The Domain Decomposition Method of Bank and Jimack as an Optimized Schwarz Method

Gabriele Ciaramella, Martin J. Gander, and Parisa Mamooler

1 Bank-Jimack Domain Decomposition Method

In 2001 Randolph E. Bank and Peter K. Jimack [1] introduced a new domain decomposition method for the adaptive solution of elliptic partial differential equations, see also [2]. The novel feature of this algorithm is that each of the subproblems is defined over the entire domain. To describe the method, we consider a linear elliptic PDE on a domain Ω, and two overlapping subdomains Ω_1 and Ω_2, $\Omega = \Omega_1 \cup \Omega_2$. Discretizing the problem on a global fine mesh leads to a linear system $K\boldsymbol{u} = \boldsymbol{f}$, where K is the stiffness matrix, $\boldsymbol{u}$ is the vector of unknown nodal values on the global fine mesh, and $\boldsymbol{f}$ is the load vector. We partition now the vector $\boldsymbol{u} = \left[\boldsymbol{u}_1, \boldsymbol{u}_s, \boldsymbol{u}_2\right]^T$, where $\boldsymbol{u}_1$ is the vector of unknowns on the nodes in $\Omega_1 \setminus \Omega_2$, $\boldsymbol{u}_s$ is the vector of unknowns on the nodes in $\Omega_1 \cap \Omega_2$, and $\boldsymbol{u}_2$ is the vector of unknowns on the nodes in $\Omega_2 \setminus \Omega_1$. We can then write the linear system in block matrix form,

$$\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^T & A_s & B_2^T \\ 0 & B_2 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_s \\ \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_s \\ \boldsymbol{f}_2 \end{bmatrix}. \tag{1}$$

The idea of the Bank-Jimack method is to consider two further meshes on Ω, one identical to the original fine mesh in Ω_1, but coarse on $\Omega \setminus \Omega_1$, and one identical to the original fine mesh in Ω_2, but coarse on $\Omega \setminus \Omega_2$. This leads to the two further linear systems

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$$\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^T & A_s & C_2 \\ 0 & \widetilde{B}_2 & \widetilde{A}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_s \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_s \\ M_2 \boldsymbol{f}_2 \end{bmatrix}, \quad \begin{bmatrix} \widetilde{A}_1 & \widetilde{B}_1 & 0 \\ C_1 & A_s & B_2^T \\ 0 & B_2 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_s \\ \boldsymbol{w}_2 \end{bmatrix} = \begin{bmatrix} M_1 \boldsymbol{f}_1 \\ \boldsymbol{f}_s \\ \boldsymbol{f}_2 \end{bmatrix}, \tag{2}$$

where we introduced the restriction matrices M_j to restrict $\boldsymbol{f}_j$ to the corresponding coarse meshes.The Bank-Jimack method is then performing the following iteration:

Algorithm 1: Bank-Jimack Domain Decomposition Method:

1: Set $k = 0$ and choose an initial guess $\boldsymbol{u}^0$.
2: Repeat until convergence

2.1 $\begin{bmatrix} \boldsymbol{r}_1^k \\ \boldsymbol{r}_s^k \\ \boldsymbol{r}_2^k \end{bmatrix} := \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_s \\ \boldsymbol{f}_2 \end{bmatrix} - \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^T & A_s & B_2^T \\ 0 & B_2 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^k \\ \boldsymbol{u}_s^k \\ \boldsymbol{u}_2^k \end{bmatrix}$

2.2 $Solve \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^T & A_s & \widetilde{B}_2^T \\ 0 & \widetilde{B}_2 & \widetilde{A}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^{k+1} \\ \boldsymbol{v}_s^{k+1} \\ \boldsymbol{v}_2^{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_1^k \\ \boldsymbol{r}_s^k \\ M_2 \boldsymbol{r}_2^k \end{bmatrix}, \quad \begin{bmatrix} \widetilde{A}_1 & \widetilde{B}_1 & 0 \\ \widetilde{B}_1^T & A_s & B_2^T \\ 0 & B_2 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1^{k+1} \\ \boldsymbol{w}_s^{k+1} \\ \boldsymbol{w}_2^{k+1} \end{bmatrix} = \begin{bmatrix} M_1 \boldsymbol{r}_1^k \\ \boldsymbol{r}_s^k \\ \boldsymbol{r}_2^k \end{bmatrix}$

2.3 $\begin{bmatrix} \boldsymbol{u}_1^{k+1} \\ \boldsymbol{u}_s^{k+1} \\ \boldsymbol{u}_2^{k+1} \end{bmatrix} := \begin{bmatrix} \boldsymbol{u}_1^k \\ \boldsymbol{u}_s^k \\ \boldsymbol{u}_2^k \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_1^{k+1} \\ \frac{1}{2}(\boldsymbol{v}_s^{k+1} + \boldsymbol{w}_s^{k+1}) \\ \boldsymbol{w}_2^{k+1} \end{bmatrix}$

2.4 $k := k + 1$

To get more insight into the Bank-Jimack method, and to relate it to Schwarz methods using optimized Schwarz theory, we consider the concrete example of the 1D Poisson equation

$$-u_{xx} = f \quad \text{in } \Omega = (0, 1), \qquad u(0) = u(1) = 0. \tag{3}$$

We define a *global fine mesh* with N mesh points (see Figure 1 (top row)), and mesh size $h := \frac{1}{N+1}$. Using a finite difference discretization, we find the linear system

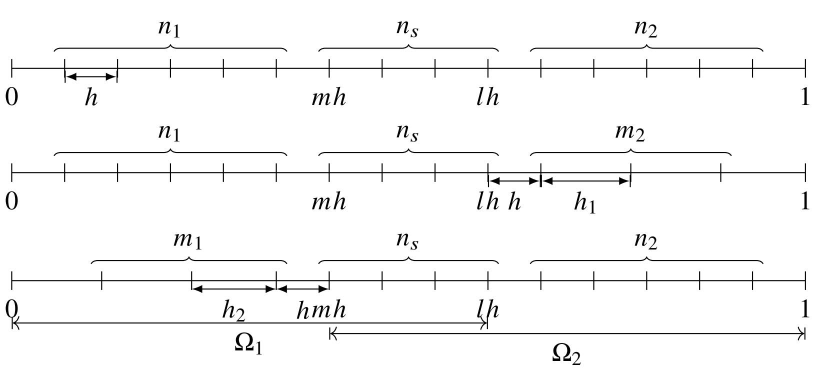

Fig. 1: Global fine mesh, and two partially coarse meshes.

$$
K\boldsymbol{u} = \boldsymbol{f}, \quad K := \begin{bmatrix} A_1 & B_1 & 0 \\ B_1^T & A_s & B_2^T \\ 0 & B_2 & A_2 \end{bmatrix} = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & \\ -1 & 2 & \ddots \\ & \ddots & \ddots \end{bmatrix},
$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $B_1 \in \mathbb{R}^{n_1 \times n_s}$, $A_s \in \mathbb{R}^{n_s \times n_s}$, $B_2 \in \mathbb{R}^{n_2 \times n_s}$ and $A_2 \in \mathbb{R}^{n_2 \times n_2}$, and $N = n_1 + n_s + n_2$ (Fig. 1). For the Bank-Jimack method, we also need the two further meshes shown in Figure 1, one with $N_1 := n_1 + n_s + m_2$ mesh points which is fine on Ω_1 with mesh size h and coarse on $\Omega \backslash \Omega_1$ with mesh size h_1, which leads to a linear system of equations of the form (2) (left), with system matrix

$$
\begin{bmatrix} A_1 & B_1 & 0 \\ B_1^T & A_s & C_2 \\ 0 & \widetilde{B}_2 & \widetilde{A}_2 \end{bmatrix} := \left[\begin{array}{ccc|ccc|cccc}
\frac{2}{h^2} & \frac{-1}{h^2} & & & & & & & & \\
\frac{-1}{h^2} & \frac{2}{h^2} & \ddots & & & & & & & \\
 & \ddots & \ddots & \frac{-1}{h^2} & & & & & & \\
\hline
 & & \frac{-1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} & & & & & \\
 & & & \frac{-1}{h^2} & \frac{2}{h^2} & \ddots & & & & \\
 & & & & \ddots & \ddots & \frac{-1}{h^2} & & & \\
\hline
 & & & & & \frac{-2}{h(h+h_1)} & \frac{2}{hh_1} & \frac{-2}{h_1(h+h_1)} & & \\
 & & & & & & \frac{-1}{h_1^2} & \frac{2}{h_1^2} & \frac{-1}{h_1^2} & \\
 & & & & & & & \frac{-1}{h_1^2} & \frac{2}{h_1^2} & \ddots \\
 & & & & & & & & \ddots & \ddots
\end{array}\right], \tag{4}
$$

and one with $N_2 = m_1 + n_s + n_2$ mesh points on Ω which is fine on Ω_2 with mesh size h, and coarse on $\Omega \backslash \Omega_2$, with coarse mesh size h_2, which leads to a linear system of equations of the form (2) (right), with system matrix

$$
\begin{bmatrix} \widetilde{A}_1 & \widetilde{B}_1 & 0 \\ C_1 & A_s & B_2^T \\ 0 & B_2 & A_2 \end{bmatrix} := \left[\begin{array}{cccc|ccc|ccc}
\frac{2}{h_2^2} & \frac{-1}{h_2^2} & & & & & & & & \\
\frac{-1}{h_2^2} & \ddots & \ddots & & & & & & & \\
 & \ddots & \ddots & -\frac{1}{h_2^2} & & & & & & \\
 & & -\frac{2}{h_2(h+h_2)} & \frac{2}{hh_2} & \frac{-2}{h(h+h_2)} & & & & & \\
\hline
 & & & \frac{-1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} & & & & \\
 & & & & \frac{-1}{h^2} & \frac{2}{h^2} & \ddots & & & \\
 & & & & & \ddots & \ddots & \frac{-1}{h^2} & & \\
\hline
 & & & & & & \frac{-1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} & \\
 & & & & & & & -\frac{1}{h^2} & \frac{2}{h^2} & \ddots \\
 & & & & & & & & \ddots & \ddots
\end{array}\right]. \tag{5}
$$

For this example, M_j are the transpose of linear interpolation matrices from the fine grid (Fig. 1, top row) to the coarse grids (Fig. 1, second and third row). We find them using an algorithm which is similar to the algorithm introduced in [6] for finding the interface matrices for non-matching grids in one dimension. Running the Bank-Jimack method on this example does not lead to a convergent method[1], see Fig. 2 (left) in the numerical experiments Section 4. This is due to the averaging used in the overlap in step 2.3 of the method, and can be fixed using a specific partition of unity given by the diagonal matrices $\widetilde{D}_1$ and $\widetilde{D}_2$ such that

$$\widetilde{D}_1 = diag(1, \times, \dots, \times, 0), \quad \widetilde{D}_2 = diag(0, \times, \dots, \times, 1), \quad \widetilde{D}_1 + \widetilde{D}_2 = I_{n_s} \tag{6}$$

One then has to replace step 2.3 in the the method of Bank-Jimack by

$$\begin{bmatrix} \boldsymbol{u}_1^{k+1} \\ \boldsymbol{u}_s^{k+1} \\ \boldsymbol{u}_2^{k+1} \end{bmatrix} := \begin{bmatrix} \boldsymbol{u}_1^{k} \\ \boldsymbol{u}_s^{k} \\ \boldsymbol{u}_2^{k} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_1^{k+1} \\ \widetilde{D}_1 \boldsymbol{v}_s^{k+1} + \widetilde{D}_2 \boldsymbol{w}_s^{k+1} \\ \boldsymbol{w}_2^{k+1} \end{bmatrix}. \tag{7}$$

We now present an important property of the Bank-Jimack method with (7):

Lemma 1 *The Bank-Jimack Algorithm with step 2.3 replaced by* (7) *produces for any initial guess* $\boldsymbol{u}^0$ *and arbitrary partitions of unity satisfying* (6) *for* $k = 1, 2, \dots$ *zero residual components outside the overlap,* $\boldsymbol{r}_1^k = \boldsymbol{r}_2^k = \boldsymbol{0}$.

Proof From step 2.1 in the Bank-Jimack method, we obtain

$$\begin{aligned} \boldsymbol{r}_1^k &= \boldsymbol{f}_1 - (A_1 \boldsymbol{u}_1^k + B_1 \boldsymbol{u}_s^k) \\ &= \boldsymbol{f}_1 - A_1(\boldsymbol{u}_1^{k-1} + \boldsymbol{v}_1^k) - B_1(\boldsymbol{u}_s^{k-1} + \widetilde{D}_1 \boldsymbol{v}_s^k + \widetilde{D}_2 \boldsymbol{w}_s^k) \quad &&\text{(step 2.3 at } k-1 \text{ and (7))} \\ &= \boldsymbol{f}_1 - A_1 \boldsymbol{u}_1^{k-1} - B_1 \boldsymbol{u}_s^{k-1} - A_1 \boldsymbol{v}_1^k - B_1(\widetilde{D}_1 \boldsymbol{v}_s^k + \widetilde{D}_2 \boldsymbol{w}_s^k) &&\text{(rearrange)} \\ &= \boldsymbol{r}_1^{k-1} - A_1 \boldsymbol{v}_1^k - B_1(\widetilde{D}_1 \boldsymbol{v}_s^k + \widetilde{D}_2 \boldsymbol{w}_s^k) &&\text{(using step 2.1)} \\ &= B_1 \boldsymbol{v}_s^k - B_1(\widetilde{D}_1 \boldsymbol{v}_s^k + \widetilde{D}_2 \boldsymbol{w}_s^k), \end{aligned}$$

since $\boldsymbol{r}_1^{k-1} - A_1 \boldsymbol{v}_1^k = B_1 \boldsymbol{v}_s^k$ because of the first system satisfied in step 2.3 at $k-1$. Now using the definition of B_1 from (4), we have

$$-B_1 \widetilde{D}_1 \boldsymbol{v}_s^k = \frac{1}{h^2} \begin{bmatrix} & & \\ & & \\ & & \\ 1 & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \times & & \\ & & \ddots & \\ & & & \times \\ & & & & 0 \end{bmatrix} \begin{bmatrix} v_{s,1}^k \\ \vdots \\ \vdots \\ v_{s,n_s}^k \end{bmatrix} = \frac{1}{h^2} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_{s,1}^k \end{bmatrix},$$

independently of the middle elements of $\widetilde{D}_1$, and thus $B_1 \boldsymbol{v}_s^k - B_1 \widetilde{D}_1 \boldsymbol{v}_s^k = \boldsymbol{0}$. On the other hand

[1] Bank and Jimack used the method as a preconditioner for a Krylov method.

$$-B_1\widetilde{D}_2\boldsymbol{w}_s^k = \frac{1}{h^2}\begin{bmatrix} & & \\ & & \\ 1 & & \end{bmatrix}\begin{bmatrix}0 & & & \\ & \times & & \\ & & \ddots & \\ & & & \times \\ & & & & 1\end{bmatrix}\begin{bmatrix}w_{s,1}^k \\ \vdots \\ \vdots \\ w_{s,n_s}^k\end{bmatrix} = \begin{bmatrix}0 \\ \vdots \\ \vdots \\ 0\end{bmatrix},$$

also independently of the middle elements of $\widetilde{D}_2$, which proves that $\boldsymbol{r}_1^k = \boldsymbol{0}$ for $k = 1, 2, \ldots$. The proof for $\boldsymbol{r}_2^k$ is similar. □

2 Optimized Schwarz Methods

Optimized Schwarz Methods (OSMs) use more effective transmission conditions than the classical Schwarz methods, for an introduction, see [4], and for their relation to sweeping and other more recent domain decomposition methods, see [7]. We now apply a parallel OSM with Robin transmission conditions to our Poisson equation (3) for two subdomains as shown in Fig. 1,

$$\begin{array}{llll} -\partial_{xx}u_1^k = f & \text{in } \Omega_1, & -\partial_{xx}u_2^k = f & \text{in } \Omega_2, \\ u_1^k = 0 & x = 0, & u_2^k = 0 & x = 1, \\ \frac{\partial u_1^k}{\partial n_1} + p_{12}u_1^k = \frac{\partial u_2^{k-1}}{\partial n_1} + p_{12}u_2^{k-1} & x = lh, & \frac{\partial u_2^k}{\partial n_2} + p_{21}u_2^k = \frac{\partial u_1^{k-1}}{\partial n_2} + p_{21}u_2^{k-1} & x = mh. \end{array} \tag{8}$$

Theorem 1 (Special case of Theorem 2 in [3]) *If* $p_{12} = \frac{1}{1-lh}$ *and* $p_{21} = \frac{1}{mh}$*, then the OSM* (8) *converges independently of the initial guess in* 2 *iterations, and is thus an optimal Schwarz method.*

Discretizing the OSM using the same mesh with N grid points as for the method of Bank-Jimack, we obtain

$$\begin{aligned} &\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 + p_{12}h \end{bmatrix}\begin{bmatrix} u_{1,1}^k \\ \vdots \\ \vdots \\ u_{1,l}^k \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ \vdots \\ f_l + (\frac{p_{12}}{h} - \frac{1}{h^2})u_{2,n_s}^{k-1} + \frac{1}{h^2}u_{2,n_s+1}^{k-1} \end{bmatrix}, \\ &\frac{1}{h^2}\begin{bmatrix} 1 + p_{21}h & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_{2,1}^k \\ \vdots \\ \vdots \\ u_{2,N-m}^k \end{bmatrix} = \begin{bmatrix} f_m + (\frac{p_{21}}{h} - \frac{1}{h^2})u_{1,m}^{k-1} + \frac{1}{h^2}u_{1,m-1}^{k-1} \\ \vdots \\ \vdots \\ f_N \end{bmatrix}. \end{aligned} \tag{9}$$

3 Bank-Jimack's Method as an Optimized Schwarz Method

We now prove that the method of Bank-Jimack is an optimized Schwarz method with a special choice of the Robin parameter. To do so, we reformulate the matrix systems in step 2.2 of the method: using Lemma 1, we have $M_2\boldsymbol{r}_2^1 = M_1\boldsymbol{r}_1^1 = \boldsymbol{0}$, and thus one can eliminate the corresponding parts from the equations to obtain

$$\begin{bmatrix} A_1 & B_1 \\ B_1^T & A_s - C_2\widetilde{A}_2^{-1}\widetilde{B}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^1 \\ \boldsymbol{v}_s^1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_1^1 \\ \boldsymbol{r}_s^1 \end{bmatrix}, \quad \begin{bmatrix} A_s - C_1\widetilde{A}_1^{-1}\widetilde{B}_1 & B_2^T \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_s^1 \\ \boldsymbol{w}_2^1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_s^1 \\ \boldsymbol{r}_2^1 \end{bmatrix}, \quad (10)$$

and we are interested in the structure of the Schur complement matrices $A_s - C_2\widetilde{A}_2^{-1}\widetilde{B}_2$ and $A_s - C_1\widetilde{A}_1^{-1}\widetilde{B}_1$.

Lemma 2 (See [9]) *The elements of the inverse of the tridiagonal matrix*

$$T = \begin{bmatrix} a_1 & b_1 & & \\ c_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & c_{n-1} & a_n \end{bmatrix} \text{ are } (T^{-1})_{ij} = \begin{cases} (-1)^{i+j} b_i \dots b_{j-1}\theta_{i-1}\phi_{j+1}/\theta_n & i < j, \\ \theta_{i-1}\phi_{j+1}/\theta_n & i = j, \\ (-1)^{i+j} c_j \dots c_{i-1}\theta_{j-1}\phi_{i+1}/\theta_n & i > j, \end{cases}$$

where $\theta_0 = 1$, $\theta_1 = a_1$, and $\theta_i = a_i\theta_{i-1} - b_{i-1}c_{i-1}\theta_{i-2}$ for $i = 2, \dots, n$, and $\phi_{n+1} = 1$, $\phi_n = a_n$, and $\phi_i = a_i\phi_{i+1} - b_i c_i \phi_{i+2}$ for $i = n-1, \dots, 1$.

Lemma 3 *The matrices $C_2\widetilde{A}_2^{-1}\widetilde{B}_2$ and $C_1\widetilde{A}_1^{-1}\widetilde{B}_1$ in the Schur complements in* (10) *are given by*

$$C_2\widetilde{A}_2^{-1}\widetilde{B}_2 = \frac{1}{h^2}\begin{bmatrix} 0 & & \\ & \ddots & \\ & & \frac{m_2 h_1}{h + m_2 h_1} \end{bmatrix}, \quad C_1\widetilde{A}_1^{-1}\widetilde{B}_1 = \frac{1}{h^2}\begin{bmatrix} \frac{m_1 h_2}{h + m_1 h_2} & & \\ & 0 & \\ & & \ddots \end{bmatrix}.$$

Proof Using the sparsity of C_2 and $\widetilde{B}_2$, we obtain

$$C_2\widetilde{A}_2^{-1}\widetilde{B}_2 = \begin{bmatrix} 0 & & \\ & \ddots & \\ \frac{-1}{h^2} & & 0 \end{bmatrix} \widetilde{A}_2^{-1} \begin{bmatrix} 0 & & \frac{-2}{h(h+h_1)} \\ & \ddots & \\ & & 0 \end{bmatrix} = \frac{1}{h^2}\begin{bmatrix} 0 & & \\ & \ddots & \\ & & \frac{2}{h(h+h_1)}(\widetilde{A}_2^{-1})_{11} \end{bmatrix},$$

and we thus need to find the first entry of $\widetilde{A}_2^{-1}$. For convenience, we find the first entry of $(h_1^2\widetilde{A}_2)^{-1}$, and then we multiply it by h_1^2. Using Lemma 2, we have $(h_1^2\widetilde{A}_2^{-1})_{11} = \frac{\theta_0\phi_2}{\theta_{m_2}}$ where $\theta_0 = 1$, and

$$\begin{aligned} \theta_{m_2} &= 2\theta_{m_2-1} - \theta_{m_2-2} = 2(2\theta_{m_2-2} - \theta_{m_2-3}) - \theta_{m_2-2} = 3\theta_{m_2-2} - 2\theta_{m_2-3} \\ &= \dots = (m_2 - 1)\left(\frac{4h_1}{h} - \frac{2h_1}{h + h_1}\right) - (m_2 - 2)\frac{2h_1}{h} = \frac{2h_1(m_2 h_1 + h)}{h(h + h_1)}, \end{aligned} \quad (11)$$

and

$$\phi_2 = 2\phi_3 - \phi_4 = 2(2\phi_4 - \phi_5 - \phi_4) = 3\phi_4 - 2\phi_5 \cdots = 2(m_2 - 1) - (m_2 - 2) = m_2.$$

We thus obtain $\frac{\theta_0 \phi_2}{\theta_{m_2}} = \frac{m_2 h(h+h_1)}{2h_1(m_2 h_1 + h)}$, which shows the first claim. The second one is proved similarly. □

Theorem 2 *The Bank-Jimack method in 1D with the partition of unity* (7) *is an optimized Schwarz method with the parameters chosen as* $p_{12} = \frac{1}{h+m_2 h_1}$ *and* $p_{21} = \frac{1}{h+m_1 h_2}$.

Proof It suffices to compare the matrix systems of the OSM (9) with the matrix systems in step 2.3 of the Bank-Jimack method, rewritten as in (10), since in stationary iterations, the standard form and the correction form are equivalent [8, Section 11.2.2]. The system matrices can be made identical by choosing p_{12} such that $1 + p_{12}h = 2 - \frac{m_2 h_1}{h+m_2 h_1}$ and p_{21} such that $1 + p_{21}h = 2 - \frac{m_1 h_2}{h+m_1 h_2}$. □

Since the parameters p_{12} and p_{21} are positive in Theorem 2, it follows from optimized Schwarz theory that the Bank-Jimack method with a partition of unity of the form (7) converges to the monodomain solution, and the convergence is independent of the particular values chosen in the partition of unity, see [5].

Corollary 1 *The Bank-Jimack method in 1D with the partition of unity* (7) *is an optimal Schwarz method: it selects the best possible Robin parameter, independently of how coarse the mesh is in the remaining parts outside of the subdomains, and thus converges in two iterations.*

Proof From Theorem 2 we can see that the Robin parameters p_{12} and p_{21} chosen by the method of Bank-Jimack are independent of the choice of the coarse grid parameters h_1 and h_2, $p_{12} = \frac{1}{h+m_2 h_1} = \frac{1}{1-lh}$ and $p_{21} = \frac{1}{h+m_1 h_2} = \frac{1}{mh}$, which are precisely the optimal choices in Theorem 1 for the OSM. □

4 Numerical Experiments

We first show numerical experiments in one spatial dimension. We discretize the Poisson equation (3) using $N = 2^i$, for $i = 4, \ldots, 7$, gridpoints on the global fine mesh (Fig.1, top row), choose $n_s = 2$ gridpoints in $\Omega_1 \cap \Omega_2$, and $m_1 = m_2 = 2$ coarse mesh points outside the subdomains (Fig. 1, middle and last rows). In Fig. 2, we show on the left that the method of Bank-Jimack using the original partition of unity is not converging. On the right, we show that the method with the new partition of unity converges in two iterations, as expected from the equivalence with the optimal Schwarz method proved in Corollary 1. In Fig. 3, we show on the left that the convergence does not depend on the number of coarse mesh points. We finally show in Fig. 3 on the right a numerical experiment in 2D, where the optimal choice of the Robin parameter in the OSM would lead to a non-local operator involving

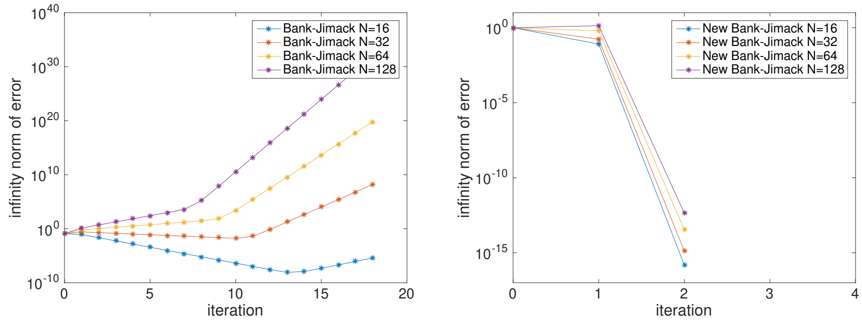

Fig. 2: Error as a function of iteration count of the method of Bank-Jimack with the original partition of unity (left) and new partition of unity (right) for various numbers of global fine mesh points.

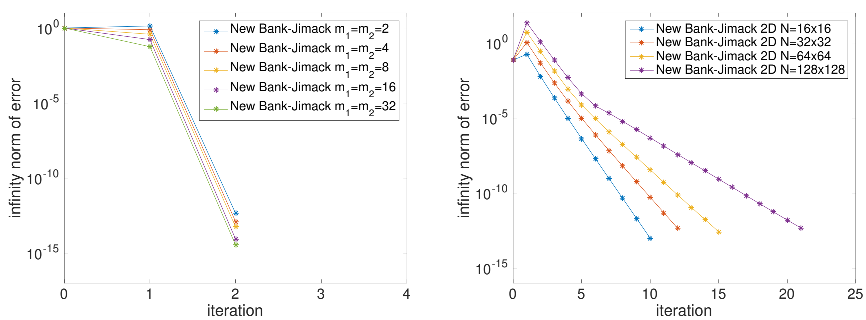

Fig. 3: Left: convergence of the method of Bank-Jimack using $N = 128$ gridpoints on the global fine mesh and various number of gridpoints on the coarse regions. Right: convergence of the method in 2D for various number of gridpoints on the global fine mesh, choosing $n_s = 2$, and $m_1 = m_2 = 2$.

a DtN map, and the method of Bank-Jimack is choosing some approximation. The study of the type of approximation chosen is our current focus of research.

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