [10].

on F, $S^\star_{iF} : S^\star_{jF}$ is the *parallel sum* [1], defined by $A : B = A(A+B)^\dagger B$ (see [17, Sect. 5.1]), and $M_F = D_{iF}^T S_{jF} D_{iF} + D_{jF}^T S_{iF} D_{jF}$, cf. [17, Sect. 5.2].

For the choice of the deluxe scaling, $D_{iF} = (S_{iF} + S_{jF})^{-1} S_{iF}$ it can be shown [17, Sect. 6.1] that $M_F = S_{iF} : S_{jF}$, such that the eigenproblem (here with no initial primal constraints) takes the form

$$(S^\star_{iF} : S^\star_{jF}) \check{w}_F = \lambda\, (S_{iF} : S_{jF}) \check{w}_F \,. \tag{32}$$

This has been implemented in a PETSc version of BDDC by Zampini [20].

For globs G shared by more than two subdomains, one can easily eliminate the dofs not on G in a first step such that one is left with an eigenproblem of size $\#\mathcal{N}_G \times \#\mathsf{dofs}(G)$ and with kernel dimension $\#\mathsf{dofs}(G)$. Getting rid of the kernel completely is possible but more tricky. But this is not so severe since the number of dofs on such a glob (e.g. an edge) is typically much less than on a face, and so one can usually afford computing with the eigenproblem from the first reduction step. Even so, some decoupling approaches have been suggested, see [2, 5] and [17, Sect. 5.5, Sect. 5.6, and Sect. 6.4].

5 Optimality of the deluxe scaling

Let F be a face and consider the reduced eigenproblem from the previous section,

$$\check{z}_F^T T_F \check{w}_F = \lambda\, \check{z}_F^T \underbrace{\left[X^T S_{jF} X + (I-X)^T S_{iF} (I-X)\right]}_{M_F(X)} \check{w}_F \,, \tag{33}$$

where $T_F = S^\star_{iF} : S^\star_{jF}$ and where we have set $D_{iF} = X$ and $D_{jF} = I - X$ in order to obtain the partition of unity. The choice of the weighting matrix, here X, can have quite an influence on the spectrum of (33). It is of course desirable to have a spectrum that has as few small eigenvalues (lower outliers) as possible. Depending on the problem, outliers can often not be avoided, but at least its number could be minimized, because this number will be the number of new primal constraints on face F if one aims at a robust method, cf. [2].

For simplicity, we first look at the case where F consists of a single dof and so S_{iF}, S_{jF}, and X are scalars. Since T_F is fixed, we see that minimizing the quadratic expression $(S_{iF} + S_{jF})X^2 - 2S_{iF}X$ is favorable, because then the (only) eigenvalue λ is maximized (such that the local bound $\omega_F = \lambda^{-1}$ is small). Minimization is achieved with the choice $X^* = \frac{S_{iF}}{S_{iF}+S_{jF}}$ which is the well-known weighted counting function (with exponent $\gamma = 1$, see [19, Sect. 6]).

If F has more than one dof, no initial primal constraints, and if T_F is non-singular,[8] then it is favorable to minimize the *trace* of the matrix on the right-hand side of (33)

[8] If one of the subdomain "Neumann" matrices $S^\star_{kF}$ is singular (e.g., corresponding to the Laplace operator on a floating subdomain), then also $T_F = S^\star_{iF} : S^\star_{jF}$ is singular.

because of the following (for details see [17, Sect. 6.2]). Firstly, the trace of a matrix equals the sum of its eigenvalues and is similarity-invariant, i.e.,

$$\operatorname{tr}(M_F(X)) = \operatorname{tr}\big(T_F^{-1/2} M_F(X) T_F^{-1/2}\big) = \sum\nolimits_{k=1}^{n} \lambda_k^{-1}, \tag{34}$$

where $\lambda_1, \ldots, \lambda_n$ are the (generalized) eigenvalues of (33). Secondly, minimizing $\sum_{k=1}^{n} \lambda_k^{-1}$ means that it is less likely that the smallest eigenvalues are very small. At the minimum, we obtain $X^* = (S_{iF} + S_{jF})^{-1} S_{iF}$, the deluxe scaling. For numerical studies comparing different scalings (including deluxe) see, e.g., [8, 7].

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Appendix

We consider the generalized eigenproblem $Ax = \lambda Bx$ with SPSD matrices A, B and call (λ, x) a *genuine* eigenpair if $\lambda \in \mathbb{R}$ and $x \notin \ker(A) \cap \ker(B)$. We call (∞, x) an eigenpair with *infinite* eigenvalue if $x \in \ker(B) \setminus \{0\}$, and (λ, x) an *ambiguous* eigenpair if $x \in \ker(A) \cap \ker(B)$.

Lemma 1 *Let us consider the generalized eigenproblem*

$$Ax = \lambda\, P^T A P x, \tag{35}$$

where $A \in \mathbb{R}^{n\times n}$ is SPSD and $P \in \mathbb{R}^{n\times n}$ a projection. Then all genuine eigenvalues λ of (35) *fulfill $\lambda \le 1$.*

Proof [17, Lemma 4.12] yields that the infinite eigenspace is

$$V_\infty = \ker(P^T A P) = \ker(P) \oplus (\ker(A) \cap \operatorname{range}(P))$$

and the ambiguous eigenspace turns out to be

$$V_{\text{amb}} := \ker(A) \cap V_\infty = (\ker(A) \cap \ker(P)) \oplus (\ker(A) \cap \operatorname{range}(P)).$$

The latter is a subspace of the above: $V_{\text{amb}} \subset V_\infty$. Since $V_\infty \subset \ker(P^T A P)$, we can eliminate V_∞ and obtain an eigenproblem that has the same finite and non-ambigious eigenvalues as (35). For such an elimination, we need a space splitting $\mathbb{R}^n = V_\infty \oplus V_c$. Here we use some complementary space V_c with the property $V_c \subset \operatorname{range}(P)$ (that is feasible because $\mathbb{R}^n = \ker(P) \oplus \operatorname{range}(P)$). We have the property that $Py = y$ on V_c because P is a projection. Following [17, Principle 4.4], the reduced eigenproblem reads: find $(\lambda, y) \in \mathbb{R} \times V_c$ such that

$$z^T S y = \lambda z^T A y \qquad \forall z \in V_c\,,$$

where S is the Schur complement w.r.t. V_∞ such that $y^T S y \le (x+y)^T A(x+y)$ for all $x \in V_\infty$ and $y \in V_c$. Note that A is definite on V_c because $V_c \subset \mathrm{range}(P)$ but $\ker(A) \cap V_c = \{0\}$ since $V_\infty \supset \ker(A) \cap \mathrm{range}(P)$. Since A is definite on V_c, the right-hand side matrix of the reduced eigenproblem is definite and so we can express the maximal eigenvalue $\lambda_{\max}$ in terms of the Rayleigh quotient. The proof is completed by using the minimizing property of the Schur complement. □

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