Optimized Schwarz Methods for Linear Elasticity and Overlumping

Kévin Santugini-Repiquet

1 Introduction

Linear Elasticity models how elastic solids deform in the presence of surface and volume forces. The model of Linear Elasticity is valid for small deformations. For large deformations, the nonlinear theory of elasticity should be used instead. For an introduction to Linear Elasticity, we refer the reader to [3]. Linear Elasticity is commonly discretized using Finite Element Methods, see [1, Chap. 11].

Domain Decomposition Methods (DDMs) have previously been applied to Linear Elasticity [2, 6, 5, 7, 10, 11, 14, 15, 13, 19, 18, 16]. However, we found no reference on applying OSMs to the equations of Linear Elasticity.

Optimized Schwarz methods(OSMs) are a family of Domain Decompositions Methods. In iterative OSMs, at each iteration, the interior equation is solved inside each subdomain with artificial conditions on each subdomain boundary. Then, data is exchanged between neighboring subdomains to update those boundary conditions. The process is reiterated until convergence. See [8] for a full analysis of OSMs. The most common transmission conditions are Robin transmission conditions and Ventcell transmission conditions. In [9], the authors showed that we should lump (and even overlump) Robin transmission conditions when applying OSMs to a FEM(Finite Element Method) discretization of Poisson Equations.

In this paper, our main goal is to apply one-level Optimized Schwarz Methods (OSM) to the Finite Element Discretization of the Linear Elasticity problems. We first present some basic definitions on Linear Elasticity in §2. To this end, we derive transmission conditions applicable to Linear Elasticity, obtain an OSM for Linear Elasticity, and establish convergence in §3 using energy estimates. Finally, in §4, we present numerical results, and observe that numerically, overlumping transmission conditions at the discrete level yields a better convergence rate.

Kévin Santugini
Bordeaux INP, IMB, UMR 5251, F-33400,Talence, France, e-mail: Kevin.Santugini-Repiquet@bordeaux-inp.fr

2 Basic Linear Elasticity Definitions

Let Ω be a domain of $\mathbb{R}^3$. Let $\boldsymbol{u}: \Omega \to \mathbb{R}^3$ be a vector field called small displacements. The strain tensor ε is defined as $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, and the stress tensor σ is defined as $\sigma_{ij}(\boldsymbol{u}) = \sum_{k\ell} \mathsf{C}_{ijk\ell}\varepsilon_{k\ell}(\boldsymbol{u})$. The tensor $\mathsf{C}_{ijk\ell}$ is called the stiffness tensor, depends on the material, and satisfy $\mathsf{C}_{ijk\ell} = \mathsf{C}_{k\ell ij}$ and $\mathsf{C}_{ijk\ell} = \mathsf{C}_{jik\ell}$. In addition, the stiffness tensor is positive definite, *i.e.*, there exists $\alpha > 0$ such that

$$\sum_{i,j,k,\ell} \mathsf{C}_{ijk\ell}(\boldsymbol{x})\varepsilon_{ij}\varepsilon_{k\ell} \geq \alpha \sum_{ij} |\varepsilon_{ij}|^2.$$

In this paper, we only consider homogenous isotropic materials. For isotropic material,

$$\mathsf{C}_{ijk\ell} = \frac{E}{1+\nu}\delta_i^k\delta_j^\ell + \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_i^j\delta_k^\ell,$$

where E is the Young modulus, and ν is the Poisson coefficient.

Let $f_v : \Omega \to \mathbb{R}^3$ be the vector field of volume forces applied to the solid body. Let $\Gamma_d \subset \partial\Omega$. Let $f_s : \Gamma_d \to \mathbb{R}^3$ be the vector field of surface forces applied to $\Gamma_f \subset \partial\Omega$. And let $\boldsymbol{d}$ be the known displacements on $\Gamma_d = \partial\Omega \setminus \Gamma_f$. In the variational formulation of Linear Elasticity, a weak solution is defined as a $\boldsymbol{u}$ in V such that for all $\boldsymbol{v}$ in V_ℓ

$$\int_\Omega \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\mathrm{d}\boldsymbol{x} = \int_\Omega f_v\boldsymbol{v}\mathrm{d}\boldsymbol{x} + \int_{\Gamma_f} f_s\boldsymbol{v}\mathrm{d}S(\boldsymbol{x}. \tag{1}$$

where

$$V = \{\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^3) : \boldsymbol{u} = \boldsymbol{d} \text{ on } \Gamma_d\}, \qquad V_\ell = \{\boldsymbol{v} \in H^1(\Omega) : \boldsymbol{v} = 0 \text{ on } \Gamma_d\}.$$

3 Optimized Schwarz Methods for Linear Elasticity

3.1 At the continuous level

In iterative OSMs, at each iteration, the interior equation is solved inside each subdomain with artificial transmission conditions at the interface between subdomains. Then, in order to update these conditions, data is exchanged between neighboring subdomains.

In order to apply OSMs to Linear Elasticity, adequate transmission conditions are needed. For Poisson equations, the simplest transmission conditions are Robin transmission conditions. Robin conditions are a linear combination of Dirichlet and Neumann boundary conditions. The Neumann conditions originates from the following integral equality:

$$\int_{\Omega} \nabla\phi \cdot \nabla\psi \mathrm{d}\boldsymbol{x} - \int_{\Omega} (-\Delta\phi)\psi \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \frac{\partial\phi}{\partial\boldsymbol{n}} \psi \mathrm{d}S(\boldsymbol{x}).$$

for all $\phi\colon \Omega \to \mathbb{R}$, and $\psi : \Omega \to \mathbb{R}$ regular enough and where $\boldsymbol{n}$ is the outer-pointing normal to Ω. Likewise, from the variational formulations of Linear Elasticity (1), we get

$$\int_{\Omega} \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \mathrm{d}\boldsymbol{x} - \int_{\Omega} (-\mathrm{div}(\sigma(\boldsymbol{u})))\boldsymbol{v} \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} (\sigma(\boldsymbol{u})\boldsymbol{n}) \cdot \boldsymbol{v} \mathrm{d}S(\boldsymbol{x}).$$

Hence, the equivalent to the Neumann boundary condition for Linear Elasticity is $\sigma(\boldsymbol{u})\boldsymbol{n}$.

To define OSMs on the equations of linear elasticity, we consider a domain Ω divided in N subdomains Ω_i $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$. Let $\boldsymbol{n}_{ij}$ be the normal to Γ_{ij} pointing from Ω_i to Ω_j. Let $\mathcal{S}_{ij}$ be operators on some functional space defined over $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$. Transmission conditions for Linear Elasticity are:

$$\sigma(\boldsymbol{u}_i^{n+1})\boldsymbol{n}_{ij} + S_{ij}\boldsymbol{u}_i^{n+1} = \sigma(\boldsymbol{u}_j^n)\boldsymbol{n}_{ij} + S_{ij}\boldsymbol{u}_j^n$$

In particular, Robin transmission conditions for Linear Elasticity are obtained when $S_{ij}(\boldsymbol{u}) = p\boldsymbol{u}$ with $p \in \mathbb{R}^+$ being the Robin parameter. In this paper, we always suppose $S_{ij} = S_{ji}$. The Optimized Schwarz Algorithm for the equations of Linear Elasticity at the continuous level is given in Algorithm 1.

Algorithm 1: (Optimized Schwarz for Linear Elasticity)

Initialize $g_{ij}^0\colon \Gamma_{ij} \to \mathbb{R}^3$, to some initial guess in $L^2(\Gamma_{ij})$.
for $n \geq 0$ and until convergence **do**
In each subdomain Ω_i, compute the iterates $\boldsymbol{u}_i^n$ in parallel as the solutions in Ω_i to the variational formulation of:

$$\begin{cases} \mathrm{div}(\sigma(\boldsymbol{u}_i^n)) + \boldsymbol{f}_v = 0 \text{ in } \Omega_i, \\ \sigma(\boldsymbol{u}_i^n)\boldsymbol{n}_{ij} + S_{ij}\boldsymbol{u}_i^n = g_{ij}^n \text{ on } \Gamma_{ij}, \\ \boldsymbol{u}_i^n = \boldsymbol{d} \text{ on } \partial\Omega_i \cap \Gamma_d, \\ \sigma(\boldsymbol{u}_i^n)\boldsymbol{n} = \boldsymbol{f}_s \text{ on } \partial\Omega_i \cap \Gamma_f . \end{cases}$$

For all neighboring subdomains Ω_i and Ω_j, set $g_{ij}^{n+1} := -g_{ji}^n + (S_{ij} + S_{ji})u_j^{n+1}$.
end for

Using Energy Estimates introduced in [17, 4] for the Poisson equation, we can prove the convergence of OSMs applied to Linear Elasticity at the continuous level.

Theorem 1 *If $S_{ij}^h = S_{ji}^h$, if each S_{ij} is symmetric positive definite, and if there is one subdomain where $\Gamma_d \cap \partial\Omega_i$ is of nonzero surface measure, then the Optimized Schwarz Method (1) at the continuous level is convergent.*

Proof Due to the linearity of the equations, we can without loss of generality suppose that the volume forces $\boldsymbol{f}_s$, surface forces $\boldsymbol{f}_v$ and known displacements $\boldsymbol{d}$ are null.

For each subdomain Ω_i, we multiply the interior equation satisfied by $\boldsymbol{u}_i^n$ by $\boldsymbol{u}_i^n$ then integrate over Ω_i. After applying Green's formulas, we get:

$$\int_{\Omega_i} \sigma(\boldsymbol{u}_i^n) : \varepsilon(\boldsymbol{u}_i^n)\mathrm{d}\boldsymbol{x} = \sum_j \int_{\Gamma_{ij}} (\sigma(\boldsymbol{u}_i^n)\boldsymbol{n}_{ij}) \cdot \boldsymbol{u}_i^n \mathrm{d}S(\boldsymbol{x}). \tag{2}$$

By [12, Theorem 3.35], S_{ij} has a symmetric definite positive square root which we denote by $M_{ij} \coloneqq S_{ij}^{1/2}$. And we have:

$$\begin{aligned}
&\int_{\Gamma_{ij}} (\sigma(\boldsymbol{u}_i^n)\boldsymbol{n}_{ij}) \cdot \boldsymbol{u}_i^n \mathrm{d}S(\boldsymbol{x}) = \int_{\Gamma_{ij}} (M_{ij}^{-1}\sigma(\boldsymbol{u}_i^n)\boldsymbol{n}_{ij}) \cdot M_{ij}\boldsymbol{u}_i^n \mathrm{d}S(\boldsymbol{x}) \\
&= \frac{1}{4}\left(\int_{\Gamma_{ij}} |M_{ij}^{-1}(\sigma(\boldsymbol{u}_i^n)\boldsymbol{n}_{ij} + S_{ij}\boldsymbol{u}_i^n)|^2 \mathrm{d}S(\boldsymbol{x}) - \int_{\Gamma_{ij}} |M_{ij}^{-1}(\sigma(\boldsymbol{u}_i^n)\boldsymbol{n}_{ij} - S_{ij}\boldsymbol{u}_i^n)|^2 \mathrm{d}S(\boldsymbol{x})\right), \\
&= \frac{1}{4}\left(\int_{\Gamma_{ij}} |M^{-1}g_{ij}^n|^2 \mathrm{d}S(\boldsymbol{x}) - \int_{\Gamma_{ij}} |M^{-1}g_{ij}^{n+1}|^2 \mathrm{d}S(\boldsymbol{x})\right)
\end{aligned}$$

Combining this equality with (2), and summing over the subdomain index i, and over the iteration index n, we get

$$\sum_{n=0}^{+\infty}\sum_{i=1}^{N} \int_{\Omega_i} \sigma(\boldsymbol{u}_i^n) : \varepsilon(\boldsymbol{u}_i^n)\mathrm{d}\boldsymbol{x} \le \frac{1}{4}\sum_{ij}\int_{\Gamma_{ij}} |M^{-1}g_{ij}^0|^2 \mathrm{d}S(\boldsymbol{x}) < +\infty.$$

Since the stiffness tensor $\mathsf{C}_{ijk\ell}$ is positive definite, this implies $\varepsilon(\boldsymbol{u}_i^{n+1})$ converges to 0 as n goes to infinity. This proves that inside each subdomain, the iterates converges to an equiprojective vector field. This implies the limit is zero on the subdomain where $\Gamma_d \cap \partial\Omega_i$ is of nonzero measure. Since a domain is always connected by definition, and using the transmission condition, one gets the limit is also zero on the other subdomains. □

3.2 FEM Discretization of OSMs for Linear Elasticity

In this section, we describe how to discretize OSMs for Linear Elasticity with Finite Element Methods. Let's consider a tetrahedral mesh $\mathcal{T}^h$ of Ω compatible with the domain decomposition of Ω in N subdomains $(\Omega_i)_{1\le i\le N}$. Let $\mathcal{T}_i$ be the restriction of mesh $\mathcal{T}$ to subdomain Ω_i. We use $\mathcal{P}^1$ elements for each component of the small displacements. So at most three degrees (one per component) of freedom per node.

Let M be the number of degree of freedoms. Let the $\boldsymbol{\phi}_k$ be the elementary basis functions of the finite element space. For any k in $[\![1, M]\!]$, $\boldsymbol{\phi}_k$ is null on every node of the mesh except one. And on this node, $\boldsymbol{\phi}_k$ belongs to the canonical basis of $\mathbb{R}^3$. Let $\mathcal{I}_i$ be the subset of $[\![1, M]\!]$ of indices corresponding to degrees of freedoms located on a node of $\mathcal{T}_i$. The $\mathcal{I}_i$ are not disjoint. For all k, ℓ in $\mathcal{I}_i$, we set:

$$(A_i^h)_{k,\ell} \coloneqq \int_{\Omega_i} \sigma(\boldsymbol{\phi}_k) : \varepsilon(\boldsymbol{\phi}_\ell), \qquad (\boldsymbol{f}_i^h)_k = \int_{\Omega_i} f \cdot \boldsymbol{\phi}_k.$$

There are multiple ways to discretize the transmission condition. For a consistent discretization, we set:

$$(S_{i,j}^{h,\text{cons}})_{k,\ell} \coloneqq \int_{\Gamma_{ij}} (\sigma(\boldsymbol{\phi}_k)\boldsymbol{n}) \cdot \boldsymbol{\phi}_\ell,$$

for all k, ℓ in $\mathcal{I}_i \cap \mathcal{I}_j$. Alternatively, we can use a lump discretization, and define $S_{i,j}^{h,\text{lump}}$ as the diagonal matrix obtained by lumping $S_{i,j}^{h,\text{cons}}$. We also set the overlumped matrix $S_{ij}^{h,\omega} \coloneqq (1-\omega)S_i^{h,\text{cons}} + \omega S_{ij}^{h,\text{lump}}$. For Poisson equation, overlumping has been shown to be beneficial in [9].

The main issue is deciding how transmission conditions should be updated, especially near cross-points (or cross-edges). This is especially true when cross-points (or cross-edges in 3d) are present. When using a FEM discretization of Linear Elasticity, the discrete value of $\sigma(\boldsymbol{u}_i)\boldsymbol{n}$ is only known as a variational quantity, as an integral over the boundary of $\partial\Omega_i$. Near cross-point, this variational quantity represents an integral over multiple surfaces each shared by $\partial\Omega_i$ with another subdomain. Ideally, this quantity must be split before being sent to the neighboring subdomains. Unfortunately, near cross-points, there is no canonical way to do so. See [9], for an explanation on how to discretize OSMs near cross-points for Poisson Equation, including the "Auxiliary Variable Method". When there are cross-points, at the discrete level, the g_{ij}^{n+1} cannot be derived from the discrete u_i^n. However, using (3b), they can be derived from both the g_{ij}^n and the u_i^n. Hence, in the Auxiliary Variable Method, the unknowns are not the discrete u_i^n, but the discrete g_{ij}^{n+1}.

The OSM iteration can be written at the discrete level as:

$$A_i^h \boldsymbol{u}_i^n = \boldsymbol{f}_i^h + \sum_j S_{ij}^h g_{ij}^n, \tag{3a}$$

$$g_{ij}^{n+1} \coloneqq -g_{ji}^n + (S_{ij}^h + S_{ji}^h) u_j^n. \tag{3b}$$

Theorem 2 *If $S_{ij} = S_{ji}$, if each S_{ij} is symmetric positive definite, and if there is one subdomain where $\Gamma_d \cap \partial\Omega_i$ is of nonzero surface measure, then the Auxiliary Variable Method OSM, Eq.* (3)*, applied to a FEM-discretization of Linear Elasticity is convergent. I.E., if $(u_i)_{1\le i\le N}$ represents the discrete mono-domain solution, $u_i - u_i^n$ converges to* 0*.*

Proof There exists a finite sequence of $(g_{ij})_{ij}$ that is a fixed point of the (3b) iterate. Hence, we can suppose $\boldsymbol{f}^h$ null. Using u_i^n as the test function, we get

$$\int_{\Omega_i} \sigma(\boldsymbol{u}_i^n) : \varepsilon(\boldsymbol{u}_i^n)\mathrm{d}\boldsymbol{x} = \sum_j \int_{\Gamma_{ij}} (g_{ij}^n - S_{ij}u_{ij}^n) \cdot u_{ij}^n \mathrm{d}\boldsymbol{x}$$

Then, we can reuse the end of the proof of Theorem1 almost verbatim. □

4 Numerical Results

We consider a cylindrical domain with a diameter of 1 and a heigh of 3.2. We subdivide this domain in two identical cylindrical subdomains. The domain is meshed using 4144 tetrahedrons. We set the Young Modulus $E = 1$ and the Poisson coefficient to either $\nu = 0.1$ or $\nu = 0.49$. We tested various values of the Robin parameter p and of the lump parameter ω. We found the best convergence for $p = 0.4$. As for Poisson equations, we found that overlumping the transmission condition substantially improves convergence, see convergence curves in Figures 1 and 2. Convergence is slower when the Poisson coefficient is near $1/2$.

We also did a similar test by subdividing the same cylindrical domain into ten identical cylindrical subdomains. We set the Young Modulus $E = 1$ and the Poisson coefficient $\nu = 0.1$. See convergence curves in Figure 3. As expected in the absence of coarse spaces, the convergence of the Optimized Schwarz Method is considerably slower with ten subdomains.

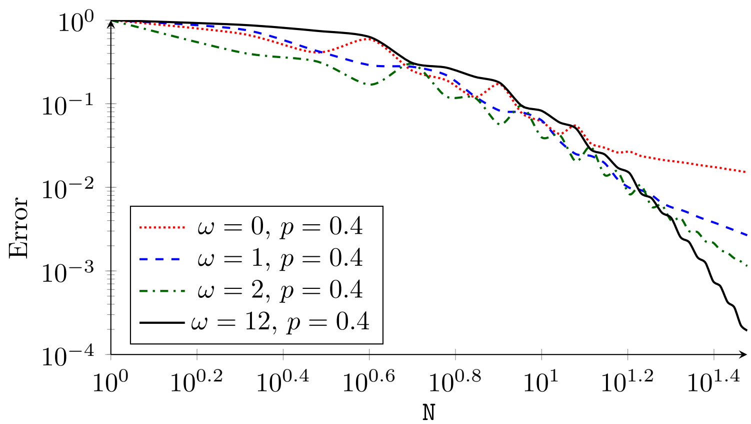

Fig. 1: Numerical Results for two subdomains with $\nu = 0.1$

5 Conclusion

In this paper, we showed how to to derive the equivalent of Robin boundary transmission for the equations of linear elasticity. Using overlumping, we improved these boundary transmission condition without the need to discretize higher order transmission conditions. We proved the theoretical convergence of Non Overlapping Optimized Schwarz Methods for linear elasticity.

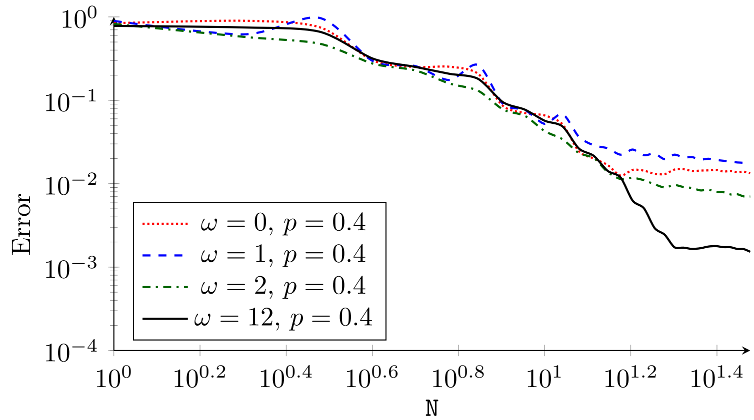

Fig. 2: Numerical Results for two cylindrical subdomains with $E = 1.0$ and $\nu = 0.49$

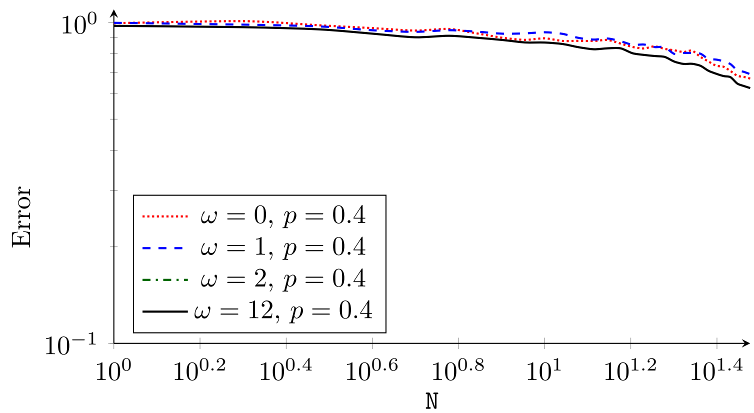

Fig. 3: Numerical Results for ten cylindrical subdomains with $E = 1.0$ and $\nu = 0.1$

As future works, we currently see three ways to expand upon this work. First, we will further study how to discretize the OSMs method for linear elasticity when cross-points are present. Then, we will generalize the Robin boundary condition for linear elasticity by replacing the scalar Robin parameter p with a 3 by 3 matrix. Finally, we are planning to add a coarse space to OSMs for linear elasticity.

References

1. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods. Springer New York (2008). DOI:10.1007/978-0-387-75934-0
2. Cai, M., Pavarino, L.F., Widlund, O.B.: Overlapping Schwarz methods with a standard coarse space for almost incompressible linear elasticity. SIAM Journal on Scientific Computing **37**(2),

A811–A830 (2015)
3. Ciarlet, P.G.: Mathematical Elasticity: Three-Dimensional Elasticity, *Studies in Mathematics and its Applications*, vol. 1. Elsevier, Academic Press (1988)
4. Després, B.: Domain decomposition method and the Helmholtz problem. In: G.C. Cohen, L. Halpern, P. Joly (eds.) Mathematical and numerical aspects of wave propagation phenomena, *Proceedings in Applied Mathematics Series*, vol. 50, pp. 44–52. Society for Industrial and Applied Mathematics (1991)
5. Dohrmann, C.R., Widlund, O.B.: An overlapping Schwarz algorithm for almost incompressible elasticity. SIAM Journal on Numerical Analysis **47**(4), 2897–2923 (2009)
6. Dohrmann, C.R., Widlund, O.B.: Hybrid domain decomposition algorithms for compressible and almost incompressible elasticity. International Journal for Numerical Methods in Engineering **82**(2), 157–183 (2010)
7. Farhat, C., Mandel, J.: The two-level FETI method for static and dynamic plate problems part i: An optimal iterative solver for biharmonic systems. Computer methods in applied mechanics and engineering **155**(1-2), 129–151 (1998)
8. Gander, M.J.: Optimized Schwarz methods. SIAM J. Numer. Anal. **44**(2), 699–731 (2006)
9. Gander, M.J., Santugini, K.: Cross-points in domain decomposition methods with a finite element discretization. Electron. Trans. Numer. Anal. **45**, 219–240 (2016)
10. Goldfeld, P., Pavarino, L.F., Widlund, O.B.: Balancing Neumann-Neumann methods for mixed approximations of linear elasticity. In: Lecture Notes in Computational Science and Engineering, pp. 53–76. Springer Berlin Heidelberg (2002)
11. Goldfeld, P., Pavarino, L.F., Widlund, O.B.: Balancing Neumann-Neumann preconditioners for mixed approximations of heterogeneous problems in linear elasticity. Numerische Mathematik **95**(2), 283–324 (2003). DOI:10.1007/s00211-002-0450-9
12. Kato, T.: Perturbation theory for linear operators, *Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen*, vol. 132. Springer-Verlag (1966)
13. Klawonn, A., Neff, P., Rheinbach, O., Vanis, S.: FETI-DP domain decomposition methods for elasticity with structural changes:p-elasticity. ESAIM: Mathematical Modelling and Numerical Analysis **45**(3), 563–602 (2010). DOI:10.1051/m2an/2010067
14. Klawonn, A., Pavarino, L.F.: An overlapping additive Schwarz method for a saddle point problem from linear elasticity. In: J. Wang, M.A. III, B. Chen, T. Mathew (eds.) Iterative Methods in Scientific Computation, pp. 219–224. IMACS Series in Computational and Applied Mathematics (1998)
15. Klawonn, A., Widlund, O.B.: A domain decomposition method with Lagrange multipliers and inexact solvers for linear elasticity. SIAM J. Sci. Comput. **22**(4), 1199–1219 (2000)
16. Le Tallec, P., De Roeck, Y.H., Vidrascu, M.: Domain-decomposition methods for large linearly elliptic three dimensional problems. J. Comput. Appl. Math. **34** (1991)
17. Lions, P.L.: On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In: T.F. Chan, R. Glowinski, J. Périaux, O. Widlund (eds.) Third International Symposium on Domain Decomposition Methods for Partial Differential Equations , held in Houston, Texas, March 20-22, 1989, pp. 202–223. SIAM, Philadelphia, PA (1990)
18. Pavarino, L.F., Widlund, O.B., Zampini, S.: BDDC preconditioners for spectral element discretizations of almost incompressible elasticity in three dimensions. SIAM Journal on Scientific Computing **32**(6), 3604–3626 (2010)
19. Smith, B.F.: An optimal domain decomposition preconditioner for the finite element solution of linear elasticity problems. SIAM J. Sci. Stat. Comput. **13**(1), 364–378 (1992)