A Two-level Overlapping Schwarz Method Using Energy Minimizing Multiscale Finite Element Functions

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1 Introduction

In this paper, a two-level overlapping Schwarz algorithm is proposed for solving finite element discretization of the following model problem,

$$\int_\Omega \rho(x)\nabla u(x)\cdot\nabla v(x)\,dx = \int_\Omega f(x)v(x)\,dx, \quad \forall v(x)\in H_0^1(\Omega), \tag{1}$$

where $u(x)$ is in the Sobolev space $H_0^1(\Omega)$, the space of integrable functions with their weak derivatives of the first order being square integrable. The coefficient $\rho(x)$ can be highly varying and random with high contrast inside Ω. For such model problems, the standard coarse problem in the two-level overlapping Schwarz algorithm often fails and a more robust coarse problem is required.

A new idea here is that we will form the coarse problem by utilizing multiscale finite element functions proposed in [2]. The multiscale finite element functions are obtained by solving certain constrained energy minimizing problems where the constraints are formed by using a set of selected eigenvectors from a generalized eigenvalue problem in each overlapping subdomain. The generalized eigenvalue problem is similar to that considered in [4]. In their work, the eigenvectors are directly used to form the coarse basis functions and the resulting preconditioner is

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shown to have a condition number robust with respect to the contrast of the coefficient but dependent on the overlapping width in the subdomain partition.

The advantage in our new method is that the resulting coarse basis functions provide a more robust coarse problem and thus the condition number of the resulting preconditioner becomes robust to the overlapping width as well as the contrast in the model coefficient. The idea was originated from [2] where it was shown that such constrained energy minimizing finite element functions can approximate the solution of the model problem with the errors dependent on the coarse mesh size but independent of the contrast in the model coefficient. One disadvantage of our approach is that the constrained minimization problem needs to be solved in the whole domain. To overcome this heavy cost, we can localize the minimization problem on each subregion and use the solution to form the coarse basis functions. In [2], it was shown that the error between the full minimization solution and the localized one decays exponentially as a function of the subregion size. From that result, we can expect that the proposed preconditioner with these localized coarse basis functions also share the same good quality, i.e., is robust with respect to the overlapping width as well as the contrast in the coefficient. More detailed analysis and extensive numerical tests will be given later in a full version of this short proceeding paper [14].

We note that the similar idea, enriching the coarse problem by using adaptively chosen eigenvectors from generalized eigenvalue problems on each subdomain or on each subdomain interface, has been also extensively developed for other types of domain decomposition algorithms, such as, FETI(-DP), BDD(C), and additive-Schwarz algorithms, see [10, 3, 9, 1, 8, 11, 13, 5, 6, 7].

2 Multiscale finite element basis functions

For finite element approximation of the solution of the model problem (1), we introduce a piecewise linear conforming finite element space V_h ($\subset H_0^1(\Omega)$) defined for a triangulation $\mathcal{T}_h$ of Ω. We assume that the triangulation is fine enough to resolve the variation in the coefficient $\rho(x)$ in the following sense,

$$\max_{\tau\in\mathcal{T}_h} \frac{\max_{x\in\tau}\rho(x)}{\min_{x\in\tau}\rho(x)} \leq C, \tag{2}$$

for a given constant C.

We partition the domain Ω into overlapping subdomains $\{\Omega_i\}_{i=1}^N$ where each Ω_i is a connected union of triangles in $\mathcal{T}_h$. For a given overlapping subdomain partition, we introduce a partition of unity $\{\theta_i(x)\}_{i=1}^N$, where $\sum_{i=1}^N \theta_i(x) = 1$ and each $\theta_i(x)$ is supported in Ω_i.

We consider the following generalized eigenvalue problem in each subdomain Ω_i:

$$a_i(\phi_j^{(i)}, w) = \lambda_j^{(i)} s_i(\phi_j^{(i)}, w), \quad \forall w \in V(\Omega_i),$$

where $V(\Omega_i)$ is the restriction of the functions in V_h to the subdomain Ω_i and the local bilinear forms are defined as

$$a_i(v,w) := \int_{\Omega_i} \rho(x)\nabla v \cdot \nabla w \, dx, \quad s_i(v,w) := \int_{\Omega_i} \rho(x)|\nabla\theta_i(x)|^2 v\, w \, dx.$$

We let the eigenvalues $\lambda_j^{(i)}$ be arranged in ascending order and choose the eigenvectors $\phi_j^{(i)}$ with their associate eigenvalues $\lambda_j^{(i)}$ smaller than a given tolerance value Λ, i.e., $\lambda_j^{(i)} < \Lambda$. We use the notation l_i for the number of such eigenvectors.

We first form an auxiliary multiscale finite element space by collecting all the selected eigenvectors

$$V_{aux} := \left\{ \phi_j^{(i)} \,|\, i = 1, \cdots, N, \;\; j = 1, \cdots, l_i \right\}.$$

We introduce the following definition for a function v in V_h: v is $\phi_j^{(i)}$-orthogonal if $s_i(v, \phi_j^{(i)}) = 1$ and $s_k(v, \phi_l^{(k)}) = 0$ for $k \neq i$, $l = 1, \cdots, l_k$, $k = i$, $l = 1, \cdots, j-1, j+1, \cdots, l_i$. We obtain a set of coarse basis functions $\psi_j^{(i)}$ as the solution of the following constrained minimization problem:

$$\psi_j^{(i)} = \text{argmin}\{a(\psi,\psi) \,|\, \psi \in V_h, \;\; \psi \text{ is } \phi_j^{(i)}\text{-orthogonal.}\}, \tag{3}$$

where

$$a(u,v) := \int_\Omega \rho(x)\nabla u \cdot \nabla v \, dx.$$

The coarse space V_{glb} defined as a span of these functions $\psi_j^{(i)}$ can be shown to have the following property: V_{glb} is the orthogonal complement of $\widetilde{V}$ with respect to the bilinear form $a(\cdot,\cdot)$, where the space $\widetilde{V}$ is defined by

$$\widetilde{V} := \{v \in V_h \,|\, s_i(v, \phi_j^{(i)}) = 0, \, i = 1, \cdots, N, j = 1, \cdots, l_i\}. \tag{4}$$

As proposed in [2], we can consider a more practical relaxed constrained energy minimizing problem:

$$\psi_{r,j}^{(i)} = \text{argmin} \left\{ a(\psi,\psi) + s(\pi\psi - \phi_j^{(i)}, \pi\psi - \phi_j^{(i)}) \,|\, \forall \psi \in V_h \right\}, \tag{5}$$

where

$$\pi\psi := \sum_{i=1}^{N} \sum_{j=1}^{l_i} s_i(\psi, \phi_j^{(i)}) \phi_j^{(i)}, \quad s(v,w) = \sum_{i=1}^{N} s_i(v,w).$$

We note that the function $\psi_{r,j}^{(i)}$ in (5) will satisfy the same orthogonal property with respect to the resulting coarse space as that from (3) and it can be found by solving the following problem: find $\psi_{r,j}^{(i)}$ in V_h such that

$$a(\psi_{r,j}^{(i)}, v) + s(\pi\psi_{r,j}^{(i)}, \pi v) = s(\phi_j^{(i)}, \pi v), \quad \forall v \in V_h. \tag{6}$$

Let V_{glb} be the coarse space obtained from the $\psi_{r,j}^{(i)}$ of the above relaxed constrained problem. From (6), the following orthogonal property holds

$$a(\psi_{r,j}^{(i)}, v) = 0, \quad \forall v \in \widetilde{V}$$

and we thus obtain

$$V_h = \widetilde{V} \oplus V_{glb}.$$

We note that $V_h = V_{glb}^{\perp} \oplus V_{glb}$ and that $\widetilde{V}$ is contained in $V_{glb}^{\perp}$. Since the dimension of $V_{glb}^{\perp}$ is equal to the dimension of $\widetilde{V}$, see (4), we have $\widetilde{V} = V_{glb}^{\perp}$. In the following, we will use the space V_{glb} defined by the $\psi_{r,j}^{(i)}$ in (6) as the coarse space of the two-level overlapping Schwarz algorithm.

3 Two-level overlapping Schwarz algorithm

In this section, we propose a two-level overlapping Schwarz preconditioner for the finite element discretization of the model problem in (1), i.e.,

$$Au = b.$$

We introduce the local finite element space $V_0(\Omega_i)$, which is the restriction of functions in V_h to Ω_i and vanishing on $\partial\Omega_i$. We define the local problem matrix by

$$\langle A_i v, w\rangle := \int_{\Omega_i} \rho(x)\nabla v \cdot \nabla w \, dx, \forall v, \ w \in V_0(\Omega_i).$$

We introduce the restriction R_i from V_h to $V_0(\Omega_i)$ and denote by R_i^T the extension from $V_0(\Omega_i)$ by zero to V_h . We define the coarse problem matrix by

$$A_0 = a(\psi_{r,j}^{(i)}, \psi_{r,q}^{(k)}), \ i, k = 1, \cdots, N, \text{ and } j = 1, \cdots, l_i, \ q = 1, \cdots, l_k.$$

We note that the size of the matrix A_0 is identical to the dimension of V_{glb}. We introduce R_0 as the matrix with rows consisting of the nodal values of $\psi_{r,j}^{(i)}$ in V_{glb} and define the two-level overlapping Schwarz preconditioner as

$$R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i. \tag{7}$$

For the overlapping Schwarz method, the upper bound estimate can be obtained from a coloring argument. We will only need to work on the following lower bound estimate, see [12]:

Lemma 1 *Let the triangulation $\mathcal{T}_h$ satisfy the assumption in* (2)*. For any given u in V_h, there exists $\{u_i\}_{i=0}^N$, $u_i \in V_0(\Omega_i)$, $i \geq 1$ and $u_0 \in V_{glb}$, such that*

$$u = u_0 + \sum_{i=1}^{N} u_i$$

and

$$a(u_0, u_0) + \sum_{i=1}^{N} a(u_i, u_i) \leq C_0^2 a(u, u)$$

with the constant C_0 independent of $\rho(x)$ and the overlapping width in the subdomain partition.

Proof For the proof we will choose u_0 as the solution of

$$a(u_0, v) = a(u, v), \quad \forall v \in V_{glb}$$

and choose u_i as

$$u_i = I^h(\theta_i(u - u_0)),$$

where $I^h(v)$ denotes the nodal interpolant of v to the space V_h. We note that $u - u_0$ is in $V_{glb}^{\perp}$ and also in $\widetilde{V}$ since $V_{glb}^{\perp} = \widetilde{V}$.

We can see that u_i is supported in Ω_i by construction and then obtain

$$\begin{aligned}
\sum_{i=1}^{N} a(u_i, u_i) &= \sum_{i=1}^{N} \int_{\Omega_i} \rho |\nabla I^h(\theta_i(u - u_0))|^2 \, dx \\
&\leq C_I \sum_{i=1}^{N} \int_{\Omega_i} \rho |\nabla(\theta_i(u - u_0))|^2 \, dx \\
&\leq 2C_I \sum_{i=1}^{N} \left(\int_{\Omega_i} \rho |\nabla(u - u_0)|^2 \, dx + \int_{\Omega_i} \rho |\nabla \theta_i|^2 (u - u_0)^2 \, dx \right) \\
&\leq 2C_I \sum_{i=1}^{N} (1 + \Lambda^{-1}) \int_{\Omega_i} \rho |\nabla(u - u_0)|^2 \, dx
\end{aligned}$$

where the constant C_I depends on the stability of the interpolation I^h and the constant C depends on the number of overlapping subdomains intersecting with Ω_i. In the above, we used the assumption (2) on $\mathcal{T}_h$ in the first inequality, and also that $u - u_0$ in $V_{glb}^{\perp} (= \widetilde{V})$ and thus get the third inequality with Λ^{-1}. Using that $a(u - u_0, u - u_0) + a(u_0, u_0) = a(u, u)$, we obtain the resulting bound. □

Theorem 1 *For the proposed preconditioner, the condition number bound is obtained as*

$$\kappa((R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i) A) \leq C_1 C_0^{-2},$$

where C_1 is the constant in the coloring argument, and C_0 is the constant in Lemma 1.

We note that the constant $C_0^2 = 2CC_I(1+\Lambda^{-1})$ is independent of $\rho(x)$ as well as the overlapping width, which is improvement over the previous work in [4].

On the other hand, the computation of $\psi_{r,j}^{(i)}$ requires us to solve the relaxed constrained minimization problem in the global space V_h. In practice, we can solve the same problem in a subspace of V_h, where the functions are restricted to a subregion $\widetilde{\Omega}_i$ containing Ω_i. In more detail, we solve

$$\psi_{j,ms}^{(i)} = \text{argmin}\left\{a(\psi,\psi) + s(\pi(\psi)-\phi_j^{(i)}, \pi(\psi)-\phi_j^{(i)}) \mid \forall \psi \in V_h \bigcap H_0^1(\widetilde{\Omega}_i)\right\}.$$

From the above minimization problem, we obtain $\psi_{j,ms}^{(i)}$ and denote by $\Psi_{j,ms}^{(i)}$, the extension of $\psi_{j,ms}^{(i)}$ by zero to a function in V_h. We then define V_{ms} by

$$V_{ms} := \text{span}\{\Psi_{ms,j}^{(i)} \mid i = 1,\cdots,N,\ j = 1,\cdots,l_i\}.$$

We can propose the following more practical preconditioner

$$M_{ms}^{-1} = \sum_{i=1}^{N} R_i^T A_i^{-1} R_i + R_{0,ms}^T A_{0,ms}^{-1} R_{0,ms}, \tag{8}$$

where $A_{0,ms}$ and $R_{0,ms}$ are defined similarly as before by replacing V_{glb} with V_{ms}.

4 Numerical results

In Table 1, we present some numerical results for a 2D model problem. We use the coarse problem obtained from the more practical space V_{ms}. Though we do not have an estimate of the condition numbers for this case, we can expect a similar performance to that with V_{glb}. The domain Ω is a unit square partitioned into $N \times N$ uniform squares. Each square is partitioned into uniform triangles with m elements on each edge of a square where the triangles form a mesh, $\mathcal{T}_h$. Each square is extended by d layers of fine triangles and the extended squares form the overlapping subdomain partition. In our experiment, we consider a random coefficient with its value varying between 10^{-3} to 10^3 inside the domain, and show the number of iterations and the number of primal unknowns for various subdomain partitions and for various overlapping width d. The minimization problem is solved in a smaller region $\widetilde{\Omega}_i$, which is obtained by extending each square by only one layer of neighboring squares. We can observe that the proposed method is robust with respect to the overlapping width d as well as the variation in $\rho(x)$.

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Table 1: Performance of the proposed method with $\Lambda = (1 + \log m)$: d (number of elements in the overlap) $iter$ (number of iterations), $\lambda_{\min}$ (minimum eigenvalues), $\lambda_{\max}$ (maximum eigenvalues), and pD (number of coarse basis functions per subdomain).

$N(m)$	d	$iter$	λ_{min}	λ_{max}	pD
3(10)	1	22	0.60	4.33	4.67
	2	23	1.00	4.78	6.67
	3	23	1.00	4.99	7.78
	4	23	1.00	4.99	11.11
	5	23	1.00	4.99	13.22
4(10)	1	24	0.65	4.30	4.63
	2	24	0.99	4.86	7.38
	3	24	1.00	4.98	9.94
	4	24	1.00	4.99	12.13
	5	24	1.00	4.99	13.94
5(10)	1	31	0.47	4.56	4.40
	2	25	0.87	4.96	6.12
	3	25	1.00	4.99	8.36
	4	26	0.87	4.99	10.52
	5	24	1.00	4.99	12.40

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