Non-geometric Convergence of the Classical Alternating Schwarz Method

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1 Introduction

Let Ω be a domain in $\mathbb{R}^n$ and $f \in L^2(\Omega)$ be a given function. Consider the Laplace problem

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

In error form, the alternating Schwarz method for the solution to (1) is

$$\begin{aligned} \Delta e_1^n &= 0 && \text{in } \Omega_1, & \Delta e_2^n &= 0 && \text{in } \Omega_2,\\ e_1^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_1, & e_2^n &= 0 && \text{on } \partial\Omega \cap \overline{\Omega}_2,\\ e_1^n &= e_2^{n-1} && \text{on } \Gamma_1, & e_2^n &= e_1^n && \text{on } \Gamma_2. \end{aligned} \tag{2}$$

Given any initial guess $e_0 \in V := H_0^1(\Omega)$ and solving iteratively (2), one obtains the sequence $(e_1^n)_{n\in\mathbb{N}^+} \subset H^1(\Omega_1)$ of errors in Ω_1 and the sequence $(e_2^n)_{n\in\mathbb{N}^+} \subset H^1(\Omega_2)$ of errors in Ω_2. Let us define the sequence $(e_k)_{k\in\mathbb{N}^+} \subset V$ as

$$e_k := \begin{cases} e_1^k & \text{in } \overline{\Omega}_1\\ e_2^{k-1} & \text{in } \overline{\Omega} \setminus \Omega_1 \end{cases} \text{ for } k \text{ odd}, \quad \text{and} \quad e_k := \begin{cases} e_2^k & \text{in } \overline{\Omega}_2\\ e_1^{k-1} & \text{in } \overline{\Omega} \setminus \Omega_2 \end{cases} \text{ for } k \text{ even.}$$

We denote by V_1 and V_2 the extensions by zero in Ω of $H_0^1(\Omega_1)$ and $H_0^1(\Omega_2)$. Their orthogonal complements $V_1^\perp$ and $V_2^\perp$ in V with respect to the inner product $\langle\cdot,\cdot\rangle := (\nabla\cdot,\nabla\cdot)_{L^2}$ are of the form

$$V_j^\perp = \{v \in H_0^1(\Omega) \colon \Delta v = 0 \text{ in } \Omega\backslash\overline{\Omega_j}\} \tag{3}$$

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for $j = 1, 2$. It is then possible to show that (2) is equivalent to the alternating projection method (APM), $e_k := P_{V_2^\perp} P_{V_1^\perp} e_{k-1}$, for $k \in \mathbb{N}^+$, where $P_{V_j^\perp}$ denote the orthogonal projections onto $V_j^\perp$, $j = 1, 2$; [11, 5].

For an arbitrary Hilbert space V and two closed subspaces V_1 and V_2, von Neumann [12] and Halperin [10] proved that $e_k \to 0$ whenever $\overline{V_1 + V_2} = V$. Moreover, if $V_1 + V_2$ is closed, then the convergence is geometric, i.e. there exists $\theta < 1$ such that for all $e_0 \in V$ it holds that $\|e_k\| \leq \theta^k \|e_0\|$. In the particular case of only two subspaces V_1 and V_2, it is proven that the optimal θ is $\mathrm{incl}(V_1, V_2)$, with $0 \leq \mathrm{incl}(V_1, V_2) \leq 1$ the inclination between the subspaces V_1 and V_2, and that $\theta = \mathrm{incl}(V_1, V_2) < 1$ if and only if $V_1 + V_2$ is closed; see, e.g., [6, 5].

In the context of Schwarz method, P.L. Lions proves in [11] that an overlapping decomposition $\Omega = \Omega_1 \cup \Omega_2$ guarantees that $\overline{V_1 + V_2} = V$, and gives sufficient conditions for $V_1 + V_2 = \overline{V_1 + V_2}$ to hold; see also [5, Lemma 2.16 and Theorem 2.17]. These conditions hold if the overlap $\Omega_1 \cap \Omega_2$ is a sufficiently regular domain. A natural question arises: what happens if $\Omega_1 \cap \Omega_2$ is not regular enough (e.g., non-Lipschitz)? Is the geometric convergence still guaranteed in this case?

We show in this paper that if $\Omega_1 \cap \Omega_2$ is non-Lipschitz, then $V_1 + V_2$ is not necessarily closed. Classical abstract results state that in this case the APM converges 'arbitrarily slowly' [7, 8, 1]:

Definition 1 (Arbitrarily slow convergence (ASC)) The APM is said to converge arbitrarily slowly if for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $f_n \to 0$ and for all $\varepsilon > 0$ there exists $e_0 \in V$ with $\|e_0\| < \sup_n f_n + \varepsilon$ and $\|e_k\| \geq f_n$ for all n.

An ASC is quite difficult to observe and characterize. Therefore, we introduce the notion of 'non-geometric' convergence:

Definition 2 (Non-geometric convergence (NGC)) The APM is said to converge non-geometrically if there is no $\theta < 1$ such that for all $e_0 \in V$ it holds that $\|e_k\| \leq \theta^k \|e_0\|$. Moreover, we say that a vector $e_0 \in V$ leads to NCG, if there exists no $\theta < 1$ such that $\|e_k\| \leq \theta^k \|e_0\|$.

To the best of our knowledge, the case of a non-closed sum $V_1 + V_2$ is not studied in the literature of classical Schwarz theory. Moreover, also the literature concerning the more general framework of the APM presents surprisingly few results for this problem. The aim of our work is to study ASC and NGC of the classical Schwarz method and hence to shed more light on the issue of 'slow convergence' of the APM. To do so, in Section 2 we present a domain decomposition example that leads to two subspaces V_1 and V_2 whose sum is not closed. Section 3 focuses on theoretical results about NGC and ASC of the APM. In Section 4, we consider again the example from Section 2 and discuss the dependence of the convergence rate on the initial function e_0. Moreover, we precisely characterize a dense subset of the set of all functions leading to geometric convergence. Finally, results of numerical experiments are presented in Section 5.

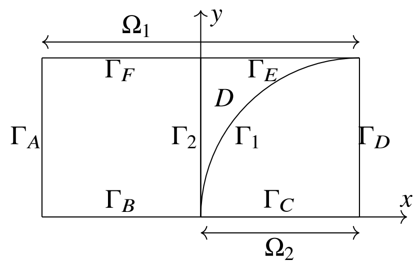

Fig. 1: Decomposition $\Omega = \Omega_1 \cup \Omega_2$ with $D = \{(x, y) \in \Omega\colon x > 0, y > x^\alpha\} = \Omega_1 \cap \Omega_2$ and $\alpha < 1$.

2 Domain decomposition with non-Lipschitz overlap

Consider a domain $\Omega = (-1, 1)\times(0, 1)$ and two subdomains $\Omega_1 = (-1, 0]\times(0, 1)\cup D$ and $\Omega_2 = (0, 1) \times (0, 1)$ with $D = \{(x, y) \in \Omega\colon x > 0, y > x^\alpha\}$ for some $\alpha > 0$. Clearly, the overlapping decomposition $\Omega_1 \cup \Omega_2 = \Omega$ holds, and D is the overlap; see Fig. 1. The following theorem shows that, if $\alpha < 1$ (hence D is a non-Lipschitz domain), then the decomposition $\Omega = \Omega_1 \cup \Omega_2$ leads to two subspaces V_1 and V_2 of V whose sum is not closed.

Theorem 1 (Non-closedness of $\mathbf{V_1 + V_2}$) *Let V_j denote the extension by zero in Ω of $H_0^1(\Omega_j)$ for $j = 1, 2$. Then $\overline{V_1 + V_2} = H_0^1(\Omega)$, but $V_1 + V_2 \neq V$ for any $\alpha < 1$.*

Proof Let $v \in \overline{V_1 + V_2}^\perp$. Then $v \in V_j^\perp$ (see (3)), for $j = 1, 2$. In particular $\Delta v = 0$ in Ω, thus $v = 0$. This proves that $\overline{V_1 + V_2}^\perp = \{0\}$ and the first claim follows.

To prove the second statement, we consider the function $v = (r^\beta \sin\phi)\psi$, where (r, ϕ) denote polar coordinates and $\psi \in C^1(\overline{\Omega})$ is a cut-off function with $\psi = 0$ on $\partial\Omega \setminus \{y = 0\}$ and $\psi = 1$ in $[-2^{-\alpha^{-1}}, 2^{-\alpha^{-1}}] \times [0, \frac{1}{2}]$. A direct calculation shows that $v \in H_0^1(\Omega)$ for $\beta > 0$, and we now prove that $v \notin \overline{V_1 + V_2}$. To do so, assume for the sake of contradiction that there are $v_1 \in V_1$ and $v_2 \in V_2$ such that $v = v_1 + v_2$. Clearly, it must hold that $v_1 = v$ on $\{x = 0\}$ and $v_1 = 0$ on $\{(x, x^\alpha)\colon 0 \le x \le 1\}$. Let $\gamma(y) := y^{\alpha^{-1}}$. Then $v_1(\gamma(y), y) = 0$ and we get $-v_1(0, y) = \int_0^{\gamma(y)} \partial_x v_1(t, y)\, dt$.[1] Hence, we have

$$\|\nabla v_1\|_{L^2}^2 \ge \int_0^{\frac{1}{2}} \int_0^{\gamma(y)} |\partial_x v_1(t, y)|^2\, dt\, dy \ge \int_0^{\frac{1}{2}} \left[\int_0^{\gamma(y)} \partial_x v_1(t, y)\, dt\right]^2 \frac{1}{\gamma(y)}\, dy$$
$$= \int_0^{\frac{1}{2}} \frac{v_1^2(0, y)}{\gamma(y)} = \int_0^{\frac{1}{2}} \frac{y^{2\beta}}{y^{\alpha^{-1}}},$$

[1] Strictly speaking this is not necessarily meaningful due to possible lack of regularity of v_1. However, it is true for smooth functions and therefore one can argue by density.

which implies that $\|\nabla v_1\|_{L^2} = \infty$ if $2\beta - \alpha^{-1} \leq -1$, i.e., if $\alpha \leq \frac{1}{1+2\beta}$. Thus, for any $\alpha < 1$, this shows that $v_1 \notin V_1$ if we choose $\beta > 0$ sufficiently small, which leads to a contradiction. Hence the second claim follows. □

Consider for any $\varepsilon \in (0,1)$ and $\lambda > 0$ the sets

$$X_{\lambda,\varepsilon} := \{u \in H_0^1(\Omega) \colon u(0,y) \geq \lambda y^{\beta} \text{ for a.e. } y \in (0,\varepsilon)\}, \quad (4)$$

where the inequality has to be understood in the sense of traces. Notice that $\cup_{\lambda>0} X_{\lambda,\varepsilon}$ is dense in V for any $0 < \varepsilon < 1$. Moreover, if $\beta = (\alpha^{-1} - 1)/2$, then according to the proof of Theorem 1 it holds that $X_{\lambda,\varepsilon} \subset V \setminus (V_1 + V_2)$. Hence, Theorem 3 implies that any $e_0 \in X_{\lambda,\varepsilon}$ leads to a NGC.

In view of Theorem 1, the geometric convergence of the Schwarz method (as APM) does not hold. This is due to results that we discuss in Section 3.

3 'Slow' convergence in the abstract framework of the APM

Consider an arbitrary Hilbert space $(V, \langle\cdot,\cdot\rangle)$ and two closed subspaces V_1 and V_2 such that $V_1 + V_2 \neq \overline{V_1 + V_2}$. Denote by $\|\cdot\|$ the norm induced by $\langle\cdot,\cdot\rangle$.[2] Does the APM, corresponding to the iteration operator $P_{V_2^\perp} P_{V_1^\perp}$, converge geometrically? The answer is negative and given in Theorem 2.

Theorem 2 (On the geometric convergence of the APM) *Let $V_1, V_2 \subset V$ be closed subspaces of a Hilbert space with $\overline{V_1 + V_2} = V$. Let $\|\cdot\|'$ be the operator norm induced by $\|\cdot\|$. The following statements are equivalent.*

(i) $V_1 + V_2 = V$.
(ii) $\|P_{V_2^\perp} P_{V_1^\perp}\|' < 1$.
(iii) There exists $\theta \in [0,1)$ such that $\forall e_0 \in V$ and $\forall k \in \mathbb{N}$ $\|(P_{V_2^\perp} P_{V_1^\perp})^k e_0\| \leq \theta^k \|e_0\|$.
(iv) For all $e_0 \in V$ there is $\theta_{e_0} \in [0,1)$ such that $\forall k \in \mathbb{N}$ $\|(P_{V_2^\perp} P_{V_1^\perp})^k e_0\| \leq \theta_{e_0}^k \|e_0\|$.

Proof The implication from (i) to (ii) is well known; see, e.g., [11, 5]. Clearly (ii) implies (iii), and (iii) implies (iv). It remains to prove that (iv) implies (i). To do so, let $e_0 \in V$, and denote $e_k = (P_{V_2^\perp} P_{V_1^\perp})^k e_0$. We observe that (iv) implies that $\lim_{k\to\infty} e_k = 0$ and that the series $y := \sum_{k=0}^{\infty} e_k$ is absolutely convergent. Moreover, we have

$$P_{V_2^\perp} P_{V_1^\perp} e_0 = (1 - P_{V_2})(1 - P_{V_1}) e_0 = e_0 - P_{V_2} P_{V_1^\perp} e_0 - P_{V_1} e_0.$$

[2] Notice that we consider here the same notation (namely the symbols V, V_1, V_2, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$) used in the other sections to describe a more abstract setting. However, it is clear from the context whether the notation refers to an abstract Hilbert space setting or to the precise domain decomposition setting.

By induction and using that $\lim_{k\to\infty} e_k = 0$ and that the series $y := \sum_{k=0}^{\infty} e_k$ converges absolutely, we obtain

$$e_0 = (P_{V_1} + P_{V_2}P_{V_1^\perp})e_0 + e_1 = (P_{V_1} + P_{V_2}P_{V_1^\perp}) \sum_{k=0}^{n} e_k + e_{n+1}$$
$$= (P_{V_1} + P_{V_2}P_{V_1^\perp})y \in V_1 + V_2.$$

Since $e_0 \in V$ was arbitrary, the claim follows. □

Theorem 2 implies that, if V_1+V_2 is not closed, then there exists an initial function e_0 such that the APM sequence $(e_k)_{k\in\mathbb{N}}$ does not converge geometrically. The issue of the rate of convergence of the APM when $V_1 + V_2$ is not closed has first been addressed by Franchetti and Light, who prove in [9] the following result.

Theorem 3 (NGC of the APM) *Let $V_1, V_2 \subset V$ be as in Theorem 2 and assume that $V_1 + V_2$ is not closed. Then, for all $e_0 \in V \setminus (V_1 + V_2)$ it holds that $\sum_{k=1}^{\infty} \frac{\|e_k\|}{\sqrt{k}} = \infty$. In particular, the convergence is NGC.*

Theorem 3 states that for any initial function $e_0 \in V \setminus (V_1 + V_2)$ the convergence of the APM is much slower than geometric. Moreover, in the same paper, the authors provide an example of a non-closed sum V_1+V_2 leading to ASC. In 1997, Bauschke, Borwein and Lewis proved in [3] that ASC holds whenever $V_1 + V_2$ is not closed. However, Bauschke, Deutsch and Hundal pointed out later in [4] that the proof of this result given in [3] is erroneous, and they give a different approach to obtain the same result:

Theorem 4 (Dichotomy between ASC and non-closedness of $\mathbf{V_1+V_2}$) *Let $V_1, V_2 \subset V$ be as in Theorem 2. Then, exactly one of the following two statements holds:*

(1) $V_1 + V_2$ is closed. Then the convergence is geometric.
(2) $V_1 + V_2$ is not closed. Then the convergence is arbitrarily slow.

In 2010, Deutsch and Hundal studied ASC for a general class of operators on Banach spaces [7, 8]. Their results include Theorem 4, also in the case of more than two subspaces. Independently, the same results have been proved in 2011 by Badea, Grivaux and Müller [1]. In the same paper it is shown that, if $V_1 + V_2$ is not closed, then, for any positive sequence $(f_n)_{n\in\mathbb{N}}$, the set $\{e_0 \in V : \|e_n\| \geq f_n$ for a. e. $n \in \mathbb{N}\}$ is dense in V.

We have seen that if $V_1 + V_2$ is not closed, then the APM converges arbitrarily slow and the convergence is much slower than geometric at least for any initial vector $e_0 \in V \setminus (V_1 + V_2)$, and that the set of all e_0 leading to ASC is dense in V. However, what is the dependence of the convergence rate on the initial vector e_0? Can one characterize the set of all e_0 leading to geometric convergence? In the papers mentioned above, there are only a few sentences hinting on the dependence of the convergence rate on the starting point e_0. In [2], Badea and Seifert have shown that one can always find a dense subset $W \subset V$ for which 'super-polynomially fast convergence' holds. However, it seems difficult to characterize such a subset in a concrete example. In the following section, we discuss the dependence of the convergence rate on the starting point e_0 for the specific example from Section 2. In particular, we provide rigorous results on the regularity that is needed for an initial function e_0 to lead to geometric convergence, and show that the set of these initial functions is a dense subset of V if the overlap of the domains is not too rough ($\alpha > 1/3$).

4 The dependence of the convergence rate on the initial function

Consider the domain decomposition studied in Section 2 with a non-Lipschitz overlap D. Recall also $V = H^1_0(\Omega)$ and the two subspaces V_1 and V_2 whose orthogonal complements are given in (3). For which initial functions $e_0 \in V$ does the Schwarz method converge geometrically?

Probably the functions that come first to the mind of the reader are the ones in V that vanish on the interface $\Gamma_1 := \overline{\partial\Omega_1 \cap \Omega}$. For these functions, the Schwarz method (2) converges in only one step. Indeed, with $F := \{v \in V : v = 0 \text{ on } \overline{\partial\Omega_1 \cap \Omega}\}$, we see that $\ker(P_{V_2^\perp} P_{V_1^\perp}) = V_1 \oplus (V_1^\perp \cap V_2) = V_1 \oplus \{v \in V : v = 0 \text{ in } \overline{\Omega_1}\} = F$, where we used (3). It is not difficult to see that, if $u \notin F$, then the iteration will not yield the exact result after any finite number of iterations. Moreover, F is not the maximal set of functions that lead to geometric convergence. This is clearly shown by Theorem 5 below. To prove it, we need the following lemma.

Lemma 1 *Let $V_1, V_2 \subset V$ be as in Theorem 2, and let $W \subset V_1 + V_2$ be a closed subspace which is invariant under $P_{V_2^\perp} P_{V_1^\perp}$. Then, there exists $\theta < 1$ such that $\|(P_{V_2^\perp} P_{V_1^\perp})^k e_0\| \leq \theta^k \|e_0\|$ for all $e_0 \in W$.*

Proof The result follows by the same arguments used in [11, Theorem I.1]. □

Theorem 5 (A set of initial functions leading to geometric convergence) *Recall the domain decomposition given in Theorem 1 and the corresponding parameter α. Consider for an arbitrary $\lambda > 0$ the set*

$$W_\lambda := \big\{ v \in V : v(x, y) \leq \lambda y \ \text{ for almost all } (x, y) \in \Omega \big\}.$$

For all $1 > \alpha > 1/3$, the sets W_λ are closed subspaces of $V_1 + V_2$ and invariant under $P_{V_2^\perp} P_{V_1^\perp}$. Moreover, $\cup_{\lambda>0} W_\lambda$ is dense in V, and for any $\lambda > 0$ there exists $\theta < 1$ such that

$$\|(P_{V_2^\perp} P_{V_1^\perp})^k e_0\| \leq \theta^k \|e_0\| \quad \text{for all } e_0 \in W_\lambda.$$

Proof Notice that W_λ are closed subspaces of V and $C_c^\infty(\Omega) \subset \cup_{\lambda>0} W_\lambda$. Hence $\cup_{\lambda>0} W_\lambda$ is dense in V.

To show that $W_\lambda \subset V_1 + V_2$, we define the cut-off function $\eta : \Omega \to \mathbb{R}$ by

$$\eta(x, y) = \begin{cases} 0 & \text{in } \Omega_1 \setminus \Omega_2, \\ 1 & \text{in } \Omega_2 \setminus \Omega_1, \\ x y^{-\alpha^{-1}} & \text{in } D = \Omega_1 \cap \Omega_2. \end{cases}$$

Then, for $(x, y) \in D$ we have

$$|\nabla\eta(x, y)| = \left|\left(y^{-\alpha^{-1}}, -\alpha^{-1}xy^{-\alpha^{-1}-1}\right)\right| \leq C(\alpha)y^{-\alpha^{-1}}. \tag{5}$$

Let now $\lambda > 0$ be fixed and let $w \in W_\lambda$. Then we claim $\eta w \in V_1$ and $(1-\eta)w \in V_2$. Using (5) and recalling that $\alpha > 1/3$, we get

$$\begin{aligned}\|(\nabla\eta)w\|^2_{L^2(\Omega)} &\leq \int_D C(\alpha)\lambda^2 \frac{y^2}{y^{2\alpha^{-1}}} = C(\alpha)\lambda^2 \int_0^1 \int_0^{y^{\alpha^{-1}}} \frac{y^2}{y^{2\alpha^{-1}}}\,dx\,dy \\ &= C(\alpha)\lambda^2 \int_0^1 \frac{y^2}{y^{\alpha^{-1}}}\,dy = C(\alpha)\lambda^2 \frac{1}{3-\alpha^{-1}}.\end{aligned}$$

Noticing $\eta \leq 1$ in Ω, the above estimate shows $\eta w \in V_1$ and $(1-\eta)w \in V_2$.

Next, we show that W_λ are invariant under $P_{V_i^\perp}$, $i = 1, 2$. Let $w \in W_\lambda$. Then $v := P_{V_1^\perp}w$ is the unique function such that $\Delta v = 0$ in Ω_1 with $v = w$ in $\Omega \setminus \Omega_1$. Therefore, since the function $\varphi(x, y) = \lambda y$ is harmonic, the maximum principle implies that $v \leq \varphi$ in Ω_1 and clearly also that $v = w \leq \varphi$ in $\Omega \setminus \Omega_1$. Hence $v \in W_\lambda$. The invariance under $P_{V_2^\perp}$ is analogous. Therefore, we obtain that W_λ is invariant under $P_{V_2^\perp}P_{V_1^\perp}$. Finally, the geometric convergence follows from Lemma 1. □

Theorem 5 says that for $\alpha > 1/3$ we have geometric convergence for all $e_0 \in \cup_{\lambda>0}W_\lambda$. The restriction $\alpha > 1/3$ is optimal. To see it, recall the sets $X_{\lambda,\varepsilon}$ defined in (4) and that $X_{\lambda,\varepsilon} \subset V \setminus (V_1 + V_2)$. Hence, Theorem 3 guarantees that any $e_0 \in X_{\lambda,\varepsilon}$ leads to a NGC. However, for $\alpha \leq 1/3$, $X_{\lambda,\varepsilon}$ and W_λ have non-trivial intersections. Therefore, if $\alpha \leq 1/3$, then there exists $e_0 \in W_\lambda$, in particular $e_0 \in W_\lambda \cap X_{\lambda,\varepsilon}$, that leads to NGC.

5 Numerical experiments

In this section, we present a numerical study of the NGC of the Schwarz method corresponding to the domain decomposition given in Fig. 1. The (monodomain) problem is discretized by linear finite elements using the software Freefem. The discrete meshes for $\Omega_1 \setminus \Omega_2$, D and $\Omega_2 \setminus \Omega_1$ are obtained by the mesh generator of Freefem where we discretized the boundary components Γ_A, Γ_D, Γ_E and Γ_F with 10 points and Γ_B, Γ_C, Γ_1 and Γ_2 with $10N$ points with a positive integer N. This choice is motivated by the higher accuracy needed close to the singularity point of ∂D. The results of our numerical experiments are shown in Fig. 2, where we plot the value $1 - \|e_n\|/\|e_{n-1}\|$ for the iteration count $n = 1, \dots, 2000$. The numerical procedure is stopped only if $\|e_n\| < 10^{-16}$ or if the value $1 - \|e_n\|/\|e_{n-1}\|$ becomes too small (or negative). Clearly, if $1 - \|e_n\|/\|e_{n-1}\|$ becomes constant as n grows, then the method reached a geometric convergence regime. On the other hand, if $1 - \|e_n\|/\|e_{n-1}\| \to 0$ as n grows, then the method converges non-geometrically. Motivated by Theorems 1 and 5, we study the numerical behavior of the Schwarz method for an overlap characterized by $\alpha = \frac{\theta}{2} + \frac{1-\theta}{3}$ for different θ in $[0, 1]$, an initial guess $e_0 \in W_1$, and different N. In particular, according to Theorem 5, we expect

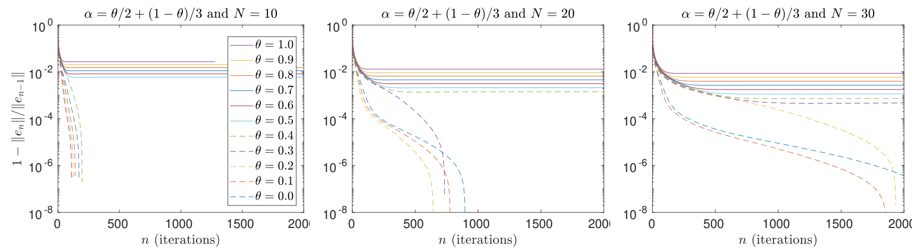

Fig. 2: Convergence behavior of the Schwarz method. The value $1 - \|e_n\|/\|e_{n-1}\|$ is shown for $N = 10$ (left), $N = 20$ (center), $N = 30$ (right).

geometric convergence for any $\theta \in (0, 1]$ and NGC for $\theta = 0$. In Fig. 2, we see that for $N = 10$ the Schwarz method is numerically geometric convergent for $\theta \in [1/2, 1]$ (solid lines), but not for $\theta < 1/2$ (dashed lines). However, if one refines the mesh with $N = 20$ and $N = 30$, then geometric convergence holds also for $\theta = 0.4$ and $\theta = 0.3$. Moreover, for bigger N also the curves for smaller θ are less steep and show a behavior closer to the proved geometric convergence. Finally, we wish to remark that, according to our experience, a more precise numerical description of the correct theoretical behavior for θ approaching zero is hard. This is mainly due to the non-Lipschitz overlap, where a correct numerical discretization is not trivial. Therefore, further studies would be needed. These are beyond the scope of this short manuscript, and we hope to consider them in future work.

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