Preconditioners for Isogeometric Analysis and Almost Incompressible Elasticity

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1 Introduction

The aim of this work is to develop a block FETI–DP preconditioner for mixed formulations of almost incompressible elasticity discretized with mixed isogeometric analysis (IGA) methods with continuous pressure. IGA is a recent technology for the numerical approximation of Partial Differential Equations (PDEs), using the highly regular function spaces generated by B-splines and NURBS not only to describe the geometry of the computational domain but also to represent the approximate solution, see e.g. [4]. For a few previous studies, focused on effective solvers for IGA of saddle point problems, see [7, 6].

Inspired by previous work by Tu and Li [10] for finite element discretizations of the Stokes system, the proposed preconditioner is applied to a reduced positive definite system involving only the pressure interface variable and the Lagrange multipliers of the FETI–DP algorithm. A novelty of our contribution consists of using BDDC with deluxe scaling for the interface pressure block and FETI–DP with deluxe scaling for the multiplier block. The numerical results reported in this paper show the robustness of this solver with respect to jumps in the elastic coefficients and the degree of incompressibility of the material.

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2 Two variational formulations of elasticity systems

Let Ω be a domain in $\mathbb{R}^3$, which can be represented exactly by the isogeometric analysis system. It is decomposed into N non-overlapping subdomains Ω_i, of diameter H_i, which are images under a geometric map $\mathbf{F}$ of a coarse element partition τ_H of a reference domain. The interface of the decomposition is given by

$$\Gamma = \left(\bigcup_{i=1}^{N} \partial\Omega_i \right) \setminus \partial\Omega.$$

The boundary $\partial\Omega$ is the union of two disjoint sets $\partial\Omega_D$ and $\partial\Omega_N$ where $\partial\Omega_D$ is of non-zero surface measure. We work with two load functions $\boldsymbol{g} \in [L^2(\Omega)]^3$ and $\boldsymbol{g}_N \in [L^2(\partial\Omega_N)]^3$, and the spaces

$$\boldsymbol{V} := \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v}|_{\partial\Omega_D} = 0\}, \quad Q := L^2(\Omega).$$

The load functions define a linear functional

$$< \boldsymbol{f}, \mathbf{v} > := \int_\Omega \boldsymbol{g} \cdot \mathbf{v} dx + \int_{\partial\Omega_N} \boldsymbol{g}_N \cdot \mathbf{v} dA.$$

If the material is compressible, we can use the variational formulation of the linear elasticity (LE) equations:

$$2 \int_\Omega \mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_\Omega \lambda \mathrm{div}\mathbf{u} \, \mathrm{div}\mathbf{v} \, dx \; = < \mathbf{f}, \mathbf{v} > \quad \forall \mathbf{v} \in \boldsymbol{V}. \tag{1}$$

Here $\boldsymbol{\varepsilon}$ is the symmetric gradient operator and $\mu(x)$ and $\lambda(x)$ the Lamé parameters of the material that for simplicity, when developing the theory, are assumed to be constant in each subdomain Ω_i, i.e. $\mu = \mu_i$ and $\lambda = \lambda_i$ in Ω_i. These parameters can be expressed in terms of the local Poisson ratio ν_i and Young's modulus E_i as

$$\mu_i := \frac{E_i}{2(1+\nu_i)}, \qquad \lambda_i := \frac{E_i \nu_i}{(1+\nu_i)(1-2\nu_i)}. \tag{2}$$

The elastic material approaches the incompressible limit when $\nu_i \to 1/2$. Our main focus will be on a mixed formulation of linear elasticity for almost incompressible (AIE) materials as, e.g., in [2, Ch. 1]: find the material displacement $\mathbf{u} \in \boldsymbol{V}$ and pressure $p \in Q$ such that

$$\begin{cases} 2 \int_\Omega \mu \, \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx - \int_\Omega \mathrm{div} \, \mathbf{v} \, p \, dx = \; < \boldsymbol{f}, \mathbf{v} > & \forall \mathbf{v} \in \boldsymbol{V}, \\ \quad - \int_\Omega \mathrm{div} \, \mathbf{u} \, q \, dx \quad - \int_\Omega \frac{1}{\lambda} \, pq \, dx = \quad 0 & \forall q \in Q. \end{cases} \tag{3}$$

Factoring out the constants μ_i and $\frac{1}{\lambda_i}$, we can define local bilinear forms in terms of integrals over the subdomains Ω_i and we obtain for the almost incompressible case

$$\mu a(\mathbf{u},\mathbf{v}) := \sum_{i=1}^{N} \mu_i a_i(\mathbf{u},\mathbf{v}) := \sum_{i=1}^{N} 2\mu_i \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, dx$$
$$b(\mathbf{v},q) := \sum_{i=1}^{N} b_i(\mathbf{v},q) := -\sum_{i=1}^{N} \int_{\Omega_i} \mathrm{div}\mathbf{v}\, q\, dx, \tag{4}$$
$$\frac{1}{\lambda} c(p,q) := \sum_{i=1}^{N} \frac{1}{\lambda_i} c_i(p,q) := \sum_{i=1}^{N} \frac{1}{\lambda_i} \int_{\Omega_i} p\, q\, dx.$$

The isogeometric approximation of the mixed elasticity problem is obtained by selecting spaces for the displacements $\mathbf{u}$ and pressure p, respectively. Following Bressan and Sangalli, [3], we select mapped NURBS functions of polynomial degree $p \geq 2$ with $p-2$ continuous derivatives for the displacement and of polynomial degree $p-1$ with $p-2$ continuous derivatives for the pressure; see, e.g., [8] for details on these Taylor–Hood spaces. The resulting pair of spaces is known to be inf-sup stable, see [3]. A major difference from finite element approximations stems from the fact that except for the lowest order case, there is no nodal basis which leads to *fat interfaces*, see [11, Sec. 4.2] and [12, Sec. 3]. This fact makes the construction of small primal spaces more urgent and complicated.

The knots of the isogeometric analysis problems are partitioned into interior knots with basis functions, with support in the subdomain interiors which do not intersect the boundaries of any subdomain, and interface knots. The latter set is partitioned into equivalence classes. These equivalence classes are associated with the subdomain vertices, edges, and faces. Thus, such a vertex class is given by the knots with basis functions with a subdomain vertex in the interior of their supports. A detailed definition of the edge and face classes are given in [8, Section 3]. These equivalence classes are important in the design, analysis, and programming of BDDC and FETI–DP as well as many other domain decomposition algorithms.

3 Dual–Primal decomposition and a FETI–DP reduced system

The interface displacement variable $\mathbf{u}$ is partitioned into a *dual* part $\mathbf{u}_{\boldsymbol{\Delta}}$ and a *primal* part $\mathbf{u}_{\boldsymbol{\Pi}}$. To be competitive, the space of primal variables, with functions which are continuous across the interface, should be of much smaller dimension than that of the space of dual variables, for which we allow jumps across the interface. The displacement variables $\mathbf{u}$ is split into interior $\mathbf{u}_{\mathbf{I}}$, dual $\mathbf{u}_{\boldsymbol{\Delta}}$, and primal $\mathbf{u}_{\boldsymbol{\Pi}}$ components, and the pressure p into interior p_I and interface p_Γ components, and we denote by λ_Δ the vector of Lagrange multipliers used to enforce the continuity of the dual displacements across the interface.

Following Tu and Li, [10], we reorder the variables as $\mathbf{u}_{\mathbf{I}}$, p_I, $\mathbf{u}_{\boldsymbol{\Delta}}$, $\mathbf{u}_{\boldsymbol{\Pi}}$, p_Γ, and λ_Δ and splitting the matrices $\mu A, B$, and $\frac{1}{\lambda}C$, defined by the the bilinear forms of (4) and the mixed method, into appropriate blocks associated with this splitting. The original saddle point system resulting from (4) is equivalent to

$$\begin{bmatrix} \mu A_{II} & B_{II}^T & \mu A_{I\Delta} & \mu A_{I\Pi} & B_{\Gamma I}^T & 0 \\ B_{II} & -\frac{1}{\lambda}C_{II} & B_{I\Delta} & B_{I\Pi} & -\frac{1}{\lambda}C_{\Gamma I}^T & 0 \\ \mu A_{\Delta I} & B_{I\Delta}^T & \mu A_{\Delta\Delta} & \mu A_{\Delta\Pi} & B_{\Gamma\Delta}^T & B_{\Delta}^T \\ \mu A_{\Pi I} & B_{I\Pi}^T & \mu A_{\Pi\Delta} & \mu A_{\Pi\Pi} & B_{\Gamma\Pi}^T & 0 \\ B_{\Gamma I} & -\frac{1}{\lambda}C_{\Gamma I} & B_{\Gamma\Delta} & B_{\Gamma\Pi} & -\frac{1}{\lambda}C_{\Gamma\Gamma} & 0 \\ 0 & 0 & B_{\Delta} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u_I} \\ p_I \\ \mathbf{u_\Delta} \\ \mathbf{u_\Pi} \\ p_\Gamma \\ \lambda_\Delta \end{bmatrix} = \begin{bmatrix} \mathbf{f_I} \\ 0 \\ \mathbf{f_\Delta} \\ \mathbf{f_\Pi} \\ 0 \\ 0 \end{bmatrix}, \tag{5}$$

where $B_\Delta = \begin{bmatrix} B_\Delta^{(1)} & B_\Delta^{(2)} & \dots & B_\Delta^{(N)} \end{bmatrix}$ is a Boolean matrix which enforces continuity, $B_\Delta \mathbf{u_\Delta} = \mathbf{0}$, of the dual displacement variables $\mathbf{u_\Delta}$ shared by neighboring subdomains. If we confine ourselves to the case where λ_Δ belongs to the range of B_Δ, this matrix, although indefinite, is nonsingular under the condition that the primal space is large enough.

If the primal space is relatively small, we can, at an acceptable cost, reduce the indefinite system (5) to a symmetric, positive definite system by eliminating the $\mathbf{u_I}$, p_I, $\mathbf{u_\Delta}$, and $\mathbf{u_\Pi}$ variables and changing the sign. We obtain a Schur complement and a reduced linear system

$$G \begin{bmatrix} p_\Gamma \\ \lambda_\Delta \end{bmatrix} = g, \tag{6}$$

which is then solved by a preconditioned conjugate gradient algorithm with a block preconditioner. Here,

$$G := \widetilde{B}_C \widetilde{A}^{-1} \widetilde{B}_C^T + \frac{1}{\lambda} \widetilde{C}, \qquad g := -\widetilde{B}_C \widetilde{A}^{-1} \begin{bmatrix} \mathbf{f_I} \\ 0 \\ \mathbf{f_\Delta} \\ \mathbf{f_\Pi} \end{bmatrix}, \tag{7}$$

and where $\widetilde{A}$ is the leading 4-by-4 principal minor of the matrix of (5) and

$$\widetilde{B}_C := \begin{bmatrix} B_{\Gamma I} & -\frac{1}{\lambda}C_{\Gamma I} & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{C} := \begin{bmatrix} C_{\Gamma\Gamma} & 0 \\ 0 & 0 \end{bmatrix}. \tag{8}$$

4 Deluxe scaling

For the Lagrange multiplier λ_Δ, we use, following Tu and Li [10], a FETI-DP preconditioner borrowed from our work on the compressible case reported in [8]. In BDDC, the average $\bar{\mathbf{u}} := E_D \mathbf{u}$ of an element in the partially discontinuous space of displacements is computed separately for the sets of interface degrees of freedom of the vertex, edge, and face equivalence classes; the operator E_D is central for both the algorithm and the analysis, see, e.g., [11]. For FETI-DP methods the complementary projection $P_D := I - E_D$ is similarly relevant. We start by defining the deluxe scaling in the simplest case of a class with only two elements, i, j, for a face $\mathcal{F}$; for more details on the fat interface and the definition of the fat equivalence classes, we refer to [11, Sec. 4.2] and [12, Sec. 3].

Let $S^{(i)}$ be the Schur interface complement of the subdomain Ω_i, and define two principal minors, $S^{(i)}_{\mathcal{F}}$ and $S^{(j)}_{\mathcal{F}}$, obtained from $S^{(i)}$ and $S^{(j)}$ by removing all rows and columns which do not belong to variables associated with $\mathcal{F}$.

With $\mathbf{u}^{(i)}_{\mathcal{F}}$ the restriction of an element in the dual space to the face $\mathcal{F}$, the deluxe average across $\mathcal{F}$ is then defined as

$$\bar{\mathbf{u}}_{\mathcal{F}} = \left(S^{(i)}_{\mathcal{F}} + S^{(j)}_{\mathcal{F}}\right)^{-1}\left(S^{(i)}_{\mathcal{F}}\mathbf{u}^{(i)}_{\mathcal{F}} + S^{(j)}_{\mathcal{F}}\mathbf{u}^{(j)}_{\mathcal{F}}\right). \tag{9}$$

We also need to define deluxe averaging operators for subdomain edges and subdomain vertices. Given the simple hexahedral subdomain geometry of the parameter space that we are considering, we find that such an equivalence class will have four and eight elements for any fat subdomain edge and vertex, respectively, in the interior of Ω. Thus, for such a fat subdomain edge $\mathcal{E}$ shared by subdomains $\Omega_i, \Omega_j, \Omega_k$, and Ω_ℓ, we use the formula

$$\bar{\mathbf{u}}_{\mathcal{E}} := \left(S^{(i)}_{\mathcal{E}} + S^{(j)}_{\mathcal{E}} + S^{(k)}_{\mathcal{E}} + S^{(\ell)}_{\mathcal{E}}\right)^{-1}\left(S^{(i)}_{\mathcal{E}}\mathbf{u}^{(i)}_{\mathcal{E}} + S^{(j)}_{\mathcal{E}}\mathbf{u}^{(j)}_{\mathcal{E}} + S^{(k)}_{\mathcal{E}}\mathbf{u}^{(k)}_{\mathcal{E}} + S^{(\ell)}_{\mathcal{E}}\mathbf{u}^{(\ell)}_{\mathcal{E}}\right).$$

An analogous formula holds for the fat vertices and involves eight operators. Edges and vertices located on the Neumann boundary of the domain will have fewer elements, depending on the number of subdomains that share them.

For each subdomain Ω_i, we then define a scaling matrix by its restriction $D^{(i)}_{\Delta}$ to subdomain Ω_i as the direct sum of diagonal blocks given by the deluxe scaling of the face, edge, and vertex terms belonging to the interface of Ω_i:

- for subdomain faces: $D^{(i)}_{\mathcal{F}} := S^{(i)}_{\mathcal{F}}\left(S^{(i)}_{\mathcal{F}} + S^{(j)}_{\mathcal{F}}\right)^{-1}$,
- for subdomain edges: $D^{(i)}_{\mathcal{E}} := S^{(i)}_{\mathcal{E}}\left(S^{(i)}_{\mathcal{E}} + S^{(j)}_{\mathcal{E}} + S^{(k)}_{\mathcal{E}} + S^{(\ell)}_{\mathcal{E}}\right)^{-1}$,
- for subdomain vertices: an analogous formula with eight operators.

These scaling matrices and their transposes provide factors of FETI-DP preconditioning operator. In terms of the complementary projection operator $P_D = I - E_D$, we have for a fat face of Ω_i :

$$P_D\mathbf{u}_{\mathcal{F}} = \left(S^{(i)}_{\mathcal{F}} + S^{(j)}_{\mathcal{F}}\right)^{-1} S^{(j)}_{\mathcal{F}}(\mathbf{u}^{(i)}_{\mathcal{F}} - \mathbf{u}^{(j)}_{\mathcal{F}}).$$

Similar formulas are easily developed for the other types of equivalence classes.

For the preconditioner block associated with the λ_Δ variable, we can borrow directly a successful preconditioner developed in [8] for compressible elasticity. We note that the bilinear form of (1) has a term additional to $\mu a(\cdot,\cdot)$ but that this does not have any real consequences.

In the present work, the pressure sub-solver $M^{-1}_{p_\Gamma}$ is chosen as the inverse of $\frac{1}{\mu}S^{C}_{\Gamma\Gamma}$ obtained from the subdomain mass matrices associated with the interface pressure variables p_Γ. This matrix is obtained by subassembling the local Schur complements $S^{C^{(i)}}_{\Gamma\Gamma}$ of the subdomain mass matrices $C^{(i)}$ weighted by $\frac{1}{\mu_i}$ and defined by

$$\frac{1}{\mu_i} S_{\Gamma\Gamma}^{C^{(i)}} := \frac{1}{\mu_i} C_{\Gamma\Gamma}^{(i)} - \frac{1}{\mu_i} C_{\Gamma I}^{(i)} C_{II}^{(i)^{-1}} C_{I\Gamma}^{(i)}.$$

To develop a competitive algorithm, we then replace the inverse of this Schur complement, defining $M_{p_\Gamma}^{-1}$, by a BDDC deluxe preconditioner built from the subdomain matrices $\frac{1}{\mu_i} S_{\Gamma\Gamma}^{C^{(i)}}$. In our experience, this has proven very successful even without a primal subspace. We note that such a preconditioner is quite helpful given that the mass matrices of the isogeometric Taylor–Hood elements are quite ill-conditioned.

5 Numerical results

We report results of some numerical experiments for the LE (1) and AIE (3) systems in two and three dimensions, discretized with isogeometric NURBS spaces with a uniform mesh size h, polynomial degree p, and regularity k. Results for much larger problems are reported in [13]. The boundary of a reference unit cube has a zero Dirichlet condition on one face, an inhomogeneous Neumann condition on the opposite face, and zero Neumann conditions on all the other faces. The domain Ω is decomposed into N non-overlapping subdomains of characteristic size H.

The tests have been performed using PetIGA-MF [5, 9] as a discretization package; the solvers used are available in the latest release, 3.10, of the PETSc library [1], and have been contributed by Stefano Zampini (see also [14]). In all experiments, the norm of the residual vector has been decreased by a factor 10^{-8}.

5.1 Checkerboard jumping coefficient test

This test is devoted to investigating the robustness of the proposed block FETI-DP preconditioners for the 2D and 3D AIE system with elastic coefficients configured in a checkerboard pattern. We consider jumps in both the Young modulus E and the Poisson ration ν. In Tables 1 and 2, the conjugate gradient (CG) iteration count (n_{it}) and the maximal (λ_M) and minimal (λ_m) eigenvalues of the preconditioned operator are reported. In the 2D test, we have fixed the number of subdomains to $N = 49 = 7 \times 7$ and the mesh size to $1/h = 128$. In the 3D test, the number of subdomains is $N = 27 = 3 \times 3 \times 3$ and the mesh size $1/h = 16$. The displacement field spline parameters of the Taylor-Hood pair are $p = 3$, $k = 1$; therefore the pressure spline parameters are $p = 2$, $k = 1$. The results show that the proposed solver is very robust with respect to all the jumps considered, since both the number of CG iterations and the extreme eigenvalues approach constant values when E becomes large or the material becomes incompressible ($\nu \to 0.5$).

2D jump test for E				2D jump test for ν			
E	n_{it}	λ_M	λ_m	ν	n_{it}	λ_M	λ_m
1e+00	23	2.91e+00	2.91e-01	0.3	10	3.16e+00	6.27e-01
1e+01	25	1.91e+00	1.83e-01	0.4	11	3.12e+00	5.04e-01
1e+02	39	2.09e+00	8.30e-02	0.45	12	3.07e+00	4.17e-01
1e+03	51	2.15e+00	3.72e-02	0.49	14	3.02e+00	3.35e-01
1e+04	52	2.16e+00	3.31e-02	0.499	14	3.01e+00	3.21e-01
1e+05	49	2.16e+00	4.23e-02	0.4999	14	3.01e+00	3.20e-01

Table 1: FETI-DP for AIE on 2D checkerboard jumping coefficient tests. Conjugate gradient iteration counts (n_{it}) and extreme eigenvalues (λ_M, λ_m) of the preconditioned operator. **Jump test for E:** $E = 1$ in black subdomains, E shown in the table in red subdomains; fixed $\nu = 0.49$. **Jump test for ν:** $\nu = 0.3$ in black subdomains, ν shown in the table in red subdomains; fixed $E = 1e+06$. In both tests: $N = 49 = 7 \times 7$ subdomains; $1/h = 128$, displacement field spline parameters $p = 3$, $k = 1$.

3D jump test for E				3D jump test for ν			
E	n_{it}	λ_M	λ_m	ν	n_{it}	λ_M	λ_m
1e+00	28	2.72e+00	2.46e-01	0.3	12	3.04e+00	5.52e-01
1e+01	39	2.83e+00	1.27e-01	0.4	13	2.82e+00	4.34e-01
1e+02	62	2.92e+00	4.35e-02	0.45	14	2.76e+00	3.69e-01
1e+03	80	2.99e+00	2.39e-02	0.49	15	2.73e+00	3.19e-01
1e+04	83	3.00e+00	2.18e-02	0.499	16	2.72e+00	3.05e-01
1e+05	83	3.00e+00	2.20e-02	0.4999	16	2.72e+00	3.04e-01

Table 2: FETI-DP for AIE on 3D checkerboard jumping coefficient tests. Conjugate gradient iteration counts (n_{it}) and extreme eigenvalues (λ_M, λ_m) of the preconditioned operator. **Jump test for E:** $E = 1$ in black subdomains, E shown in the table in red subdomains; fixed $\nu = 0.49$. **Jump test for ν:** $\nu = 0.3$ in black subdomains, ν shown in the table in red subdomains; fixed $E = 1e+06$. In both tests: $N = 27 = 3 \times 3 \times 3$ subdomains; $1/h = 16$, displacement field spline parameters $p = 3$, $k = 1$.

5.2 A comparison between FETI-DP for LE and for AIE

The aim of this test is to compare the FETI-DP preconditioner for 3D LE developed previously in [8] with the block FETI-DP solver for 3D AIE proposed in the current project, in terms of the robustness with respect to incompressibility of the material, i.e., when $\nu \to 0.5$.

In Table 3, the CG iteration count (n_{it}) and the maximal (λ_M) and minimal (λ_m) eigenvalues of the preconditioned operator are reported. The Young modulus is kept fixed to $E = 1$ in the whole domain, while ν varies as detailed in the tables. We fix the number of subdomains to $N = 27 = 3 \times 3 \times 3$ and the mesh size to $1/h = 16$. The displacement field spline parameters are $p = 3$, $k = 1$. Using a Taylor-Hood pair for the case of AIE, this results in pressure spline parameters of $p = 2$, $k = 1$.

The results show, as expected, that the FETI-DP solver for LE degenerates when the material approaches the incompressible limit, while the FETI-DP solver for

AIE is very robust in terms of both CG iterations and extreme eigenvalues of the preconditioned operator.

3D comparison						
ν	FETI-DP for LE			FETI-DP for AIE		
	n_{it}	λ_M	λ_m	n_{it}	λ_M	λ_m
0.3	16	8.73e+00	1.03e+00	20	3.04e+00	5.07e-01
0.4	19	1.30e+01	1.03e+00	23	2.73e+00	3.59e-01
0.45	25	2.02e+01	1.02e+00	25	2.73e+00	2.94e-01
0.49	43	5.49e+01	1.03e+00	28	2.72e+00	2.46e-01
0.499	100	2.48e+02	1.02e+00	28	2.72e+00	2.36e-01
0.4999	283	1.85e+03	1.02e+00	28	2.72e+00	2.35e-01

Table 3: 3D comparison between FETI-DP for LE and for AIE. Conjugate gradient iteration counts (n_{it}) and extreme eigenvalues (λ_M, λ_m) of the preconditioned operator. $E = 1$, $N = 27 = 3 \times 3 \times 3$ subdomains; $1/h = 16$, displacement field spline parameters $p = 3$, $k = 1$.

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