Coarse Spaces for Nonlinear Schwarz Methods on Unstructured Grids

Alexander Heinlein and Martin Lanser

1 Introduction

We are concerned with the solution of nonlinear problems

$$F(u) = 0 \tag{1}$$

in some finite element space V. The function $F : V \to V'$ is obtained by a finite element discretization of a nonlinear partial differential equation (PDE) on a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. To solve (1), we consider nonlinear domain decomposition methods of the Schwarz type, e.g., ASPIN (Additive Schwarz Preconditioned Inexact Newton) [1, 10] or RASPEN (Restricted Additive Schwarz Preconditioned Exact Newton) [3]. More precisely, we suggest a new approach to implement a second level or coarse level into RASPEN, which is different to FAS-RASPEN (Full Approximation Scheme - RASPEN) introduced in [3]. The coarse space is applied multiplicatively, similar to the application of multiplicative nonlinear corrections in MSPIN (Multiplicative Schwarz Preconditioned Inexact Newton); see, e.g., [9]. Therefore, we consider a standard Lagrangian coarse space as well as multiscale coarse spaces that can also be constructed for unstructured meshes and unstructured domain decompositions, e.g., decompositions obtained using METIS [8]. We compare our new approaches for the example of homogeneous and heterogeneous p-Laplace equations; see section 2. In section 3, we first describe the one level RASPEN method and our approach to implement a multiplicative second level for ASPIN and RASPEN. Second, we define three different coarse spaces - one based on a $\mathbb{P}1$ discretization on a coarse mesh and the other two based on MsFEM (Multiscale

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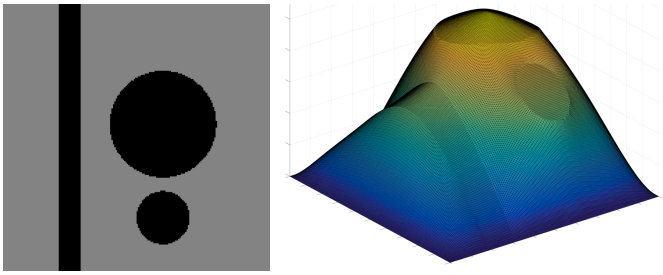

Fig. 1: Left: Definition of Ω_R (black part); **Right:** Solution of equation (2) with coefficient functions defined in (3).

Finite Element Method) [7] type discretizations on the subdomains. The MsFEM coarse spaces can easily be used in the case of unstructured decompositions and differ only in the chosen extensions from the interface to the interior parts of the nonoverlapping domain decomposition. Finally, we present numerical results considering homogeneous and heterogeneous model problems in section 4.

2 Model Problem

We consider the nonlinear model problem:

$$\begin{aligned} -\alpha\Delta_p u - \beta\Delta_2 u &= 1 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2}$$

with the scaled p-Laplace operator $\alpha\Delta_p u := \operatorname{div}(\alpha|\nabla u|^{p-2}\nabla u)$ for $p \geq 2$ and the coefficient functions $\alpha, \beta : \Omega \to \mathbb{R}$. For all computations in this paper, we always use the unit square $\Omega = [0,1] \times [0,1]$ as the computational domain. However, our approach is not restricted to this case. We consider two different coefficient distributions: a homogeneous p-Laplace equation, i.e., $\alpha(x) = 1$ and $\beta(x) = 0$ for all $x \in \Omega$, and a heterogeneous problem with a channel and two circular inclusions carrying different coefficients than the remainder of Ω, i.e.,

$$\alpha(x) = \begin{cases} 1\,000 \text{ if } x \in \Omega_R, \\ 0 \quad \text{elsewhere}, \end{cases} \qquad \beta(x) = \begin{cases} 0 \text{ if } x \in \Omega_R, \\ 1 \text{ elsewhere}. \end{cases} \tag{3}$$

The set Ω_R and the solution of the corresponding heterogeneous model problem are depicted in Figure 1. If not stated otherwise, p is always chosen as 4.

With a standard finite element discretization of a variational formulation of (2), we can derive the nonlinear discrete problem

$$K(u) - f = 0 :\Leftrightarrow F(u) = 0. \tag{4}$$

Let us remark that (4) is linear for $p = 2$. We define the corresponding equation

$$K^{lin}u - f = 0, \tag{5}$$

where K^{lin} is equivalent to the stiffness matrix of the (scaled) diffusion equation.

3 The RASPEN Method

In this section, we provide a brief description of the RASPEN method, which is based on the ASPIN algorithm; see [1, 3] for a more detailed description and a local convergence analysis. As all nonlinear domain decomposition approaches, RASPEN is based on a reformulation of (1) using a decomposition of the underlying nonlinear PDE. In the case of RASPEN, a nonlinear system

$$G(F(u)) =: \mathcal{F}(u) = 0 \tag{6}$$

is derived, where the nonlinear left-preconditioner G is given implicitly. We consider a decomposition of Ω into nonoverlapping subdomains Ω_i, $i = 1, ..., N$, and, by adding layers of finite elements, we obtain overlapping subdomains Ω_i', $i = 1, ..., N$. We denote the local finite element spaces associated with the overlapping subdomains by V_i, $i = 1, ..., N$. With standard restriction operators $R_i : V \to V_i$ and corresponding prolongation operators $P_i := R_i^T$ we can define nonlinear local corrections $T_i(u)$ by

$$R_i F(u - P_i T_i(u)) = 0, \ i = 1, ..., N. \tag{7}$$

Using restricted prolongation operators $\widetilde{P}_i$, $i = 1, ..., N$, which fulfill the condition $\sum_{i=1}^{N} \widetilde{P}_i R_i = I$, we can define the nonlinear reformulation

$$\mathcal{F}_{RA}(u) := \sum_{i=1}^{N} \widetilde{P}_i T_i(u). \tag{8}$$

of (1). Let us remark that (8) and (1) have the same solution; see [1, 3]. In the RASPEN method, (8) is solved using Newton's method, i.e., using the iteration

$$u^{(k+1)} = u^{(k)} - \left(D\mathcal{F}_{RA}(u^{(k)})\right)^{-1} \mathcal{F}_{RA}\left(u^{(k)}\right), \tag{9}$$

with the jacobian

$$D\mathcal{F}_{RA}(u) = \sum_{i=1}^{N} \widetilde{P}_i DT_i(u) = \sum_{i=1}^{N} \widetilde{P}_i \left(R_i DF(u_i) P_i\right)^{-1} R_i DF(u_i) =: \sum_{i=1}^{N} Q_i(u_i). \tag{10}$$

Here, we have $u_i = u - P_i T_i(u)$ and $DT_i(u)$ is obtained by deriving (7). Let us remark that, in each Newton iteration and on each overlapping subdomain, the local nonlinear problem (7) has to be solved for $T_i(u^{(k)})$. This can again be done using Newton's method. The necessary local Newton iterations can be carried out in parallel. We

distinguish in this paper between outer iterations, i.e., global Newton iterations as in (9), and inner iterations, i.e., local Newton iterations on the subdomain problems to compute the local nonlinear corrections $T_i(u_i)$.

3.1 A Multiplicative Coarse Space

In general, there are several approaches to implement a second level for RASPEN or ASPIN. A simple additive coarse space is suggested in [10] for ASPIN, and a multiplicative coarse space using an FAS approach is used in [3]. We choose a slightly different multiplicative approach not relying on FAS. Our coarse correction is applied after the local corrections, but different variants, i.e., applying the coarse correction before the local corrections as well as a symmetric variant doing both are suggested in [6]. All these variants can analogously be applied to ASPIN, but, for the moment, we restrict ourselves to RASPEN due to space limitations. In [6], we also discuss the differences between our proposed methods and, e.g., FAS-RASPEN, in detail. Now, let V_0 be a discrete coarse space, $R_0 : V \to V_0$ a corresponding restriction, and $P_0 := R_0^T$. Note that the columns of P_0 are just representations of the coarse basis functions on the fine mesh. The nonlinear coarse problem is given by $R_0 F(P_0 u_0)$ using a simple Galerkin approach. The nonlinear coarse correction $T_0(u)$ is then implicitly given by

$$R_0 F(u - P_0 T_0(u)) = 0. \tag{11}$$

Let us remark that the coarse correction $T_0(u)$ is computed using Newton's method in our implementation. The corresponding residual and tangential matrix of equation (11) have to be assembled on the fine grid, which can of course be done in parallel on the subdomains. Also the restriction of the residual as well as the Galerkin product necessary to form the coarse tangential matrix can be efficiently computed in parallel; see, e.g., [5, Sections 4.4 and 4.5]. There, it is also described how the coarse basis functions, i.e., the columns of P_0, can be computed in a scalable fashion; in particular, [5, Section 4.4] deals with GDSW coarse basis functions, however the coarse basis functions introduced in section 3.2 can be computed in parallel in the same way.

We can now define the two-level RASPEN method by

$$\mathcal{F}_{2l}(u) := \sum_{i=1}^{N} \widetilde{P}_i T_i(u) + P_0 T_0(u - \sum_{i=1}^{N} \widetilde{P}_i T_i(u)). \tag{12}$$

Note that the coarse correction is here applied multiplicatively after the local corrections $T_i(u_i)$. A linearization with Newton's method leads to

$$u^{(k+1)} = u^{(k)} - \left(D\mathcal{F}_{2l}(u^{(k)}) \right)^{-1} \mathcal{F}_{2l}\left(u^{(k)} \right),$$

where

$$
\begin{aligned}
D\mathcal{F}_{2l}(u) &= \sum_{i=1}^{N} \widetilde{P}_i DT_i\,(u) + P_0 DT_0(u - \sum_{i=1}^{N} \widetilde{P}_i T_i(u)) \left(I - \sum_{i=1}^{N} \widetilde{P}_i DT_i(u)\right) \\
&= \sum_{i=1}^{N} Q_i(u_i) + Q_0(v_0)(I - \sum_{i=1}^{N} Q_i(u_i)) \\
&= Q_0(v_0) + (I - Q_0(v_0)) \sum_{i=1}^{N} Q_i(u_i).
\end{aligned} \tag{13}
$$

Here, we have $v_0 = u - \sum_{i=1}^{N} P_i T_i(u) - P_0 T_0(u - \sum_{i=1}^{N} P_i T_i(u))$ and $u_i = u - P_i T_i(u)$. The projection operators $Q_i(u_i)$, $i = 1, ..., N$ are defined in (10) and

$$
Q_0(v_0) := P_0 \left(R_0 DF(v_0) P_0\right)^{-1} R_0 DF(v_0)
$$

is defined analogously and obtained by deriving (11). Additionally to the local Newton iterations, Newton's method is used to compute the coarse correction (11) in each outer iteration. We refer to this iterations as coarse iterations.

3.2 Different Coarse Basis Functions

We consider three different coarse spaces. The simplest one is a Lagrangian coarse space based on a coarse triangular mesh. Therefore, for a structured domain decomposition into square subdomains, each subdomain is split into two triangular finite elements. The coarse basis functions are just piecewise linear ($\mathbb{P}1$) nodal basis functions corresponding to this triangulation. In general, this coarse space relies on the availability of a suitable coarse triangulation. Therefore, we only use it for structured domain decompositions.

For arbitrary domain decompositions, we consider energy-minimizing coarse spaces of MsFEM [7] type. They are also related to reduced dimension GDSW coarse spaces [2]. As in those approaches, we use a nodal basis, i.e., containing one basis function $\Phi^{(j)}$, $j = 1, ..., N_V$, corresponding to each of the N_V vertices of the domain decomposition. Collecting the vectors $\Phi^{(j)}$ as columns in the matrix Φ, we obtain the restriction to the coarse space $R_0 := \Phi^T$. In particular, we construct the coarse basis functions such that they form a partition of unity on all subdomains which do not touch the Dirichlet boundary. This can be achieved by building a partition of unity on the interface of those subdomains and then extending the interface values to the interior in an energy-minimizing way.

To define the interface part $\Phi_\Gamma^{(j)}$ of the basis function $\Phi^{(j)T} = (\Phi_I^{(j)T}, \Phi_\Gamma^{(j)T})$ corresponding to a vertex $\mathcal{V}_j$, let $\mathcal{E}_k$ be one of the adjacent open edges and $\mathcal{V}_l$ the other vertex adjacent to $\mathcal{E}_k$. Then, we set $\Phi_\Gamma^{(j)}(\mathcal{V}_j) = 1$ and $\Phi_\Gamma^{(j)}(x) = 1 - \frac{||x-\mathcal{V}_j||}{||x-\mathcal{V}_j||+||x-\mathcal{V}_l||}$ for any $x \in \mathcal{E}_k$. We proceed equivalently with all other edges

adjacent to $\mathcal{V}_j$ and define $\Phi_\Gamma^{(j)}$ as zero on the remaining interface. This results in a partition of unity on the interface, even for a METIS decomposition.

As already stated, the interior values $\Phi_I^{(j)}$ are then computed by energy-minimizing extensions. In order to do so, we propose the use of energy functionals corresponding to related linear problems. In the first alternative, we compute discrete harmonic extensions with respect to the linear operator K^{lin}; see (5). Therefore, we consider the block structure

$$K^{lin} = \begin{pmatrix} K_{II}^{lin} & K_{I\Gamma}^{lin} \\ K_{\Gamma I}^{lin} & K_{\Gamma\Gamma}^{lin} \end{pmatrix}$$

and compute the values in the interior degrees of freedom by

$$\Phi_I^{(i)} = -\left(K_{II}^{lin}\right)^{-1} K_{I\Gamma}^{lin} \Phi_\Gamma^{(i)},\ i = 1, ..., N_V\,.$$

Alternatively, we use the tangential matrix for the initial value $u^{(0)}$, i.e.,

$$DK(u^{(0)}) = \begin{pmatrix} DK(u^{(0)})_{II} & DK(u^{(0)})_{I\Gamma} \\ DK(u^{(0)})_{\Gamma I} & DK(u^{(0)})_{\Gamma\Gamma} \end{pmatrix},$$

to compute the energy-minimizing extensions. In particular, we then define the extension to the interior of the subdomains by

$$\Phi_I^{(i)} = -\left(DK(u^{(0)})_{II}\right)^{-1} DK(u^{(0)})_{I\Gamma} \Phi_\Gamma^{(i)},\ i = 1, ..., N_V\,.$$

In general, this is advantageous since it only depends on the nonlinear operator F and no linear Laplacian has to be assembled additionally.

Let us remark that the energy-minimizing basis functions can be computed locally by the solution of linear problems on the interior part of the nonoverlapping subdomains. Also, they are zero on all subdomains not adjacent the corresponding vertex by construction, and therefore, no extensions have to be computed on the remaining subdomains. All three coarse spaces build a partition of unity on all subdomains which do not touch the Dirichlet boundary. This property is crucial for a good linear coarse space. All coarse spaces have the same size and therefore have the same computational cost per nonlinear or linear iteration; only the costs for the construction of the energy-minimizing coarse basis functions are higher.

4 Numerical Results

For all tests and all methods, we choose the same initial value $u^{(0)}(x, y) = xy(x-1)(y-1)$ and the same relative stopping tolerance, i.e., we stop the outer iteration if $F(u^{(k)})/F(u^{(0)}) < 1e-6$. All inner or, respectively, coarse iterations

Table 1: Homogeneous p-Laplace: Comparison of different coarse spaces for regular and METIS domain decompositions; best results for the largest experiment are marked in bold; *outer it.* gives the number of global Newton iterations; *inner it.* gives the number of local Newton iterations summed up over the outer Newton iterations (average over subdomains); *coarse it.* gives the number of nonlinear iterations on the second level summed up over the outer Newton iterations; *GMRES it.* gives the number of GMRES iterations summed up over the outer Newton iterations.

p-Laplace homogeneous $p = 4$; $H/h = 32$ for regular domains; overlap $\delta = 2$;									
		Regular				METIS			
N	**RASPEN Coarse Space**	**outer it.**	**inner it. (avg.)**	**coarse it.**	**GMRES it. (sum)**	**outer it.**	**inner it. (avg.)**	**coarse it.**	**GMRES it. (sum)**
9	-	5	25.9	-	99	7	41.4	-	238
	$\mathbb{P}1$	5	30.2	17	88	-	-	-	-
	$DK(u^{(0)})$ ext.	5	30.7	16	83	5	31.3	22	123
	K^{lin} ext.	5	29.9	16	83	5	30.7	19	121
16	-	14	73.8	-	358	11	62.8	-	458
	$\mathbb{P}1$	6	32.4	20	122	-	-	-	-
	$DK(u^{(0)})$ ext.	7	38.9	30	140	7	36.8	27	180
	K^{lin} ext.	5	30.6	18	99	6	32.5	21	152
25	-	6	28.4	-	201	12	57.6	-	578
	$\mathbb{P}1$	5	27.4	18	116	-	-	-	-
	$DK(u^{(0)})$ ext.	5	27.6	19	108	5	28.6	20	126
	K^{lin} ext.	5	27.2	18	108	6	31.4	22	151
36	-	15	66.9	-	563	11	53.1	-	617
	$\mathbb{P}1$	6	30.6	21	145	-	-	-	-
	$DK(u^{(0)})$ ext.	7	34.3	30	164	6	30.4	23	155
	K^{lin} ext.	5	28.7	19	117	6	30.0	21	152
49	-	6	29.0	-	268	13	60.9	-	811
	$\mathbb{P}1$	**5**	27.3	**18**	126	-	-	-	-
	$DK(u^{(0)})$ ext.	**5**	27.4	19	**121**	7	32.0	27	178
	K^{lin} ext.	**5**	**27.2**	**18**	122	**6**	**29.4**	**21**	**152**

are stopped with an equivalent relative residual criterion in the corresponding local or, respectively, coarse finite element space, after a reduction of $1e-3$ is reached. This is sufficient since the inner and coarse initial values get more and more accurate while the outer loop converges. As a linear solver for the tangential systems, we use GMRES (Generalized Minimal RESidual) iterations with a relative stopping tolerance of $1e-8$. Of course, in particular, in the first Newton steps, we might over-solve the linear systems, and choosing the forcing terms correctly could be beneficial for all methods; see [4].

We first consider a numerical scalability study for the homogeneous p-Laplace for $p = 4$; see Table 1. Here, for regular domain decompositions, we choose $H/h = 32$ and therefore 2 048 triangular finite elements per nonoverlapping subdomain. For the METIS decompositions, the global problem sizes are identical to the corresponding regularly decomposed problems. We present the number of outer or global Newton iterations, which is up to 2.5 times higher in the one level RASPEN method compared with the best of the two-level approaches. All three coarse levels show a similar performance for the regular domain decomposition and both extension based coarse spaces perform well for the METIS decompositions. In general, the two-level RASPEN method needs less inner iterations and significantly less GMRES iterations, especially for irregular domain decompositions.

Table 2: Heterogeneous p-Laplace: See Table 1 for description of column labels and Fig. 1 for the coefficient distribution.

p-Laplace heterogeneous (channel + 2 circles) $p = 4$; $H/h = 32$ for regular domains; overlap $\delta = 2$;									
		Regular				METIS			
N	**RASPEN Coarse Space**	**outer it.**	**inner it. (avg.)**	**coarse it.**	**GMRES it. (sum)**	**outer it.**	**inner it. (avg.)**	**coarse it.**	**GMRES it. (sum)**
36	-	5	14.3	-	321	**5**	**14.2**	-	346
	$\mathbb{P}1$	5	15.6	17	139	-	-	-	-
	$DK(u^{(0)})$ ext.	5	15.1	16	139	**5**	15.2	**18**	**125**
	K^{lin} ext.	**4**	**12.7**	**13**	**108**	**5**	15.5	**18**	128

For the chosen heterogeneous problem (see Table 2), the number of outer Newton iterations is similar for all methods. Nevertheless, the linear convergence, i.e. the number of GMRES iterations, is superior in the two-level variants. All in all, our experiments show that our multiplicative second level with the chosen coarse basis functions has a superior linear convergence and, in some cases, also a better nonlinear convergence - regardless if regular or METIS decompositions are used.

In general, the discrete extension using K_{lin} shows a slightly better performance than the extension with the tangent $DK(u^{(0)})$, but the latter one will always be available, also for different nonlinear model problems where a suitable linear operator K_{lin} cannot be found easily. Considering, e.g., nonlinear hyperlelasticity or elasto-plasticity problems, the linear elasticity model or a multi-dimensional Laplacian could be used to form K_{lin}, but for large loads or highly plastic behavior, $DK(u^{(0)})$ might be a better choice.

5 Conclusion

We have presented a new approach to implement a multiplicative coarse space for AS-PIN or RASPEN, which is robust for the considered model problems. Additionally, we presented two different coarse spaces usable for irregular domain decompositions and compared both against the one level RASPEN method and, for regular domain decompositions, also against a classical $\mathbb{P}1$ coarse space. Both coarse spaces are competitive and cheap to compute.

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