BDDC for a Saddle Point Problem with an HDG Discretization

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1 Introduction

The Balancing Domain Decomposition by Constraints (BDDC) algorithms, introduced in [4], are nonoverlapping domain decomposition methods. The coarse problems in the BDDC algorithms are given in terms of a set of primal constraints. An important advantage with such a coarse problem is that the Schur complements that arise in the computation will all be invertible. The BDDC algorithms have been extended to many different applications with different discretizations such as [9, 10, 13, 14, 2] and [11, 12].

In this paper, the BDDC algorithm is developed for the incompressible Stokes equation with an Hybridizable Discontinuous Galerkin (HDG) discretization. The HDG discretization for incompressible Stokes flow was introduced in [7] and analyzed in [3]. The main features of the HDG is that it reduces the globally coupled unknowns to the numerical trace of the velocity on the element boundaries and the mean of the pressure on the element. The size of the reduced saddle point problem is significantly smaller compared to the original one. In [7], the reduced saddle point problem is solved by an augmented Lagrange approach. An additional time dependent problem is introduced and solved by a backward-Euler method. Here, we solve the reduced saddle point problem directly using the BDDC methods. Similar to the earlier domain decomposition works on saddle point problems such as [8, 5, 6], and [9], we reduce the saddle point problem to a positive definite problem in a benign subspace and therefore the conjugate gradient (CG) method can be used to solve the resulting system. Due to the discontinuous pressure basis functions in this HDG

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discretization, the complicated no-net-flux condition, which is needed to make sure all CG iterates are in the benign subspace, can be ensured by edge and face average constraints for each velocity component in two and three dimensions, respectively. These required constraints are the same as those for the elliptic problems with the HDG discretizations, cf. [11].

The rest of the paper is organized as follows. The HDG discretization for the Stokes problem are described in Section 2. In Section 3, the original system is reduced to an interface problem and a BDDC preconditioner is then introduced. The condition number estimate for the system with the BDDC preconditioner is provided in Section 4. Finally, we give some computational results in Section 5.

2 A Stokes problem and an HDG Discretization

The following Stokes problem is defined on a bounded polygonal domain Ω, in two or three dimensions, with a Dirichlet boundary condition:

$$\begin{cases} -\triangle \mathbf{u} + \nabla p = \mathbf{f}, \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, \text{ in } \Omega, \\ \mathbf{u} = g, \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{f} \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Without loss of generality, we assume that $g = 0$. The solution of (1) is unique with the pressure p determined up to a constant. Here we will look for the solution with the pressure p having a zero average over the domain Ω.

We follow the approach in [7] and rewrite (1) as follows:

$$\begin{cases} \mathbf{L} - \nabla \mathbf{u} = 0, & \text{in} \quad \Omega, \\ -\nabla \cdot \mathbf{L} + \nabla p = \mathbf{f}, & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in} \quad \Omega, \\ \mathbf{u} = 0, & \text{in} \quad \partial\Omega. \end{cases} \tag{2}$$

Let $P_k(D)$ be the space of polynomials of order at most k on D. We set $\mathbf{P}_k(D) = [P_k(D)]^n$ ($n = 2$ and 3 for two and three dimensions, respectively) and $\mathcal{P}_k(D) = [P_k(D)]^{n\times n}$. $\mathbf{L}$, $\mathbf{u}$, and p will be approximated by these discontinuous finite element spaces defined on a shape-regular and quasi-uniform triangulation of Ω, denoted by $\mathcal{T}_h$. Let h be the characteristic element size h of $\mathcal{T}_h$ and κ be an element in $\mathcal{T}_h$. The union of edges of elements κ is denoted by $\mathcal{E}$. $\mathcal{E}_i$ and $\mathcal{E}_\partial$ are two subsets of $\mathcal{E}$, for the edges in the interior of the domain and on its boundary, respectively. Define the following finite element spaces: $\mathbf{G}_k = \{\mathbf{G}_h \in \left[L^2(\Omega)\right]^{n\times n} : \mathbf{G}_h|_\kappa \in \mathcal{P}_k(\kappa),\ \forall\kappa \in \Omega\}$, $\mathbf{V}_k = \{\mathbf{v}_h \in \left[L^2(\Omega)\right]^n : \mathbf{v}_h|_\kappa \in \mathbf{P}_k(\kappa),\ \forall\kappa \in \Omega\}$, $W_k = \{p_h \in L^2(\Omega) : p_h|_\kappa \in P_k(\kappa),\ \int_\Omega p_h = 0,\ \forall\kappa \in \Omega\}$, $\mathbf{M}_k = \{\mu_h \in \left[L^2(\mathrm{e})\right]^n : \mu_h|_e \in \mathbf{P}_k(e),\ \forall e \in \mathcal{E}\}$, and $\mathbf{\Lambda}_k = \{\mu_h \in \mathbf{M}_k : \mu_h|_e = 0,\ \forall e \in \partial\Omega\}$. To make our notation simple, we drop the subscript k from now on. The discrete problem resulting from the HDG discretization can be written as: to find $(\mathbf{L}_h, \mathbf{u}_h, p_h, \lambda_h) \in (\mathbf{G}, \mathbf{V}, W, \mathbf{\Lambda})$ such that for all $(\mathbf{G}_h, \mathbf{v}_h, q_h, \mu_h) \in (\mathbf{G}, \mathbf{V}, W, \mathbf{\Lambda})$

$$\begin{cases} (\mathbf{L}_h, \mathbf{G}_h)_{\mathcal{T}_h} + (\mathbf{u}_h, \nabla \cdot \mathbf{G}_h)_{\mathcal{T}_h} - \langle \lambda_h, \mathbf{G}_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} & = 0, \\ (\mathbf{L}_h, \nabla \mathbf{v}_h)_{\mathcal{T}_h} - (p_h, \nabla \cdot \mathbf{v}_h)_{\mathcal{T}_h} - \langle \mathbf{L}_h \mathbf{n} - p_h \mathbf{n} - \tau_\kappa (\mathbf{u}_h - \lambda_h), \mathbf{v}_h \rangle_{\partial \mathcal{T}_h} & = (\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h}, \\ - \langle \mathbf{L}_h \mathbf{n} - p_h \mathbf{n} - \tau_\kappa (\mathbf{u}_h - \lambda_h), \mu_h \rangle_{\partial \mathcal{T}_h} & = 0, \\ -(\mathbf{u}_h, \nabla q_h)_{\mathcal{T}_h} + \langle \lambda_h \cdot \mathbf{n}, q_h \rangle_{\partial \mathcal{T}_h} & = 0. \end{cases} \tag{3}$$

where τ_κ is a local stabilization parameter, see [7] for details.

The matrix form of (3) can be written as

$$\begin{bmatrix} A_{\mathbf{LL}} & A_{\mathbf{uL}}^T & A_{\lambda\mathbf{L}}^T & 0 \\ A_{\mathbf{uL}} & A_{\mathbf{uu}} & A_{\lambda\mathbf{u}}^T & B_{p\mathbf{u}}^T \\ A_{\lambda\mathbf{L}} & A_{\lambda\mathbf{u}} & A_{\lambda\lambda} & B_{p\lambda}^T \\ 0 & B_{p\mathbf{u}} & B_{p\lambda} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{L} \\ \mathbf{u} \\ \lambda \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_h \\ \mathbf{0} \\ 0 \end{bmatrix}, \tag{4}$$

where $\mathbf{F}_h = -(\mathbf{f}, \mathbf{v}_h)_{\mathcal{T}_h}$ and we use $\mathbf{L}$, $\mathbf{u}$, λ, and p to denote the unknowns associated with $\mathbf{L}_h$, $\mathbf{u}_h$, λ_h, and p_h, respectively. In each κ, we decompose the pressure degrees of freedom p into the element average pressure p_{0e} and the rest called the element interior pressure p_i and let $W = W_i \oplus W_{0e}$, correspondingly. We can easily eliminate $\mathbf{L}$, $\mathbf{u}$ and p_i element-wise from (4) and obtain the system for λ and p_{0e} only

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ p_{0e} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}. \tag{5}$$

The global problem (4) can also be written as the following saddle point problem

$$\begin{bmatrix} A_a & B_a^T \\ B_a & 0 \end{bmatrix} \begin{bmatrix} u_a \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{F}_a \\ 0 \end{bmatrix}, \tag{6}$$

where

$$A_a = \begin{bmatrix} A_{\mathbf{LL}} & A_{\mathbf{uL}}^T & A_{\lambda\mathbf{L}}^T \\ A_{\mathbf{uL}} & A_{\mathbf{uu}} & A_{\lambda\mathbf{u}}^T \\ A_{\lambda\mathbf{L}} & A_{\lambda\mathbf{u}} & A_{\lambda\lambda} \end{bmatrix}, \quad B_a^T = \begin{bmatrix} 0 \\ B_{p\mathbf{u}}^T \\ B_{p\lambda}^T \end{bmatrix}, \quad u_a = \begin{bmatrix} \mathbf{L} \\ \mathbf{u} \\ \lambda \end{bmatrix}, \quad \text{and } \mathbf{F}_a = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_h \\ \mathbf{0} \end{bmatrix}. \tag{7}$$

We note that A_a is the same as the matrix obtained using HDG discretization for elliptic problem as discussed in [11].

3 The BDDC algorithm

We decompose Ω into N nonoverlapping subdomain Ω_i with diameters H_i, $i = 1, \ldots, N$, and set $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse triangles and that the number of such elements forming an individual subdomain is uniformly bounded. We define edges/faces as open sets shared by two subdomains. Two nodes belong to the same edge/face when they are associated with the same pair of subdomains. Let Γ be the interface between the subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h := \left(\cup_{i\neq j}\partial\Omega_{i,h} \cap \partial\Omega_{j,h}\right) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is that of $\partial\Omega$. We assume the triangulation of each subdomain is quasi-uniform.

We decompose the velocity numerical trace $\mathbf{\Lambda} = \mathbf{\Lambda}_I \oplus \widehat{\mathbf{\Lambda}}_\Gamma$ and the element average pressure $W_{0e} = W_I \oplus W_0$, where $\widehat{\mathbf{\Lambda}}_\Gamma$ denotes the degrees of freedom associated with Γ. $\mathbf{\Lambda}_I = \prod_{i=1}^N \mathbf{\Lambda}_I^{(i)}$ and $W_I = \prod_{i=1}^N W_I^{(i)}$ are products of subdomain interior velocity numerical trace spaces $V_I^{(i)}$ and subdomain interior pressure spaces $W_I^{(i)}$, respectively. The elements of $\mathbf{\Lambda}_I^{(i)}$ are supported in the subdomain Ω_i and vanishes on its interface Γ_i, while the elements of $W_I^{(i)}$ are the restrictions of the pressure variables to Ω_i which satisfy $\int_{\Omega_i} p_I^{(i)} = 0$. $\widehat{\mathbf{\Lambda}}_\Gamma$ is the subspace of edge/face functions on Γ in $\mathbf{\Lambda}$, and W_0 is the subspace of W with constant values $p_0^{(i)}$ in the subdomain Ω_i that satisfy $\sum_{i=1}^N p_0^{(i)} m\left(\Omega_i\right) = 0$, where $m\left(\Omega_i\right)$ is the measure of the subdomain Ω_i.

We denote the space of interface velocity numerical trace variables of the subdomain Ω_i by $\mathbf{\Lambda}_\Gamma^{(i)}$, and the associated product space by $\mathbf{\Lambda}_\Gamma = \prod_{i=1}^N \mathbf{\Lambda}_\Gamma^{(i)}$; generally edge/face functions in $\mathbf{\Lambda}_\Gamma$ are discontinuous across the interface. We define the restriction operators $R_\Gamma^{(i)} : \widehat{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Gamma^{(i)}$ to be an operator which maps functions in the continuous global interface velocity numerical trace variable space $\widehat{\mathbf{\Lambda}}_\Gamma$ to the subdomain component space $\mathbf{\Lambda}_\Gamma^{(i)}$. Also, $R_\Gamma : \widehat{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Gamma$ is the direct sum of $R_\Gamma^{(i)}$.

The global interface problem is assembled from the subdomain interface problems, and can be written as: find $(\lambda_\Gamma,\ p_0) \in \left(\widehat{\mathbf{\Lambda}}_\Gamma, W_0\right)$ such that

$$\widehat{S}\begin{bmatrix}\lambda_\Gamma\\ p_0\end{bmatrix} = \begin{bmatrix}g_\Gamma\\ 0\end{bmatrix}, \qquad \text{where } \widehat{S} = \begin{bmatrix}\widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T\\ \widehat{B}_{0\Gamma} & 0\end{bmatrix}. \tag{8}$$

Here $\widehat{S}_\Gamma$, $\widehat{B}_{0\Gamma}$, and g_Γ are assembled from the subdomain matrices.

In order to introduce the BDDC preconditioner, we first introduce a partially assembled interface space $\widetilde{\mathbf{\Lambda}}_\Gamma = \widehat{\mathbf{\Lambda}}_\Pi \oplus \mathbf{\Lambda}_\Delta = \widehat{\mathbf{\Lambda}}_\Pi \oplus \prod_{i=1}^{N} \mathbf{\Lambda}_\Delta^{(i)}$. Here, $\widehat{\mathbf{\Lambda}}_\Pi$ is the coarse level, primal interface velocity space and the space $\mathbf{\Lambda}_\Delta$ is the direct sum of the $\mathbf{\Lambda}_\Delta^{(i)}$, which are spanned by the remaining interface degrees of freedom. In the space $\widetilde{\mathbf{\Lambda}}_\Gamma$, we relax most continuity constraints across the interface but retain the continuity at the primal unknowns, which makes all the linear systems nonsingular.

We need to introduce several restriction, extension, and scaling operators between different spaces. $\overline{R}_\Gamma^{(i)} : \widetilde{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Gamma^{(i)}$ restricts functions in the space $\widetilde{\mathbf{\Lambda}}_\Gamma$ to the components $\mathbf{\Lambda}_\Gamma^{(i)}$ of the subdomain Ω_i. $\overline{R}_\Gamma : \widetilde{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Gamma$ is the direct sum of $\overline{R}_\Gamma^{(i)}$. $R_\Delta^{(i)} : \widehat{\mathbf{\Lambda}}_\Gamma \to \mathbf{\Lambda}_\Delta^{(i)}$ maps the functions from $\widehat{\mathbf{\Lambda}}_\Gamma$ to $\mathbf{\Lambda}_\Delta^{(i)}$, its dual subdomain components. $R_{\Gamma\Pi} : \widehat{\mathbf{\Lambda}}_\Gamma \to \widehat{\mathbf{\Lambda}}_\Pi$ is a restriction operator from $\widehat{\mathbf{\Lambda}}_\Gamma$ to its subspace $\widehat{\mathbf{\Lambda}}_\Pi$. $\widetilde{R}_\Gamma : \widehat{\mathbf{\Lambda}}_\Gamma \to \widetilde{\mathbf{\Lambda}}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and $R_\Delta^{(i)}$. We define the positive scaling factor $\delta_i^{\dagger}(x)$ as follows:

$$\delta_i^{\dagger}(x) = \frac{1}{card\,(\mathcal{I}_x)}, \qquad x \in \partial\Omega_{i,h} \cap \Gamma_h,$$

where $\mathcal{I}_x$ is the set of indices of the subdomains that have x on their boundaries, and $card\,(\mathcal{I}_x)$ counts the number of the subdomain boundaries to which x belongs. We note that $\delta_i^{\dagger}(x)$ is constant on each edge/face. Multiplying each row of $R_{\Delta}^{(i)}$ with the scaling factor gives us $R_{D,\Delta}^{(i)}$. The scaled operators $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{D,\Delta}^{(i)}$.

We denote the direct sum of the local interface velocity Schur complement by S_Γ and the partially assembled interface velocity Schur complement is defined by $\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma$. Correspondingly, we define an operator $\widetilde{B}_{0\Gamma}$, which maps the partially assembled interface velocity space $\widetilde{\mathbf{\Lambda}}_\Gamma$ into the space of right-hand sides corresponding to W_0. $\widetilde{B}_{0\Gamma}$ is obtained from the subdomain operators by assembling them with respect to the primal interface velocity part. Using the following notation

$$\widetilde{R}_D = \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix}, \qquad \widetilde{S} = \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix}, \tag{9}$$

and the preconditioned BDDC algorithm is then of the form: find $(\lambda_\Gamma, p_0) \in \left(\widehat{\mathbf{\Lambda}}_\Gamma, W_0\right)$, such that

$$\widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \widehat{S} \begin{bmatrix} \lambda_\Gamma \\ p_0 \end{bmatrix} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D \begin{bmatrix} g_\Gamma \\ 0 \end{bmatrix}. \tag{10}$$

Note that $\widetilde{R}_{D,\Gamma}$ is of full rank and that the preconditioner is nonsingular.

Definition 1 (Benign Subspaces) We will call

$$\widehat{\mathbf{\Lambda}}_{\Gamma,B} = \{\lambda_\Gamma \in \widehat{\mathbf{\Lambda}}_\Gamma \mid \widehat{B}_{0\Gamma}\lambda_\Gamma = 0\}, \quad \widetilde{\mathbf{\Lambda}}_{\Gamma,B} \quad = \{\lambda_\Gamma \in \widetilde{\mathbf{\Lambda}}_\Gamma \mid \widetilde{B}_{0\Gamma}\lambda_\Gamma = 0\}$$

the benign subspaces of $\widehat{\mathbf{\Lambda}}_\Gamma$ and $\widetilde{\mathbf{\Lambda}}_\Gamma$, respectively.

It is easy to see that the operators $\widehat{S}$ and $\widetilde{S}$, defined in (8) and (10), are symmetric positive definite on $\left(\widehat{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right)$ and $\left(\widetilde{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right)$, respectively. A preconditioned conjugate gradient method can then be used to solve the global BDDC preconditioned interface problem (10).

4 Condition number estimate for the BDDC preconditioner

In this section, we only consider the case that the stabilization parameter $\tau_\kappa = O(\frac{1}{h_\kappa})$, where h_κ the diameter of the element κ. Other choices of τ_κ will be considered elsewhere.

Similar to the inf-sup condition of the weak Galerkin finite element methods [15, Lemma 4.3], we have the following lemma:

Lemma 1 *There exists a positive constant β independent of h and H, such that*

$$\sup_{u_a \in (\mathbf{G},\mathbf{V},\mathbf{\Lambda})} \frac{u_a^T B_a^T p}{\left(u_a^T A_a u_a\right)^{1/2}} \geq \beta \|p\|_{L^2(\Omega)}, \tag{11}$$

for all $p \in W$. Here A_a, B_a are defined in (7). The theorem is also hold when Ω is replaced by a subdomain Ω_i.

Using Lemma 1 for each subdomain, we can prove a well-known relation between the harmonic extension and Stokes extension when the subdomain boundary velocity is given. Similar results for the standard finite element discretization can be found in [1]. Then we can prove a bound of the averaging operator E_D for the Stokes problem.

Lemma 2 *There exists a positive constant C, which is independent of H and h, such that*

$$|E_D w|_{\widetilde{S}}^2 \leq C \frac{(1+\beta)^2}{\beta^2} \left(1 + \log \frac{H}{h}\right)^2 |w|_{\widetilde{S}}^2, \qquad \forall\, w = (\lambda_\Gamma,\ p_0) \in \left(\widetilde{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right),$$

where β is the inf-sup stability constant.

With the help of Lemma 2, we can obtain our main result

Theorem 1 *The preconditioned operator $M^{-1}\widehat{S}$ is symmetric, positive definite with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\widehat{S}}$ on the space $\left(\widehat{\mathbf{\Lambda}}_{\Gamma,B}, W_0\right)$.The condition number of $M^{-1}\widehat{S}$ is bounded by $C\frac{(1+\beta)^2}{\beta^2}\left(1+\log\left(\frac{H}{h}\right)\right)^2$, where C is a constant, which is independent of H and h, and β is the inf-sup stability constant, defined in Lemma 1.*

5 Numerical Experiments

We have applied our BDDC algorithms to the model problem (1), where $\Omega = [0,1]^2$. Zero Dirichlet boundary conditions are used. The right-hand side function $\mathbf{f}$ is chosen such that the exact solution is

$$\mathbf{u} = \begin{bmatrix} \sin^3(\pi x)\sin^2(\pi y)\cos(\pi y) \\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x) \end{bmatrix} \quad \text{and} \quad p = x - y.$$

We decompose the unit square into $N \times N$ subdomains with the sidelength $H = 1/N$. Equation (1) is discretized, in each subdomain, by the kth-order HDG method with an element diameter h. The preconditioned conjugate gradient iteration is stopped when the relative l_2-norm of the residual has been reduced by a factor of 10^6.

Table 1: Performance of solving (10) with HDG discretization ($\tau_\kappa = 1/h_\kappa$)

		$k=0$		$k=1$		$k=2$	
H/h	#sub	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8	4×4	4.21	10	4.72	12	12.72	14
	8×8	5.12	12	8.81	17	11.52	20
	16×16	5.00	13	10.43	21	13.44	24
	24×24	5.14	13	10.83	20	13.96	25
	32×32	5.14	13	10.84	20	14.09	25
#sub	H/h	Cond.	Iter.	Cond.	Iter.	Cond.	Iter.
8×8	4	2.56	9	6.23	14	8.52	17
	8	5.12	12	8.81	17	11.52	20
	16	7.59	15	11.86	20	17.86	24
	24	9.22	17	13.86	22	20.32	25
	32	10.48	19	15.37	23	22.21	26

We consider the choice of the stabilization constant $\tau_\kappa = \frac{1}{h_\kappa}$. We have carried out two sets of experiments to obtain iteration counts and condition number estimates. In the first set of the experiments, we fixed $\frac{H}{h} = 8$, the subdomain local problem size, and change the number of subdomains to test the scalability of the algorithms (the condition number is independent of the number of subdomains). In the second set of experiments, we fixed the number of subdomains to 64 and change $\frac{H}{h}$, the subdomain local problem size. The performance of the algorithms for the Stokes problem is similar to those for the elliptic problems. The experimental results are fully consistent with our theory.

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References

1. Bramble, J., Pasciak, J.: A domain decomposition technique for Stokes problems. Appl. Numer. Math. **6**(4), 251–261 (1990)
2. Canuto, C., Pavarino, L.F., Pieri, A.B.: BDDC preconditioners for continuous and discontinuous Galerkin methods using spectral/hp elements with variable local polynomial degree. IMA J. Numer. Anal. **34**(3), 879–903 (2014)
3. Cockburn, B., Gopalakrishnan, J., Nguyen, N.C., Peraire, J., Sayas, F.: Analysis of HDG methods for Stokes flow. Math. Comp. **80**(274), 723–760 (2011)

4. Dohrmann, C.: A preconditioner for substructuring based on constrained energy minimization. SIAM J. Sci. Comput. **25**(1), 246–258 (2003)
5. Li, J.: A Dual-Primal FETI method for incompressible Stokes equations. Numer. Math. **102**, 257–275 (2005)
6. Li, J., Widlund, O.: BDDC algorithms for incompressible Stokes equations. SIAM J. Numer. Anal. **44**(6), 2432–2455 (2006)
7. Nguyen, N.C., Peraire, J., Cockburn, B.: A hybridizable discontinuous Galerkin method for Stokes flow. Comput. Methods Appl. Mech. Engrg. **199**(9-12), 582–597 (2010)
8. Pavarino, L., Widlund, O.: Balancing Neumann-Neumann methods for incompressible Stokes equations. Comm. Pure Appl. Math. **55**(3), 302–335 (2002)
9. Tu, X.: A BDDC algorithm for a mixed formulation of flows in porous media. Electron. Trans. Numer. Anal. **20**, 164–179 (2005)
10. Tu, X.: A BDDC algorithm for flow in porous media with a hybrid finite element discretization. Electron. Trans. Numer. Anal. **26**, 146–160 (2007)
11. Tu, X., Wang, B.: A BDDC algorithm for second-order elliptic problems with hybridizable discontinuous Galerkin discretizations. Electron. Trans. Numer. Anal. **45**, 354–370 (2016)
12. Tu, X., Wang, B.: A BDDC algorithm for the Stokes problem with weak Galerkin discretizations. Comput. Math. Appl. **76**(2), 377–392 (2018)
13. Beirão da Veiga, L., Cho, D., Pavarino, L.F., Scacchi, S.: BDDC preconditioners for isogeometric analysis. Math. Models Methods Appl. Sci. **23**(6), 1099–1142 (2013)
14. Beirão da Veiga, L., Pavarino, L., Scacchi, S., Widlund, O., Zampini, S.: Isogeometric BDDC preconditioners with deluxe scaling. SIAM J. Sci. Comput. **36**(3), A1118–A1139 (2014)
15. Wang, J., Ye, X.: A weak Galerkin finite element method for the Stokes equations. Adv. Comput. Math. **42**(1), 155–174 (2016)