A Balancing Domain Decomposition by Constraints Preconditioner for a C^0 Interior Penalty Method

Susanne C. Brenner, Eun-Hee Park, Li-Yeng Sung, and Kening Wang

1 Introduction

Consider the following weak formulation of a fourth order problem on a bounded polygonal domain Ω in $\mathbb{R}^2$:
Find $u \in H_0^2(\Omega)$ such that

$$\int_\Omega \nabla^2 u : \nabla^2 v\,dx = \int_\Omega f v\,dx \qquad \forall\, v \in H_0^2(\Omega), \tag{1}$$

where $f \in L_2(\Omega)$, and $\nabla^2 v : \nabla^2 w = \sum_{i,j=1}^2 (\partial^2 v/\partial x_i \partial x_j)(\partial^2 w/\partial x_i \partial x_j)$ is the inner product of the Hessian matrices of v and w.

For simplicity, let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω consisting of rectangles and take $V_h \subset H_0^1(\Omega)$ to be the Q_2 Lagrange finite element space associated with $\mathcal{T}_h$. (Results also hold for quadrilateral meshes.) Then the model problem (1) can be discretized by the following C^0 interior penalty Galerkin method [7, 3]:
Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v\,dx \qquad v \in V_h,$$

where

Susanne C. Brenner and Li-Yeng Sung
Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, USA, e-mail: brenner@math.lsu.edu, e-mail: sung@math.lsu.edu

Eun-Hee Park
School of General Studies, Kangwon National University, Samcheok, Gangwon 25913, Republic of Korea, e-mail: eh.park@kangwon.ac.kr

Kening Wang
Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA, e-mail: kening.wang@unf.edu

$$a_h(v,w) = \sum_{D\in\mathcal{T}_h}\int_T \nabla^2 v : \nabla^2 w\,dx + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}\int_e \left[\!\left[\frac{\partial v}{\partial \mathbf{n}}\right]\!\right]\left[\!\left[\frac{\partial w}{\partial \mathbf{n}}\right]\!\right] ds$$
$$+ \sum_{e\in\mathcal{E}_h}\int_e \left(\left\{\!\!\left\{\frac{\partial^2 v}{\partial \mathbf{n}^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial w}{\partial \mathbf{n}}\right]\!\right] + \left\{\!\!\left\{\frac{\partial^2 w}{\partial \mathbf{n}^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial v}{\partial \mathbf{n}}\right]\!\right]\right) ds.$$

Here η is a positive penalty parameter, $\mathcal{E}_h$ is the set of edges of $\mathcal{T}_h$, and $|e|$ is the length of the edge e. The jump $[\![\cdot]\!]$ and the average $\{\!\{\cdot\}\!\}$ are defined as follows.

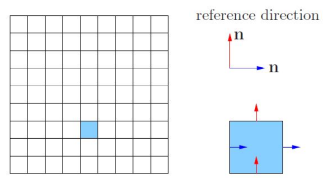

Fig. 1: (a) A triangulation of Ω. (b) A reference direction of normal vectors on the edges of $T \in \mathcal{T}_h$.

Let $\mathbf{n}_e$ be the unit normal chosen according to a reference direction shown in Fig. 1. If e is an interior edge of $\mathcal{T}_h$ shared by two elements D_- and D_+, we define on e,

$$\left[\!\left[\frac{\partial v}{\partial \mathbf{n}}\right]\!\right] = \frac{\partial v_+}{\partial \mathbf{n}_e} - \frac{\partial v_-}{\partial \mathbf{n}_e} \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial \mathbf{n}^2}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_+}{\partial \mathbf{n}_e^2} + \frac{\partial^2 v_-}{\partial \mathbf{n}_e^2}\right),$$

where $v_\pm = v|_{D_\pm}$. On an edge of $\mathcal{T}_h$ along $\partial\Omega$, we define

$$\left[\!\left[\frac{\partial v}{\partial \mathbf{n}}\right]\!\right] = \pm\frac{\partial v}{\partial \mathbf{n}_e} \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial \mathbf{n}^2}\right\}\!\!\right\} = \frac{\partial^2 v}{\partial \mathbf{n}_e^2},$$

in which the negative sign is chosen if $\mathbf{n}_e$ points towards the outside of Ω, and the positive sign otherwise.

It is noted that for $\eta > 0$ sufficiently large (Lemma 6 in [3]), there exist positive constants C_1 and C_2 independent of h such that

$$C_1 a_h(v,v) \le |v|^2_{H^2(\Omega,\mathcal{T}_h)} \le C_2 a_h(v,v) \quad \forall v \in V_h,$$

where

$$|v|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{D\in\mathcal{T}_h}|v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\left\|\left[\!\left[\frac{\partial v}{\partial \mathbf{n}}\right]\!\right]\right\|^2_{L_2(e)}.$$

Compared with classical finite element methods for fourth order problems, C^0 interior penalty methods have many advantages [3, 5, 7]. However, due to the nature of fourth order problems, the condition number of the discrete problem resulting from C^0 interior penalty methods grows at the rate of h^{-4} [8]. Thus a good preconditioner is essential for solving the discrete problem efficiently and accurately. In this paper, we develop a nonoverlapping domain decomposition preconditioner for C^0 interior penalty methods that is based on the balancing domain decomposition by constraints (BDDC) approach [6, 4, 1].

The rest of the paper is organized as follows. In Section 2 we introduce the subspace decomposition. We then design a BDDC preconditioner for the reduced problem in Section 3, followed by condition number estimates in Section 4. Finally, we report numerical results in Section 5 that illustrate the performance of the proposed preconditioner and corroborate the theoretical estimates.

2 A Subspace Decomposition

We begin with a nonoverlapping domain decomposition of Ω consisting of rectangular (open) subdomains $\Omega_1, \Omega_2, \cdots, \Omega_J$ aligned with $\mathcal{T}_h$ such that $\partial\Omega_j \bigcap \partial\Omega_\ell = \emptyset$, a vertex, or an edge, if $j \neq \ell$.

We assume the subdomains are shape regular and denote the typical diameter of the subdomains by H. Let $\Gamma = \left(\bigcup_{j=1}^J \partial\Omega_j\right) \setminus \partial\Omega$ be the interface of the subdomains, and $\mathcal{E}_{h,\Gamma}$ be the subset of $\mathcal{E}_h$ containing the edges on Γ.

Since the condition that the normal derivative of v vanishes on Γ is implicit in terms of the standard degrees of freedom (dofs) of the Q_2 finite element, it is more convenient to use the modified Q_2 finite element space (Fig. 2) as V_h. Details of the modified Q_2 finite element space can be found in [5].

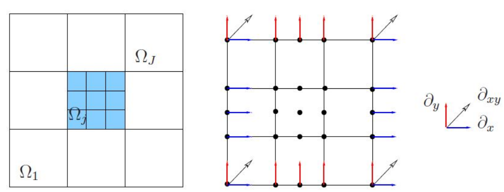

Fig. 2: (a) A nonoverlapping decomposition of Ω into $\Omega_1, \cdots, \Omega_J$ and a triangulation of the subdomain Ω_j. (b) Dofs of $V_h|_{\Omega_j}$. (c) Reference directions for the first order and mixed derivatives.

First of all, we decompose V_h into two subspaces

$$V_h = V_{h,C} \oplus V_{h,D},$$

where

$$V_{h,C} = \left\{ v \in V_h : \left[\left[\frac{\partial v}{\partial \mathbf{n}} \right]\right] = 0 \text{ on the edges in } \mathcal{E}_h \text{ that are subsets of } \bigcup_{j=1}^{J} \partial\Omega_j \right\}$$

and

$$V_{h,D} = \left\{ v \in V_h : \left\{\!\!\left\{ \frac{\partial v}{\partial \mathbf{n}} \right\}\!\!\right\} = 0 \text{ on edges in } \mathcal{E}_{h,\Gamma}, \text{ and} \right.$$
$$\left. v \text{ vanishes at all interior nodes of each subdomain} \right\}.$$

Let $A_h : V_h \to V_h'$ be the symmetric positive definite (SPD) operator defined by

$$\langle A_h v, w\rangle = a_h(v, w) \qquad \forall\, v, w \in V_h,$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form between a vector space and its dual. Similarly, we define $A_{h,C} : V_{h,C} \to V_{h,C}'$ and $A_{h,D} : V_{h,D} \to V_{h,D}'$ by

$$\langle A_{h,C} v, w\rangle = a_h(v, w)\ \forall\, v, w \in V_{h,C} \quad \text{and} \quad \langle A_{h,D} v, w\rangle = a_h(v, w)\ \forall\, v, w \in V_{h,D}.$$

Then we have the following lemma.

Lemma 1 *For any $v \in V_h$, there is a unique decomposition $v = v_C + v_D$, where $v_C \in V_{h,C}$ and $v_D \in V_{h,D}$. In addition, it holds that*

$$\langle A_h v, v\rangle \approx \langle A_{h,C} v_C, v_C\rangle + \langle A_{h,D} v_D, v_D\rangle \quad \forall\, v \in V_h.$$

Remark 1 Since the subspace $V_{h,D}$ only contains dofs on the boundary of subdomains, the size of the matrix $A_{h,D}$ is of order J/h. We can implement the solve $A_{h,D}^{-1}$ directly. Therefore, it is crucial to have an efficient preconditioner for $A_{h,C}$.

Because functions in $V_{h,C}$ have continuous normal derivatives on the edges in $\mathcal{E}_{h,\Gamma}$ and vanishing normal derivatives on $\partial\Omega$, it is easy to observe that

$$a_h(v, w) = \sum_{j=1}^{J} a_{h,j}(v_j, w_j) \qquad \forall\, v, w \in V_{h,C},$$

where $v_j = v\big|_{\Omega_j}, w_j = w\big|_{\Omega_j}$, and $a_{h,j}(\cdot,\cdot)$ is the analog of $a_h(\cdot,\cdot)$ defined on elements and interior edges of Ω_j. Note that $a_{h,j}(\cdot,\cdot)$ is a localized bilinear form.

Next we define

$$V_{h,C}(\Omega\backslash\Gamma) = \left\{v \in V_{h,C} : v \ \text{ has vanishing derivatives up to order 1 on } \ \Gamma\right\}$$
$$V_{h,C}(\Gamma) = \left\{v \in V_{h,C} : a_h(v, w) = 0, \ \ \forall\, w \in V_{h,C}(\Omega\backslash\Gamma)\right\}.$$

Functions in $V_{h,C}(\Gamma)$ are referred to as discrete biharmonic functions. They are uniquely determined by the dofs associated with Γ.

For any $v_C \in V_{h,C}$, there is a unique decomposition $v_C = v_{C,\Omega\backslash\Gamma} + v_{C,\Gamma}$, where $v_{C,\Omega\backslash\Gamma} \in V_{h,C}(\Omega\backslash\Gamma)$ and $v_{C,\Gamma} \in V_{h,C}(\Gamma)$. Furthermore, let $A_{h,C,\Omega\backslash\Gamma} : V_{h,C}(\Omega\backslash\Gamma) \to V_{h,C}(\Omega\backslash\Gamma)'$ and $S_h : V_{h,C}(\Gamma) \to V_{h,C}(\Gamma)'$ be SPD operators defined by

$$\langle A_{h,C,\Omega\backslash\Gamma} v, w\rangle = a_h(v, w) \quad \forall\, v, w \in V_{h,C}(\Omega\backslash\Gamma),$$
$$\langle S_h v, w\rangle = a_h(v, w) \quad \forall\, v, w \in V_{h,C}(\Gamma),$$

then it holds that for all $v_C \in V_{h,C}$ with $v_C = v_{C,\Omega\backslash\Gamma} + v_{C,\Gamma}$,

$$\langle A_{h,C} v_C, v_C\rangle = \langle A_{h,C,\Omega\backslash\Gamma} v_{C,\Omega\backslash\Gamma}, v_{C,\Omega\backslash\Gamma}\rangle + \langle S_h v_{C,\Gamma}, v_{C,\Gamma}\rangle.$$

Remark 2 It is noted that $A^{-1}_{h,C,\Omega\setminus\Gamma}$ can be implemented by solving the localized biharmonic problems on each subdomain in parallel. Hence, a preconditioner for S_h^{-1} needs to be constructed.

3 A BDDC Preconditioner

In this section a preconditioner for the Schur complement S_h is constructed by the BDDC methodology.

Let $V_{h,C,j}$, $1 \le j \le J$ be the restriction of $V_{h,C}$ on the subdomain Ω_j. We define $\mathcal{H}_j$, the space of local discrete biharmonic functions, by

$$\mathcal{H}_j = \left\{ v \in V_{h,C,j} : a_{h,j}(v,w) = 0 \quad \forall\, w \in V_{h,C}(\Omega_j) \right\},$$

where $V_{h,C}(\Omega_j)$ is the subspace of $V_{h,C,j}$ whose members vanish up to order 1 on $\partial\Omega_j$. The space $\mathcal{H}_C$ is then defined by gluing the spaces $\mathcal{H}_j$ together at the cross points such that

$$\mathcal{H}_C = \left\{ v \in L_2(\Omega) : v\big|_{\Omega_j} \in \mathcal{H}_j \text{ and } v \text{ has continuous dofs at subdomain corners} \right\}.$$

We equip $\mathcal{H}_C$ with the bilinear form:

$$a_h^C(v,w) = \sum_{1 \le j \le J} a_{h,j}(v_j, w_j) \qquad \forall\, v, w \in \mathcal{H}_C,$$

where $v_j = v\big|_{\Omega_j}$ and $w_j = w\big|_{\Omega_j}$.

Next we introduce a decomposition of $\mathcal{H}_C$,

$$\mathcal{H}_C = \mathring{\mathcal{H}} \oplus \mathcal{H}_0$$

where

$$\mathring{\mathcal{H}} = \{ v \in \mathcal{H}_C : \text{ the dofs of } v \text{ vanish at the corners of the subdomains } \Omega_1, \ldots, \Omega_J \},$$

$$\mathcal{H}_0 = \left\{ v \in \mathcal{H}_C : a_h^C(v,w) = 0 \quad \forall\, w \in \mathring{\mathcal{H}} \right\}.$$

Let $\mathring{\mathcal{H}}_j$ be the restriction of $\mathring{\mathcal{H}}$ on Ω_j. We then define SPD operators $S_0 : \mathcal{H}_0 \longrightarrow \mathcal{H}_0'$ and $S_j : \mathring{\mathcal{H}}_j \longrightarrow \mathring{\mathcal{H}}_j'$ by

$$\langle S_0 v, w \rangle = a_h^C(v,w) \quad \forall\, v, w \in \mathcal{H}_0 \quad \text{and} \quad \langle S_j v, w \rangle = a_{h,j}(v,w) \quad \forall\, v, w \in \mathring{\mathcal{H}}_j.$$

Now the BDDC preconditioner B_{BDDC} for S_h is given by

$$B_{BDDC} = (P_\Gamma I_0)\, S_0^{-1}\, (P_\Gamma I_0)^t + \sum_{j=1}^{J} \left(P_\Gamma \mathbb{E}_j\right) S_j^{-1} \left(P_\Gamma \mathbb{E}_j\right)^t ,$$

where $I_0 : \mathcal{H}_0 \to \mathcal{H}_C$ is the natural injection, $\mathbb{E}_j : \mathring{\mathcal{H}}_j \to \mathcal{H}_C$ is the trivial extension, and $P_\Gamma : \mathcal{H}_C \longrightarrow V_{h,C}$ is a projection defined by averaging such that for all $v \in \mathcal{H}_C$, $P_\Gamma v$ is continuous on Γ up to order 1.

Remark 3 A preconditioner $B : V_h{}' \longrightarrow V_h$ for A_h can then be constructed as follows:

$$B = I_D A_{h,D}^{-1} I_D^t + I_{h,C,\Omega\backslash\Gamma} A_{h,C,\Omega\backslash\Gamma}^{-1} I_{h,C,\Omega\backslash\Gamma}^t + I_\Gamma B_{BDDC} I_\Gamma^t,$$

where $I_D : V_{h,D} \to V_h$, $I_{h,C,\Omega\backslash\Gamma} : V_{h,C}(\Omega\backslash\Gamma) \to V_h$, and $I_\Gamma : V_{h,C}(\Gamma) \to V_h$ are natural injections.

4 Condition Number Estimates

In this section we present the condition number estimates of $B_{BDDC} S_h$. Let us begin by noting that

$$V_{h,C}(\Gamma) = P_\Gamma I_0 \mathcal{H}_0 + \sum_{j=1}^{J} P_\Gamma \mathbb{E}_j \mathring{\mathcal{H}}_j .$$

Then it follows from the theory of additive Schwarz preconditioners (see for example [10, 11, 9, 2]) that the eigenvalues of $B_{BDDC} S_h$ are positive, and the extreme eigenvalues of $B_{BDDC} S_h$ are characteristic by the following formulas

$$\lambda_{\min}\left(B_{BDDC} S_h\right) = \min_{\substack{v \in V_{h,C}(\Gamma) \\ v \neq 0}} \frac{\langle S_h v, v \rangle}{\displaystyle\min_{\substack{v = P_\Gamma I_0 v_0 + \sum_{j=1}^J P_\Gamma \mathbb{E}_j \mathring{v}_j \\ v_0 \in \mathcal{H}_0, \mathring{v}_j \in \mathring{\mathcal{H}}_j}} \left(\langle S_0 v_0, v_0 \rangle + \sum_{j=1}^{J} \langle S_j \mathring{v}_j, \mathring{v}_j \rangle \right)},$$

$$\lambda_{\max}\left(B_{BDDC} S_h\right) = \max_{\substack{v \in V_{h,C}(\Gamma) \\ v \neq 0}} \frac{\langle S_h v, v \rangle}{\displaystyle\min_{\substack{v = P_\Gamma I_0 v_0 + \sum_{j=1}^J P_\Gamma \mathbb{E}_j \mathring{v}_j \\ v_0 \in \mathcal{H}_0, \mathring{v}_j \in \mathring{\mathcal{H}}_j}} \left(\langle S_0 v_0, v_0 \rangle + \sum_{j=1}^{J} \langle S_j \mathring{v}_j, \mathring{v}_j \rangle \right)},$$

from which we can establish a lower bound for the minimum eigenvalue of $B_{BDDC} S_h$, an upper bound for the maximum eigenvalue of $B_{BDDC} S_h$, and then an estimate on the condition number of $B_{BDDC} S_h$.

Theorem 1 *It holds that* $\lambda_{\min}(B_{BDDC} S_h) \geq 1$ *and* $\lambda_{\max}(B_{BDDC} S_h) \leq (1 + \ln(H/h))^2/C$*, which imply*

$$\kappa(B_{BDDC}S_h) = \frac{\lambda_{\max}(B_{BDDC}S_h)}{\lambda_{\min}(B_{BDDC}S_h)} \le C(1 + \ln(H/h))^2,$$

where the positive constant C is independent of h, H, and J.

5 Numerical Results

In this section we present some numerical results to illustrate the performance of the preconditioners B_{BDDC} and B. We consider our model problem (1) on the unit square $(0,1) \times (0,1)$. By taking the penalty parameter η in $a_h(\cdot,\cdot)$ and $a_{h,j}(\cdot,\cdot)$ to be 5, we compute the maximum eigenvalue, the minimum eigenvalue, and the condition number of the systems $B_{BDDC}S_h$ and BA_h for different values of H and h.

The eigenvalues and condition numbers of $B_{BDDC}S_h$ and BA_h for 16 subdomains are presented in Tables 1 and 2, respectively. They confirm our theoretical estimates. In addition, the corresponding condition numbers of A_h are provided in Table 2.

Moreover, to illustrate the practical performance of the preconditioner, we present in Table 3 the number of iterations required to reduce the relative residual error by a factor of 10^{-6} for the preconditioned system and the un-preconditioned system, from which we can observe the dramatic improvement in efficiency due to the preconditioner, especially as h gets smaller.

Table 1: Eigenvalues and condition numbers of $B_{BDDC}S_h$ for $H = 1/4$ (J = 16 subdomains)

	$\lambda_{\max}(B_{BDDC}S_h)$	$\lambda_{\min}(B_{BDDC}S_h)$	$\kappa(B_{BDDC}S_h)$
h=1/8	3.6073	1.0000	3.6073
h=1/12	2.9197	1.0000	2.9197
h=1/16	3.0908	1.0000	3.0908
h=1/20	3.2756	1.0000	3.2756
h=1/24	3.4535	1.0000	3.4535

Table 2: Eigenvalues and condition numbers of BA_h, and condition numbers of A_h for $H = 1/4$ (J = 16 subdomains)

	$\lambda_{\max}(BA_h)$	$\lambda_{\min}(BA_h)$	$\kappa(BA_h)$	$\kappa(A_h)$
h=1/8	4.0705	0.2148	18.9490	1.1064e+03
h=1/12	3.4107	0.2507	13.6054	1.3426e+04
h=1/16	3.4866	0.2578	13.5244	6.1689e+04
h=1/20	3.5947	0.2590	13.8787	1.8215e+05
h=1/24	3.7123	0.2593	14.3181	4.2288e+05

Acknowledgements The work of the first and third authors was supported in part by the National Science Foundation under Grant No. DMS-16-20273.

Table 3: Number of iterations for reducing the relative residual error by a factor of 10^{-6} for $H = 1/4$ (J = 16 subdomains)

	$Niter(A_h x = b)$	$Niter(BA_h x = Bb)$
h=1/8	95	27
h=1/12	235	23
h=1/16	434	23
h=1/20	704	23
h=1/24	1026	23

References

1. S. C. Brenner and E.-H. Park and L. -Y. Sung. A BDDC preconditioner for a symmetric interior penalty method. *Electron. Trans. Numer. Anal.*, 46, 190–214, 2017.
2. S. C. Brenner and L. R. Scott. The Mathematical Theory of Finite Element Methods (Third Edition). *Springer-Verlag*, New York, 2008.
3. S. C. Brenner and L.-Y. Sung. C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. *J. Sci. Comput.*, 22/23, 83–118, 2005.
4. S. C. Brenner and L.-Y. Sung. BDDC and FETI-DP without matrices or vectors. *Comput. Methods Appl. Mech. Engrg.*, 196, 1429–1435, 2007.
5. S. C. Brenner and K. Wang. An iterative substructuring algorithm for a C^0 interior penalty method. *ETNA*, 39, 313–332, 2012.
6. C. R. Dohrmann. A preconditioner for substructuring based on constrained energy minimization. *SIAM J. Sci. Comput.*, 25, 246–258, 2003.
7. G. Engel, K. Garikipati, T. Hughes, M. Larson, L. Mazzei, and R. Taylor. Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. *Comput. Methods Appl. Mech. Engrg.*, 191, 3669-3750, 2002.
8. S. Li and K. Wang. Condition number estimates for C^0 interior penalty methods. *Domain Decomposition Methods in Science and Engineering XVI*, 55, 675-682, 2007.
9. T. Mathew. Domain Decomposition Methods for the Numerical Solutions of Partial Differential Equations. *Springer-Verlag*, Berlin, 2008.
10. B. Smith and P. Bjørstad and W. Gropp. Domain Decomposition. *Cambridge University Press*, Cambridge, 1996.
11. A. Toselli and O. B. Widlund. Domain Decomposition Methods - Algorithms and Theory. *Springer-Verlag*, Berlin, 2005.