Adaptive Schwarz Method for DG Multiscale Problems in 2D

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1 Introduction

In many real physical phenomena, there is heterogeneity, e.g., in some ground flow problems in heterogeneous media. When some finite element discretization method is applied to a physical model, one usually obtains a discrete problem which is very hard to solve by a preconditioned iterative method like, e.g., Preconditioned Conjugate Gradient (PCG) method. One of the most popular methods of constructing parallel preconditioners are domain decomposition methods, in particular, non-overlapping or overlapping additive Schwarz methods (ASM), cf. e.g., [16]. In Schwarz methods, a crucial role is played by carefully constructed coarse spaces. For multiscale problems with heterogeneous coefficients standard overlapping Schwarz methods with classical coarse spaces fail often to be fast and robust solvers. Therefore we need new coarse spaces which are adaptive to the jumps of the coefficients, i.e. the convergence of the ASM method is independent of the distribution and the magnitude of the coefficients of the original problem. We refer to [6], [15] and the references therein for similar earlier works on domain decomposition methods which used adaptivity in the construction of the coarse spaces.

In our paper, we consider the Symmetric Interior Penalty Galerkin (SIPG) finite element discretization, i.e., a symmetric version of the interior penalty discontinuous Galerkin (DG) method. DG methods became increasingly popular in recent years,

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since they allow that the finite element functions can be completely discontinuous across the element edges, cf. e.g. [14] for an introduction to DG methods.

In the case when the coefficients are discontinuous only across the interfaces between subdomains and are homogeneous inside them, then Schwarz methods with standard coarse spaces are fast and efficient, cf. e.g., [3, 16]. This is however not true in the case when the coefficients may be highly varying and discontinuous almost everywhere, what has in recent years brought many researchers' interest to the construction of new coarse spaces, cf. e.g. [11, 5, 7, 8, 9, 12, 13, 15, 10].

2 Discrete Problem

Let us consider the following elliptic second order boundary value problem in 2D: Find $u^* \in H_0^1(\Omega)$

$$\int_\Omega \alpha(x)\nabla u^* \nabla v \, dx = \int_\Omega f v \, dx, \qquad \forall v \in H_0^1(\Omega), \tag{1}$$

where Ω is a polygonal domain in $\mathbb{R}^2$, $\alpha(x) \geq \alpha_0 > 0$ is the coefficient, and $f \in L^2(\Omega)$.

We introduce $\mathcal{T}_h$ the quasi-uniform triangulation of Ω consisting of closed triangles such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$. Further h_K denotes the diameter of K, and let $h = \max_{K \in \mathcal{T}_h} h_K$ be the mesh parameter for the triangulation.

We will further assume that α is piecewise constant on $\mathcal{T}_h$. Let be given a coarse non-overlapping partitioning of Ω into the open, connected Lipschitz polytopes Ω_i, called substructures or subdomains, such that $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$. We also assume that those substructures are aligned with the fine triangulation, i.e. any fine triangle of $\mathcal{T}_h$ is contained in one substructure. For the simplicity of presentation, we further assume that these substructures form a coarse triangulation of the domain which is shape regular in the sense of [1]. Let Γ_{ij} denote the open edge common to subdomains Ω_i and Ω_j not in $\partial\Omega$ and let Γ be the union of all $\partial\Omega_k \setminus \partial\Omega$.

Further let us define a discrete space S_h as the piecewise linear finite element space defined on the triangulation $\mathcal{T}_h$,

$$S_h = S_h(\Omega) := \{u \in L^2(\overline{\Omega}) : \; u_{|K} \in P_1, \; K \in \mathcal{T}_h\}.$$

Note that the functions in S_h are multivalued on boundaries of all fine triangles of $\mathcal{T}_h$ except on $\partial\Omega$. Therefore we introduce a set of all edges of elements of $\mathcal{T}_h$ as $\mathcal{E}_h$. Let the $\mathcal{E}_h^\partial \subset \mathcal{E}_h$ be the subset of boundary edges i.e. the edges contained in $\partial\Omega$, and $\mathcal{E}_h^I = \mathcal{E}_h \setminus \mathcal{E}_h^\partial$ be the subset of interior edges, i.e. the edges interior to Ω. We define the L^2-inner products over the elements and the edges respectively as follows, $(u, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_K uv \, dx$ and $(u, v)_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \int_e uv \, ds$ for $u, v \in S_h$.

The following weights, cf. e.g. [2], are introduced $\omega_+^e = \alpha_-/(\alpha_+ + \alpha_-)$ and $\omega_-^e = \alpha_+/(\alpha_+ + \alpha_-)$, $e \in \mathcal{E}_h^I$, where e is the common edge between two neighboring triangles K_+ and K_-, α_+ and α_- are the restrictions of α to K_+ and K_-, respectively. We have $\omega_+^e + \omega_-^e = 1$. We also need the following notations: $[u] = u_+ n_+ + u_- n_-$ and $\{u\} = \omega_+^e u_+ + \omega_-^e u_-$, where u_+ and u_- are the traces of $u_{|K_+}$ and $u_{|K_-}$ on $e \subset \mathcal{E}_h^I$, while n_+ and n_- are the unit outer normal to ∂K_+ and ∂K_-, respectively. On the boundary we introduce $[u] = u\, n$ and $\{u\} = u$, where n is the unit outer normal to the edge $e \subset \partial K \cap \partial\Omega$, and u is the trace of $u_{|K}$ onto e. We consider SIPG method discrete problems: (cf. [2]). Find $u_h^* \in S_h$

$$a(u_h^*, v) = f(v) \qquad \forall v \in S_h, \tag{2}$$

where $a(u,v) = (\alpha\nabla u, \nabla v)_{\mathcal{T}_h} - (\{\alpha\nabla u\}, [v])_{\mathcal{E}_h} - (\{\alpha\nabla v\}, [u])_{\mathcal{E}_h} + \gamma(\Psi_h[u],[v])_{\mathcal{E}_h}$. Here Ψ_h is a piecewise constant function over the edges of $\mathcal{E}_h$, and γ is a constant positive penalty parameter. The function Ψ_h when restricted to $e \in \mathcal{E}_h^I$, is defined as follows, cf. [2], $\Psi_{h|e} = h_e^{-1}(\omega_+^e\alpha_+ + \omega_-^e\alpha_-) = h_e^{-1}\frac{2}{\frac{1}{\alpha_+}+\frac{1}{\alpha_-}}$ on $\overline{e} = \partial K_+ \cap \partial K_-$,with h_e being the length of the edge $e \in \mathcal{E}_h$.

We have, cf. e.g. [2], $h_e^{-1}\alpha_{min} \le \Psi_{h|e} \le 2h_e^{-1}\alpha_{min}$, $\alpha_{min} = \min(\alpha_+, \alpha_-)$. On a boundary edge $e \in \mathcal{E}_h^\partial$ we define $\Psi_{h|e} = h_e^{-1}\alpha_{|K}$. Note that ∇u_h^* is piecewise constant over the fine elements. The discrete problem has a unique solution provided the penalty parameter is sufficiently large, cf. [2]. Let us define a patch around an interface (edge) Γ_{kl}, denoted by Γ_{kl}^δ, as the interior of the union of all closed fine triangles having at least a vertex on Γ_{kl}. For the simplicity of the presentation let us assume that the patches cannot share a fine triangle. We divide any patch Γ_{kl}^δ into two disjoint open domains - subpatches, $\Gamma_{kl}^{\delta,i} = \Gamma_{kl}^\delta \cap \Omega_i$ for $i = k, l$.

The discrete boundary layer of Ω_k: Ω_k^δ, is defined as the sum of all subpatches and parts of their boundaries belonging to a subdomain Ω_k, i.e. we have $\overline{\Omega}_k^\delta = \bigcup_{\Gamma_{kl}\subset\partial\Omega_k\cap\Gamma}\overline{\Gamma}_{kl}^{\delta,k}$. Each subdomain inherits a local triangulation $\mathcal{T}_h(\Omega_i)$ from $\mathcal{T}_h$, thus we can define a local subspace extended by zero to the remaining substructures: $S_i := \{u \in S_h : u_{|K} = 0 \quad K \not\subset \Omega_i\}$ and its subspace S_i^δ formed by the functions from S_i which are also zero on the patch Ω_i^δ.

Since the form $a(u,v)$ is positive definite over S_i^δ we can introduce a local projection operator $\mathcal{P}_i : S_h \to S_i^\delta$: find $\mathcal{P}_i u \in S_i^\delta$ such that for $u \in S_h$

$$a(\mathcal{P}_i u, v) = a(u, v), \quad \forall v \in S_i^\delta.$$

Note that $\mathcal{P}_i u$ can be computed by solving a local problem over Ω_i.

The discrete harmonic part of $u \in S_i$ is defined as $\mathcal{H}_i u := u_{|\Omega_i} - \mathcal{P}_i u \in S_i$. We say that a function $u \in S_h$ is discrete harmonic if it is discrete harmonic in each subdomain, i.e. $u_{|\Omega_i} = \mathcal{H}_i u$ for $i = 1, \ldots, N$. Knowing the values of discrete harmonic $u \in S_i$ on the patch Ω_i^δ allows us to compute u over the remaining triangles contained in Ω_i by solving a local problem. We also introduce spaces related to an edge patch Γ_{kl}^δ. Let $S_{kl} \subset S_h$ be the space formed by all discrete harmonic functions which are zero on the all patches except Γ_{kl}^δ. We see that $S_{kl} \subset S_k \cup S_l$.

3 Additive Schwarz Method

In this section, we present our overlapping additive Schwarz method for solving (2). Our method is based on the abstract Additive Schwarz Method framework, cf. e.g., [16] for details.

The space S_h is decomposed into the local sub-spaces and a global coarse space. For the local spaces we take $\{S_i\}_{i=1}^N$. We have $S_h = \sum_{i=0}^N S_i$. The global coarse space S_0 is defined in (7), cf. § 3.1, below. Note that the supports of two functions $u_i \in S_i, u_j \in S_j$ for $i \neq j$ with $i, j > 0$ are disjoint, but $a(u_i, u_j)$ may be nonzero due to the edge terms in the bilinear form $a(u, v)$. Thus we see that $S_h = \sum_{i=0}^N S_i$ is a direct sum, but not an orthogonal one in terms of $a(u, v)$. We can interpret this space decomposition as an analog of a classical P_1 continuous finite element decomposition into overlapping subspaces with the minimal overlap.

Next we define the projection like operators $T_i : S_h \to S_i$ as

$$a(T_i u, v) = a(u, v), \quad \forall v \in S_i, \qquad i = 0, \ldots, N. \tag{3}$$

Note that to compute $T_i u$ $i = 1, \ldots, N$ we have to solve N independent local problems, but to get $T_0 u$ we have to solve a global one, cf. § 3.1. Let $T := T_0 + \sum_{i=1}^N T_i$, be the Additive Schwarz operator. We further replace (2) by the following equivalent problem: Find $u_h^* \in S_h$ such that

$$T u_h^* = g, \tag{4}$$

where $g = \sum_{i=0}^N g_i$ and $g_i = T_i u_h^*$. Note that g_i may be computed without knowing the solution u_h^* of (2), cf. e.g., [16]. The following theoretical estimated of the condition number can be derived:

Theorem 1 *For all $u \in S_h$, the following holds,*

$$c\left(1 + \max_{\Gamma_{kl}} \frac{1}{\lambda_{n_{kl}+1}^{\Gamma_{kl}}}\right)^{-1} a(u, u) \leq a(Tu, u) \leq C\, a(u, u),$$

where C, c are positive constants independent of the coefficient α, the mesh parameter h and the subdomain size H and $\lambda_{n_{kl}+1}^{\Gamma_{kl}}$ is defined in (6).

Below, in § 3.2 we give a sketch of the proof.

3.1 Adaptive patch coarse space

We introduce our adaptive patch based coarse space in this section.

First, we introduce a DG analog of the classical multiscale space, see e.g. [7]. Let $S_{ms} \subset S_h$ be the space of discrete harmonic functions such that for each patch Γ_{kl}^δ a function $u \in S_{ms}$ satisfies

$$a_{kl}(u,v)=0 \qquad \forall v\in S_{kl}^{v}, \tag{5}$$

where $a_{kl}(u,v)=\sum_{K\subset\Gamma_{kl}^{\delta}}\int_K \alpha\nabla u\cdot\nabla v\,dx+\sum_{e\subset\Gamma_{kl}^{\delta}\cup(\partial\Omega\cap\partial\Gamma_{kl}^{\delta})}\Psi_h\int_e[u][v]\,ds$, and $S_{kl}^{v}\subset S_{kl}$ is formed by the functions which are zero at all degrees of freedom which are at the geometrical ends (crosspoints) of the edge Γ_{kl}. Note that the second sum in the definition of $a_{kl}(u,v)$ is over the fine edges that are either interior to the patch or are on the boundary of Ω.

We introduce the edge generalized eigenvalue problem, which is to find the eigenvalue and its eigenfunction: $(\lambda_j^{kl},\psi_j^{kl})\in\mathbb{R}_+\times S_{kl}^{v}$ such that

$$a_{kl}(\psi_j^{kl},v)=\lambda_j^{\Gamma_{kl}}b_{kl}(\psi_j^{kl},v), \qquad \forall v\in S_{kl}^{v}, \tag{6}$$

where $a_{kl}(u,v)$ is introduced above. The form $b_{kl}(u,v)$ may be equal to $b_{kl}^{(0)}(u,v)=a(u,v)$ or as in [4] it can be equal to $b_{kl}^{(1)}(u,v)=h^{-2}\int_{\Gamma_{kl}^{\delta}}\alpha uv\,dx$ or equals the scaled discrete L^2-version of the $b_{kl}^{(1)}$ form, namely, $b_{kl}^{(2)}(u,v)=\sum_{K\in\Gamma_{kl}^{\delta}}\alpha_{|K}\sum_{j=1}^{3}|u(v_j)|^2$. Here in the last sum v_j, for $j=1,2,3$, denote the vertices of the fine triangle K. Thus we get three different versions of the eigenproblem. Note that the last discrete form $b_{kl}^{(2)}$ can be represented by a diagonal matrix in a matrix form of the eigenproblem. Hence we see that this generalized eigenproblem can be rewritten as a standard eigenproblem, which makes the computations cheaper, cf. also § 4.3 in [9].

We order the eigenvalues in the increasing way as follows $0<\lambda_1^{kl}\le\ldots\le\lambda_{M_{kl}}^{kl}$ for $M_{kl}=\dim(S_{kl}^{v})$. We now can define the local face spectral component of the coarse space for all $\Gamma_{kl}\subset\Gamma$ and the whole coarse space V_0 as follows

$$S_{kl}^{eig}=\mathrm{Span}(\psi_j^{kl})_{j=1}^{n_{kl}}, \qquad S_0=S_{ms}+\sum_{\Gamma_{kl}\subset\Gamma}S_{kl}^{eig}, \tag{7}$$

where $n_{kl}\le M_{kl}$ is the number of eigenfunctions ψ_j^{kl} chosen by us, e.g. in such a way that the eigenvalue $\lambda_{n_{kl}}^{kl}$, is below a given threshold.

3.2 The sketch of the proof of Theorem 1

The proof follows the lines of the proof of Theorem 3 in [4] and is based on the abstract framework of Additive Schwarz Method, cf. e.g. § 2.3 in [16]. Below C denotes a generic constant independent of the mesh parameters and the problem coefficients. We have to check three key assumptions. The latter two ones, namely, the Strengthened Cauchy Inequalities and Local Stability are verified in a standard way with constants independent of coefficients or mesh parameters. It remains to verify the Stable Splitting assumption. Let $u\in S_h$ and we first define $u_0\in S_0$ as $u_{ms}+\sum_{\Gamma_{kl}\subset\Gamma}u_{kl}$ where $u_{ms}\in S_{ms}$ takes the values of u at all DOFs at crosspoints. Next on any patch Γ_{kl}^{δ} let u_{kl} be the b_{kl}-orthogonal projection of $u-u_{ms}$ onto S_{kl}^{eig}, i.e.

$$u_{kl} = \sum_{j \le n_{kl}} \frac{b_{kl}(\psi_j^{kl}, u - u_{ms})}{b_{kl}(\psi_j^{kl}, \psi_j^{kl})} \psi_j^{kl} \in S_{kl}^{eig}.$$

Finally, we define

$$u_j := (u - u_0)_{|\overline{\Omega}_j} \in S_j \qquad j = 1, \ldots, N,$$

what gives us the splitting: $u = u_0 + \sum_{j=1}^N u_j$.

Then we estimate the discrete harmonic part $w = \mathcal{H}(u - u_0) = \mathcal{H}u - u_0$, which is zero at crosspoints. Namely, we have the following splitting: $w = \sum_{\Gamma_{kl}} w_{kl}$, where $w_{kl} \in S_{kl}^{v}$ is a discrete harmonic function, which is equal to $u - u_0$ on the respective patch. Note that w_{kl} is b_{kl}-orthogonal to S_{kl}^{eig}. Next we can show that $a(w, w) \le C \sum_{\Gamma_{kl}} b_{kl}(w_{kl}, w_{kl})$ for all types of the bilinear form b_{kl}.

Using the classical theory of the eigenvalue problems, and some technical tools related to SIPG discretizations we can show the stable splitting

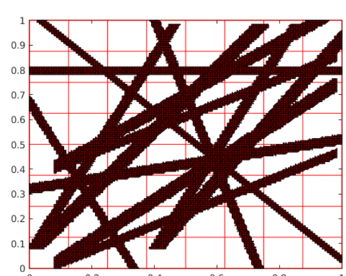
$$a(u_0, u_0) + \sum_{k=1}^{N} a(u_k, u_k) \le C \left(1 + \max_{\Gamma_{kl}} \frac{1}{\lambda_{n_{kl}+1}^{\Gamma_{kl}}} \right) a(u, u).$$

The statement in Theorem 1 follows from the abstract ASM theory.

4 Numerical tests

In the tests, our model problem is defined on the unit square with zero Dirichlet boundary condition and a constant force function. We solve it using the SIPG discretization, and the PCG iteration with our additive Schwarz preconditioner. The RHS form in the eigenvalue problem is the scaled L^2 one, i.e., $b_{kl}^{(1)}$. We decompose the domain into 8×8 non-overlapping square sub-domains. We have $H/h = 16$. The penalty parameter γ is equal to four, and the iterations are stopped when the relative residual norm became less than 10^{-6}.

Fig. 1: The coefficient is equal to one on the background and α_0 on the channels. A domain with $8x8$ subdomains. The channels are crossing each other.



For the adaptive coarse space the threshold for including an eigenfunction is $\lambda \le 0.18$.

	DG on distribution Fig. 1			
	#Enrichments= 0	#Enrichments= 2	#Enrichments= 4	Adaptive
α_0	Cond.	Cond.	Cond.	Cond.
10^0	57.31(53)	15.65(31)	9.64(24)	15.65(31)
10^2	1.41×10^2(83)	27.03(44)	12.01(31)	26.77(44)
10^4	2.12×10^2(97)	46.71(57)	12.12(32)	27.05(45)
10^6	2.13×10^2(102)	46.78(59)	11.39(35)	26.98(48)

Table 1: Numerical results showing condition number estimates and iteration counts (in parentheses). #Enrichments is per patch (edge).

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