Does the Partition of Unity Influence the Convergence of Schwarz Methods?

Martin J. Gander

1 Which Schwarz Methods Need a Partition of Unity?

The classical alternating Schwarz method does not need a partition of unity in its definition [3]: one solves one subdomain after the other, stores subdomain solutions, and always uses the newest data available from the neighboring subdomains. In the parallel Schwarz method introduced by Lions in [10], where all subdomains are solved simultaneously, one also stores subdomain solutions, but one has to distinguish two cases: if in the decomposition there are never more than two subdomains that intersect, which we call the *no crosspoint assumption*, then one also does not need a partition of unity to define the method, one simply takes data from the neighboring subdomains with which the subdomain intersects, and in that case the parallel Schwarz method has a variational interpretation [10]. If however points of the boundary of a subdomain are contained in more than one neighboring subdomain, then one has to decide from which neighboring subdomain to take data, or one can use a linear combination. In this case, the parallel Schwarz method does not have a variational interpretation [10], for an example, see Figure 2.2 in [3]. The decision from which of the neighboring subdomains data should be taken has to be made only on the boundary of each subdomain, and by the maximum principle, it is better to take data as far away from the boundary of the neighboring subdomains as possible to benefit from the largest error decay. This can be achieved if the overlapping domain decomposition is obtained from a non-overlapping one by enlarging the non-overlapping subdomains equally, and then using the subdomain solutions restricted to the non-overlapping subdomains to define a global approximation from which data is taken for the next iteration, see [3] for more details.

The situation for the algebraic Schwarz methods is more delicate, since these methods define approximate iterates over the entire domain only, so in the overlap, where necessarily more than one iterate is available, one has to decide which one or

M. J. Gander
Université de Genève, Switzerland e-mail: `Martin.Gander@unige.ch`

which combination should be stored. For the multiplicative Schwarz method, which was proved to be equivalent to the discretization of the alternating Schwarz method, see [3], one also does not need a partition of unity, since the method stores the most recently updated values in the overlaps. Additive Schwarz does also not use a partition of unity, it adds all contributions in the overlap, which however leads to a non convergent stationary iterative method [3]. Additive Schwarz is thus not equivalent to a discretization of the parallel Schwarz method of Lions [3], it has to be used as a preconditioner for a Krylov method, which corrects the error made by Additive Schwarz adding all contributions in the overlap. In RAS [1], implicitly a partition of unity was defined by "neglecting part of the communication routine", but any other partition of unity could be used as well. A natural question is if the choice of the partition of unity influences the convergence properties of RAS. It was proved in [3] that RAS is equivalent to the discretization of the parallel Schwarz method of Lions, if the parallel Schwarz method of Lions uses as partition of unity the restriction to the non-overlapping domain decomposition, as described above. Similarly, it was shown in [9] that Additive Schwarz with Harmonic extension [1] is also equivalent to the discretization of the parallel Schwarz method of Lions, but only under certain restrictions on the decomposition. Finally, also a variant called Restricted Additive Schwarz with Harmonic extension (RASH) was introduced in [1], but it was found to have less good convergence properties, even though RASH is symmetric for symmetric problems, while RAS and ASH are not.

Optimized transmission conditions [2] were introduced for RAS in [12, 11] leading the Optimized Restricted Additive Schwarz method (ORAS), and also for ASH leading to OASH [9], and in both cases a direct equivalence to discretized optimized Schwarz methods was proved. A symmetric variant ORASH was proposed in [5] (under the name SORAS), which needs a special coarse correction to permit a convergence analysis of the method using the abstract Schwarz framework. The symmetrized version ORASH has also been studied again with radiation transmission conditions for the Helmholtz case in [4], see also the earlier work for Helmholtz in [7, 8] for a BDD variant with overlap.

We are interested in understanding if the choice of partition of unity influences the convergence of RAS and RASH and their optimized variants ORAS and ORASH. We will prove that in the two subdomain case the choice of partition of unity has no influence on the convergence properties of RAS, and ORAS under an additional condition on the partition of unity, while RASH and ORASH are extremely sensitive to the choice of the partition of unity. The main reason for this is that RAS and ORAS are equivalent to classical and optimized parallel Schwarz methods, while RASH and ORASH have no such interpretation as iterative domain decomposition methods, and generate an extra residual term which we compute explicitly. We also investigate the many subdomain case, including cross points, and show numerically that the partition of unity in the presence of cross points has the same weak influence on the functioning of RAS as on the equivalent discretized parallel Schwarz method of Lions. RASH however is extremely sensitive, and its convergence properties are much less favorable than the convergence properties of RAS.

2 Partitions of Unity for RAS, RASH, ORAS and ORASH

To get a better understanding on how information is transmitted between subdomains in RAS, ORAS, RASH and ORASH, and how this is influenced by the partition of unity used, we consider as our model problem

$$(\eta - \Delta)u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\eta \geq 0$ is a parameter, $\Omega \subset \mathbb{R}^2$ and f is some given source function $f : \Omega \to \mathbb{R}$. We discretize (1) using a finite element or finite difference method and obtain the linear system

$$A\mathbf{u} = \mathbf{f}, \tag{2}$$

where $A \in \mathbb{R}^{m\times m}$ is the system matrix, $\mathbf{f} \in \mathbb{R}^m$ is the discretization of the source term, and $\mathbf{u} \in \mathbb{R}^m$ is an approximation of the solution at the grid points. Schwarz methods are based on a decomposition of the domain Ω into overlapping subdomains Ω_j, $j = 1, 2, \ldots, J$. At the discrete level this decomposition can be identified with a decomposition of the degrees of freedom of the discrete system (2) into a set of overlapping or non-overlapping subsets[1], and is most easily represented by restriction matrices R_j of size $m_j \times m$ which are restrictions of the identity matrix to the rows corresponding to the degrees of freedom in subdomain Ω_j. The restriction operators R_j can also be used to define the subdomain matrices $A_j := R_j A R_j^T$, which correspond to subdomain problems with Dirichlet transmission conditions. We next define a discrete partition of unity represented by diagonal matrices $\chi_j \in \mathbb{R}^{m\times m}$ such that $\sum_{j=1}^{J} \chi_j = I$, the identity, and which equal one on the diagonal for degrees of freedom that belong to one subdomain only. This discrete partition of unity can conveniently be used to define also modified restriction matrices $\widetilde{R}_j := R_j \chi_j$, and the classical choice we have seen in Section 1 is to use a non-overlapping partition to define the χ_j, which leads to $\widetilde{R}_j$ matrices that still only contain zeros and ones, see [1]. There are however also other possibilities, and we define in particular the five partition of unity functions χ_j^ℓ, $\ell = 1, 2, 3, 4, 5$ shown in Figure 1. The first one is the one used in RAS. The second one computes the average of the two subdomain solutions in the overlap. The third one takes a linear combination, weighted by a linear function depending on the distance from the interfaces, and the fourth and fifth one are spline functions, the last one staying longer close to one on the boundary then the former. Each partition of unity function χ_j^ℓ leads to an associated restriction matrix $\widetilde{R}_j^\ell := R_j \chi_j^\ell$ for $\ell = 1, 2, 3, 4, 5$.

We can now define the discrete Schwarz methods RAS, RASH, ORAS and ORASH by defining the preconditioning matrix M^{-1} in the stationary iterative method

$$\mathbf{u}^n = \mathbf{u}^{n-1} + M^{-1}(\mathbf{f} - A\mathbf{u}^{n-1}), \tag{3}$$

[1] For a detailed explanation why a non-overlapping decomposition at the algebraic level still implies an overlapping decomposition at the continuous level for classical finite element and finite difference methods, see [3, Section 3]

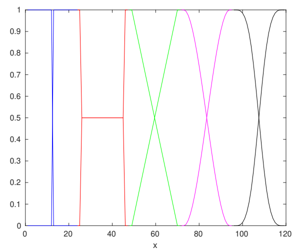

Fig. 1: Five partitions of unity functions we will test, shown in one dimension across a typical overlap size.

$$M^{-1}_{RAS_\ell} = \sum_{j=1}^{J} (\widetilde{R}^\ell_j)^T A_j^{-1} R_j, \;\; M^{-1}_{RASH_\ell} = \sum_{j=1}^{J} (\widetilde{R}^\ell_j)^T A_j^{-1} \widetilde{R}^\ell_j,$$

$$M^{-1}_{ORAS_\ell} = \sum_{j=1}^{J} (\widetilde{R}^\ell_j)^T \widetilde{A}_j^{-1} R_j, \text{ and } M^{-1}_{ORASH_\ell} = \sum_{j=1}^{J} (\widetilde{R}^\ell_j)^T \widetilde{A}_j^{-1} \widetilde{R}^\ell_j,$$

where the subdomain matrices $\widetilde{A}_j$ correspond to subdomain problems with Robin transmission conditions, see [11].

3 Influence of the Partition of Unity on RAS and RASH in 1D

We start by a numerical experiment in one spatial dimension: we use $\Omega := (0, 1)$, two subdomains $\Omega_1 := (0, \beta)$ and $\Omega_2 := (\alpha, 1)$ with $\alpha < \beta$ and solve the model problem in (1) for $\eta = 0$ with boundary conditions $u(0) = 0$ and $u(1) = 1$, so that the solution is a straight line going from zero to one. We discretize the problem using centered finite differences with $m = 100$ interior mesh points, which leads to the mesh size $h = \frac{1}{m+1}$, and we assign the first b mesh points to the first subdomain matrix, and the last $m-a$ mesh points to the second subdomain matrix, which implies that $A_1 \in \mathbb{R}^{b\times b}$, $A_2 \in \mathbb{R}^{m-a\times m-a}$, $R^\ell_1 \in \mathbb{R}^{b\times m}$ $R^\ell_2 \in \mathbb{R}^{m-a\times m}$ and that $\alpha = ah$ and $\beta = (b + 1)h$. We choose $a = 40$ and $b = 60$. The parallel Schwarz method of Lions, which does not need a partition of unity in the case of two subdomains, would then compute

$$A_1 \begin{bmatrix} u^n_{1,1} \\ \vdots \\ u^n_{1,b-1} \\ u^n_{1,b} \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_{b-1} \\ f_b - \frac{1}{h^2} u^{n-1}_{2,b+1} \end{bmatrix}, \quad A_2 \begin{bmatrix} u^n_{2,a+1} \\ u^n_{2,a+2} \\ \vdots \\ u^n_{2,m} \end{bmatrix} = \begin{bmatrix} f_{a+1} - \frac{1}{h^2} u^{n-1}_{1,a} \\ f_{a+2} \\ \vdots \\ f_m \end{bmatrix}. \tag{4}$$

We show in Figure 2 in the first five panels the iterates of RAS and RASH when using the five partition of unity functions χ^ℓ. Note that the iterates of the parallel Schwarz

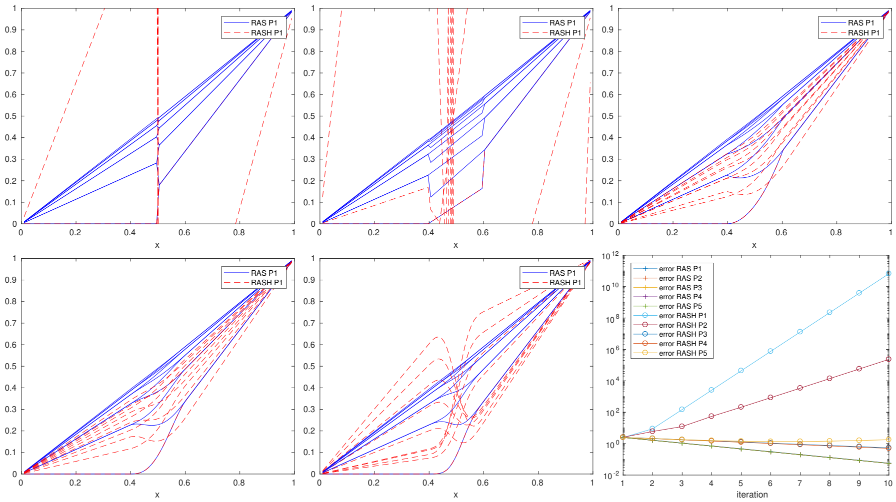

Fig. 2: Iterates of RAS (blue) and RASH (red) for the five partitions of unity, and corresponding convergence curves.

method would just converge monotonically from below to the solution which is a straight line from zero to one. We see that RAS (in solid blue) is converging in the same way for all five partition of unity functions, we only see a difference in the overlap depending on the partition of unity used. RASH however (in dashed red) is diverging violently for the first two partition of unity functions χ^1 and χ^2, converging, albeit more slowly than RAS for the partition of unity functions χ^3 and χ^4, and diverging again for the partition of unity function χ^5. The corresponding convergence curves are shown in the last panel in Figure 2, and we see indeed that RAS converges at the same rate for all partition of unity functions, while RASH only converges for two, and is substantially slower than RAS.

We now prove that the convergence of RAS does not depend on the choice of the partition of unity function, and that RAS is a faithful implementation of the parallel Schwarz method of Lions.

Theorem 1 (The convergence of RAS$_\ell$ does not depend on the partition of unity used) *If the initial iterate* $\mathbf{u}^0$ *of RAS$_\ell$ satisfies* $u_a^0 = u_{2,a}^0$ *and* $u_{b+1}^0 = u_{1,b+1}^0$*, where* $u_{2,a}^0$ *and* u_{b+1}^0 *are the initial guess of the parallel Schwarz method of Lions* (4)*, then the iterates of RAS$_\ell$ outside the overlap coincide with the iterates of the discretized parallel Schwarz method of Lions* (4)*,* $u_j^n = u_{1,j}^n$*,* $j \in \{1, 2, \dots a\} \cup \{b+1, b+2, \dots, m\}$*, independently of the partition of unity* χ^ℓ *used in RAS$_\ell$.*

Proof The proof is by induction: according to the iteration formula for RAS$_\ell$ in (3), one first computes the residual $\mathbf{f} - A\mathbf{u}^0$, which can be written partitioned into two parts in two different ways,

$$\begin{bmatrix} f_1 \\ \vdots \\ f_b \\ f_{b+1} \\ \vdots \\ f_m \end{bmatrix} - \begin{bmatrix} A_1 \begin{bmatrix} u_1^0 \\ \vdots \\ u_b^0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \frac{1}{h^2} u_{b+1}^0 \end{bmatrix} \\ B_1 \begin{bmatrix} u_{b+1}^0 \\ \vdots \\ u_m^0 \end{bmatrix} + \begin{bmatrix} \frac{1}{h^2} u_b^0 \\ 0 \\ \vdots \end{bmatrix} \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_a \\ f_{a+1} \\ \vdots \\ f_m \end{bmatrix} - \begin{bmatrix} B_2 \begin{bmatrix} u_1^0 \\ \vdots \\ u_a^0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \frac{1}{h^2} u_{a+1}^0 \end{bmatrix} \\ A_2 \begin{bmatrix} u_{a+1}^0 \\ \vdots \\ u_m^0 \end{bmatrix} + \begin{bmatrix} \frac{1}{h^2} u_a^0 \\ 0 \\ \vdots \end{bmatrix} \end{bmatrix},$$

where B_j is the remaining diagonal block for the subdomain matrix A_j and of no importance, since the following restriction step in RAS$_\ell$ removes it,

$$R_1(\mathbf{f} - A\mathbf{u}^0) = \begin{bmatrix} f_1 \\ \vdots \\ f_b - \frac{1}{h^2} u_{b+1}^0 \end{bmatrix} - A_1 \begin{bmatrix} u_1^0 \\ \vdots \\ u_b^0 \end{bmatrix}, \; R_2(\mathbf{f} - A\mathbf{u}^0) = \begin{bmatrix} f_{a+1} - \frac{1}{h^2} u_a^0 \\ \vdots \\ f_m \end{bmatrix} - A_2 \begin{bmatrix} u_{a+1}^0 \\ \vdots \\ u_m^0 \end{bmatrix}. \tag{5}$$

Next the subdomain solves A_j^{-1} are applied in parallel, which cancel the remaining A_j matrices,

$$A_1^{-1} R_1(\mathbf{f} - A\mathbf{u}^0) = A_1^{-1} \begin{bmatrix} f_1 \\ \vdots \\ f_b - \frac{1}{h^2} u_{b+1}^0 \end{bmatrix} - \begin{bmatrix} u_1^0 \\ \vdots \\ u_b^0 \end{bmatrix}, \; A_2^{-1} R_2(\mathbf{f} - A\mathbf{u}^0) = A_2^{-1} \begin{bmatrix} f_{a+1} - \frac{1}{h^2} u_a^0 \\ \vdots \\ f_m \end{bmatrix} - \begin{bmatrix} u_{a+1}^0 \\ \vdots \\ u_m^0 \end{bmatrix}.$$

We now see that due to the assumption of identical starting values, $u_a^0 = u_{2,a}^0$ and $u_{b+1}^0 = u_{1,b+1}^0$, precisely the subdomain solves of the parallel Schwarz method of Lions (4) appeared,

$$\begin{bmatrix} u_{1,1}^1 \\ \vdots \\ u_{1,b}^1 \end{bmatrix} = A_1^{-1} \begin{bmatrix} f_1 \\ \vdots \\ f_b - \frac{1}{h^2} u_{b+1}^0 \end{bmatrix}, \qquad \begin{bmatrix} u_{2,a+1}^1 \\ \vdots \\ u_{2,m}^1 \end{bmatrix} = A_2^{-1} \begin{bmatrix} f_{a+1} - \frac{1}{h^2} u_a^0 \\ \vdots \\ f_m \end{bmatrix},$$

and we therefore obtain in the last combination step of RAS$_\ell$

$$\begin{aligned} \mathbf{u}^1 &= \begin{bmatrix} u_1^0 \\ \vdots \\ u_{m-1}^0 \end{bmatrix} + (\widetilde{R}_1^\ell)^T \left(\begin{bmatrix} u_{1,1}^1 \\ \vdots \\ u_{1,b}^1 \end{bmatrix} - \begin{bmatrix} u_1^0 \\ \vdots \\ u_{b-1}^0 \end{bmatrix} \right) + (\widetilde{R}_2^\ell)^T \left(\begin{bmatrix} u_{2,a+1}^1 \\ \vdots \\ u_{2,m}^1 \end{bmatrix} - \begin{bmatrix} u_{a+1}^0 \\ \vdots \\ u_m^0 \end{bmatrix} \right) \\ &= (\widetilde{R}_1^\ell)^T \begin{bmatrix} u_{1,1}^1 \\ \vdots \\ u_{1,b}^1 \end{bmatrix} + (\widetilde{R}_2^\ell)^T \begin{bmatrix} u_{2,a+1}^1 \\ \vdots \\ u_{2,m}^1 \end{bmatrix}, \end{aligned} \tag{6}$$

because the old iterate cancels due to the partition of unity used in the $\widetilde{R}_j^\ell$, and the same property also shows that the new iterate $\mathbf{u}^1$ of RAS$_\ell$ coincides outside the overlap with the parallel Schwarz iterates from (4), and this independently of the partition of unity used. Induction now concludes the proof. □

So why does RASH$_\ell$ fail? This can be seen in step (5), where in RASH$_\ell$ the $\widetilde{R}_j^\ell$ operators would be applied containing the partition of unity. Adding and subtracting $R_j(\mathbf{f} - A\mathbf{u}^0)$, we obtain in RASH$_\ell$

$$\widetilde{R}_1(\mathbf{f} - A\mathbf{u}^0) = \begin{bmatrix} f_1 \\ \vdots \\ f_b - \frac{1}{h^2}u_{b+1}^0 \end{bmatrix} - A_1 \begin{bmatrix} u_1^0 \\ \vdots \\ u_b^0 \end{bmatrix} + (\widetilde{R}_1 - R_1)(\mathbf{f} - A\mathbf{u}^0), \tag{7}$$

$$\widetilde{R}_2(\mathbf{f} - A\mathbf{u}^0) = \begin{bmatrix} f_{a+1} - \frac{1}{h^2}u_a^0 \\ \vdots \\ f_m \end{bmatrix} - A_2 \begin{bmatrix} u_{a+1}^0 \\ \vdots \\ u_m^0 \end{bmatrix} + (\widetilde{R}_2 - R_2)(\mathbf{f} - A\mathbf{u}^0). \tag{8}$$

This implies that in the last combination step of RASH$_\ell$ artificial source terms are left,

$$\mathbf{u}^1 = (\widetilde{R}_1^\ell)^T \begin{bmatrix} u_{1,1}^1 \\ \vdots \\ u_{1,b}^1 \end{bmatrix} + (\widetilde{R}_2^\ell)^T \begin{bmatrix} u_{2,a+1}^1 \\ \vdots \\ u_{2,m}^1 \end{bmatrix} + \left(\sum_{j=1}^{2} \widetilde{R}_j^T A_j^{-1} (\widetilde{R}_j - R_j) \right) (\mathbf{f} - A\mathbf{u}^0). \tag{9}$$

These source terms modify the correct Schwarz iterates, and even though these artificial source terms go to zero when the residual goes to zero, they greatly affect the convergence, and can even lead to divergence, see Figure 2.

4 RAS, RASH, ORAS and ORASH in 2D

The generalization of Theorem 1 to higher spatial dimensions and more than two subdomains does not present any difficulties under the no-crosspoint assumption[2]. As an illustration, we show numerical experiments on the unit square, solving the model problem (1) for $\eta = 0$ using a uniform mesh size $h = \frac{1}{40}$ and two equal subdomains which overlap by $11h$ and $\mathbf{f} = 0$, which means that we simulate directly the error equations. We show in Figure 3 the third iteration starting with the same random initial guess for RAS$_\ell$ (left) and the corresponding results for RASH$_\ell$ (right). As in one spatial dimension, RAS$_\ell$ converges outside of the overlap like the parallel Schwarz method of Lions, only in the overlap one can see the influence of the partition of unity, which does not affect the convergence. This is very different for RASH$_\ell$, where convergence can be completely destroyed by the partition of unity.

[2] Even in the presence of cross points, the equivalence of the discretized parallel Schwarz method of Lions and RAS is proved in [3, Theorem 3.5] for a partition of unity of the form χ^1, the proof for other partitions of unity can be obtained following the arguments in the proof of Theorem 1.

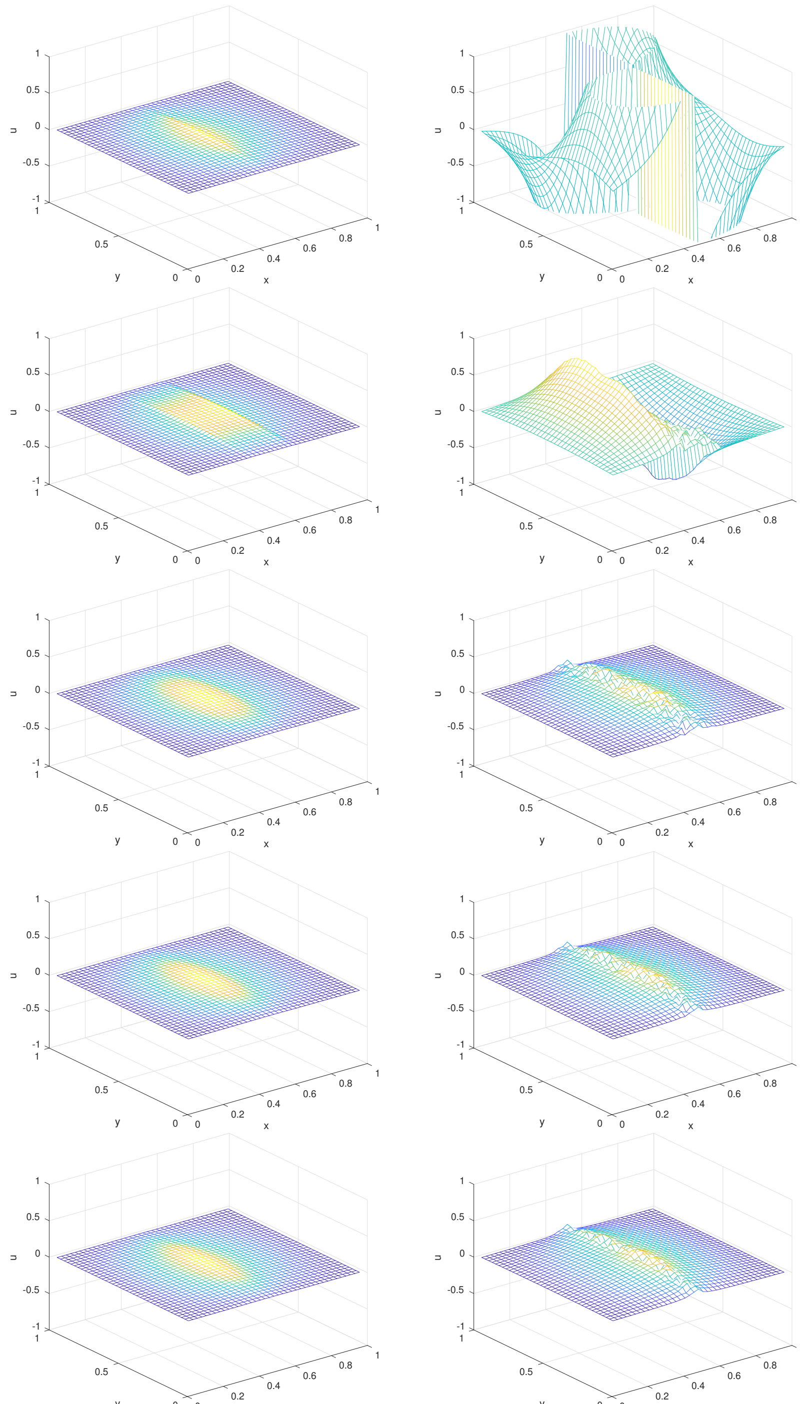

Fig. 3: Third iterate of RAS_ℓ (left) and RASH_ℓ (right) for $\ell = 1, 2, 3, 4, 5$ corresponding to the five different partitions of unity.

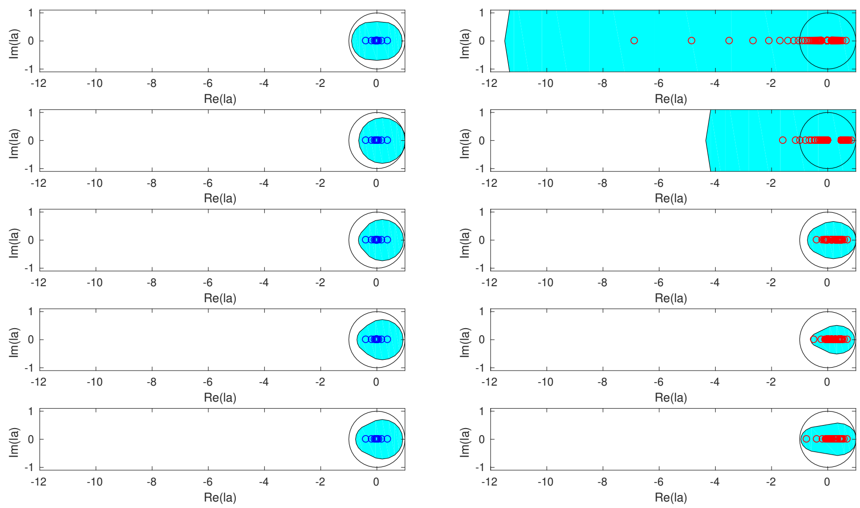

Fig. 4: Spectra and numerical range of the preconditioned operators with RAS_ℓ (left) and RASH_ℓ (right) for $\ell = 1, 2, 3, 4, 5$ corresponding to the five different partitions of unity.

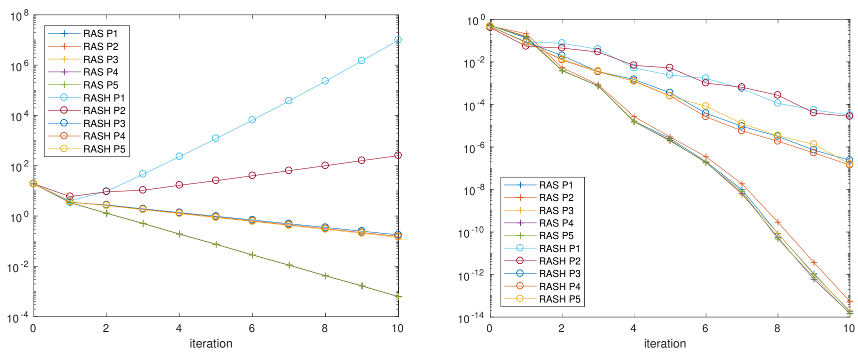

Fig. 5: Convergence of RAS_ℓ and RASH_ℓ as iterative solvers (left) and when used as preconditioners for GMRES (right), for for $\ell = 1, 2, 3, 4, 5$ corresponding to the five different partitions of unity.

In Figure 4 we show the spectra and numerical range of the preconditioned operators. As expected, we see that the spectra of RAS_ℓ are not affected by the partition of unity, while the spectra of RASH_ℓ are: the first two partitions of unity cause large negative eigenvalues, which explain the divergence of the iterative method in this case. The smoother partitions of unity lead to convergent methods, but the spectra are clearly less favorable for convergence. A similar observation holds also for the numerical range which can be related to the convergence of preconditioned GMRES: for RAS_ℓ, the numerical range is very similar, which indicates similar convergence for GMRES, whereas for RASH_ℓ, the first two partitions of unity lead to a much larger numerical range which is unfavorable for GMRES. This is illustrated in Figure 5, where we see on the left that as iterative solver, RAS_ℓ faithfully produces the same convergence behavior of the parallel Schwarz method

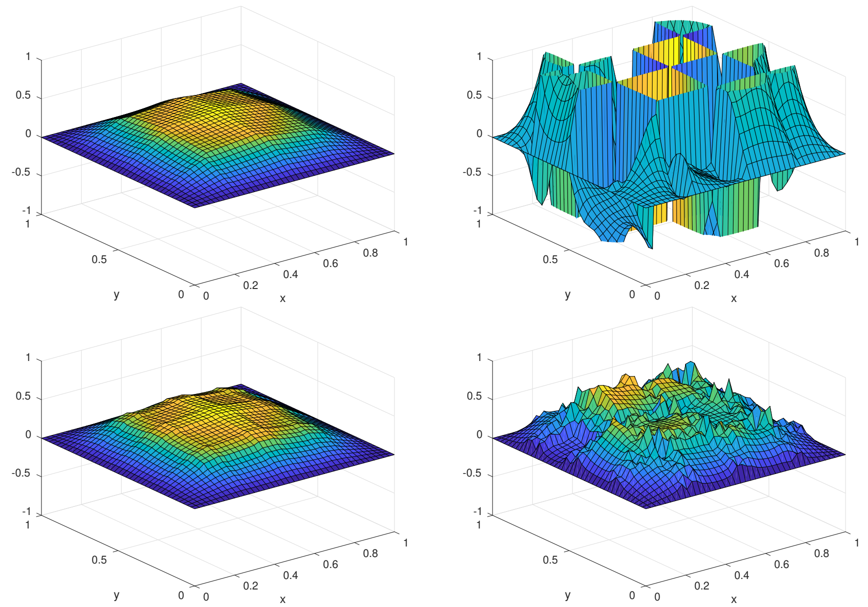

Fig. 6: Third iterate of RAS_ℓ (left) and RASH_ℓ (right) for $\ell = 1, 2$ and the 4×4 subdomain case.

of Lions independently of the partition of unity used, which leads on the right when used as preconditioner to rapid convergence of the residuals, not identical, since the residuals are also minimized in the overlap, where the partition of unity has a slight influence on the numerical range as seen in Figure 4 on the left. This is very different for RASH_ℓ, which can both converge and diverge as an iterative solver, see Figure 5 on the left. When used as a preconditioner, RASH_ℓ is much less effective than RAS_ℓ, and the convergence depends on the partition of unity used: as indicated by the numerical range in Figure 4 on the right, the first two partitions of unity lead to worse convergence of GMRES for RASH_ℓ, see Figure 5 on the right.

We next investigate for the first two partitions of unity the case where cross points are present, namely a decomposition of the unit square into 4×4 subdomains, using the same mesh size $h = \frac{1}{40}$ but a smaller overlap $3h$ to still clearly see the subdomains, see Figure 6. Like the parallel Schwarz method of Lions, RAS depends only little on the partition of unity used, while RASH depends very strongly. In Figure 7 we show the spectra and numerical range of the preconditioned operators, and we see that while RAS_ℓ also is convergent in the presence of cross points, RASH_ℓ is not for the two partitions of unity. We show in Figure 8 (left) the corresponding convergence plots. We observe that in the presence of cross points, the convergence of RAS depends a little on the partition of unity, exactly like the parallel Schwarz method of Lions: the first partition of unity is better than the second one, since it takes data further away from the interfaces, which is better for the Schwarz method by the maximum principle. The dependence of RASH is however very strong: we see violent divergence for the first partition of unity, and also slow divergence for the

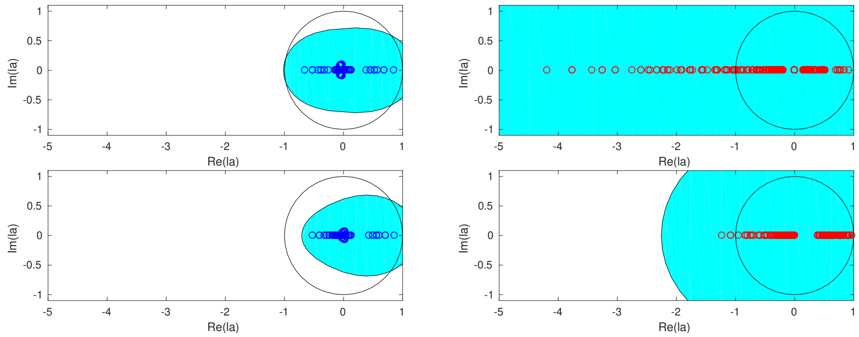

Fig. 7: Spectra and numerical range of the preconditioned operators with RAS_ℓ (left) and RASH_ℓ (right) for $\ell = 1, 2$ and the 4×4 subdomain case.

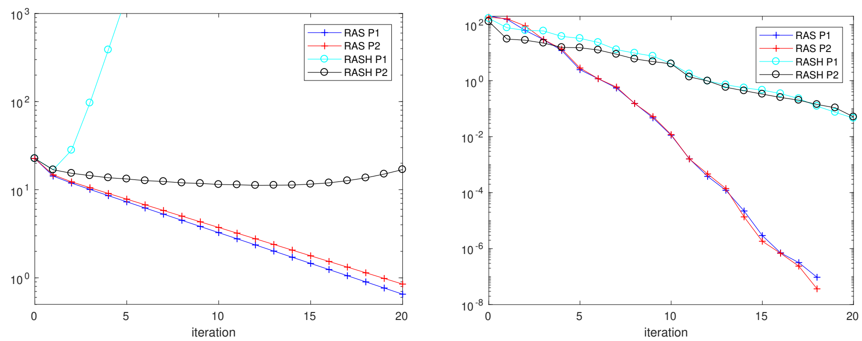

Fig. 8: Convergence of RAS_ℓ and RASH_ℓ as iterative solvers (left) and when used as preconditioners for GMRES (right), for $\ell = 1, 2$ and the 4×4 subdomain case.

second one. The spectrum and numerical range in Figure 7 (right) explains their less favorable properties as preconditioners, see Figure 8 (right).

We finally test the optimized variants ORAS and ORASH: it was shown in [11] that ORAS is a discretization of the optimized Schwarz method with Robin transmission conditions for the partition of unity function χ^1, and this result holds provided the partition of unity equals one at least for the first layer inside the overlap, which is almost satisfied by χ^5 as well, but not for the other partitions of unity. We show in Figure 9 the results corresponding to Figure 5 but now using ORAS_ℓ and ORASH_ℓ. We see that ORAS_1 performs indeed best, like an optimized Schwarz method and much better than RAS_ℓ. ORAS_5 also works, but ORAS_2, ORAS_3 and ORAS_4 are now not functioning properly, since the partition of unity overwrites the location where derivative information needs to be extracted. ORASH_ℓ never works properly, which then also leads to very poor performance when used as a preconditioner, see Figure 9 on the right, even worse than RAS_ℓ, and only marginally better than RASH_ℓ. It is therefore delicate to use the symmetrized versions RASH_ℓ and ORASH_ℓ, and for ORAS_ℓ the partition of unity needs to satisfy a constraint. Note

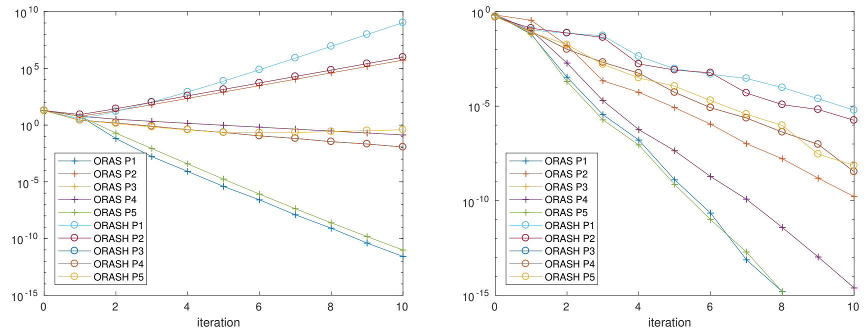

Fig. 9: Convergence of ORAS_ℓ and ORASH_ℓ as iterative solvers (left) and when used as preconditioners for GMRES (right), for $\ell = 1, 2, 3, 4, 5$ corresponding to the five different partitions of unity.

that similar problems were also observed in an alternating version in [6, subsection 6.1] when not keeping the correct Robin interface data.

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