Dispersion Correction for Helmholtz in 1D with Piecewise Constant Wavenumber

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1 Introduction

The Helmholtz equation is the simplest model for time harmonic wave propagation, and it contains already all the fundamental difficulties such problems pose when trying to compute their solution numerically. Since time harmonic wave propagation has important applications in many fields of science and engineering, the numerical solution of such problems has been the focus of intensive research efforts, see [15, 16, 20, 10, 5, 23, 28], and the review [17] and references therein for domain decomposition approaches, and [3, 11, 13, 25, 14, 7] and references therein for multigrid techniques. The main problem is that all grid based numerical methods like finite differences or finite elements are losing accuracy because of what is called the *pollution effect* [2, 1]. It is not sufficient to just choose a number of grid points large enough to resolve the wave length determined by the wave number to obtain an accurate solution; the larger the wave number, the more grid points per wave length are needed. This leads to extremely large linear systems that need to be solved when the wave number becomes large, which is hard using classical iterative methods, see [12] and references therein. The pollution effect is due to the *numerical dispersion*, a property which unfortunately all grid based methods have, see also [22, 26] and references therein. In the case of a constant wave number, to reduce the numerical dispersion of the standard 5-point finite difference scheme, a rotated 9-point FDM was proposed in [19] which minimizes the numerical dispersion, see also [4, 24, 27, 6] for more recent such approaches. In particular, in [8] a new approach was introduced which does not only modify the finite difference stencil, but also

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the wave number itself in the discrete scheme to minimize dispersion. This led to a new finite difference scheme in two spatial dimensions with much smaller dispersion error than all previous approaches, see also [18]. Minimizing numerical dispersion is also important for effective coarse grid corrections in domain decomposition and for constructing efficient multigrid solvers [25]: for a constant wave number in 1D it is even possible to obtain perfect multigrid efficiency using standard components and dispersion correction, it suffices to use a suitably modified numerical wave number on each level [13], see also [21]. Note that this is very different from [9] where a large complex shift is used in the spirit of [11].

In all previous work on dispersion correction, a main assumption is that the wave number is constant throughout the domain. We propose and study here a new dispersion correction for the Helmholtz equation in 1D in the case where the wave number is only piecewise constant, and allowed to jump in between. Using the exact solution of a transmission problem, we determine for a finite difference discretization a dispersion correction at the interface where the wave number is jumping by introducing a modified numerical wave number there. We show by numerical experiments that this dispersion correction leads to much more accurate solutions than the scheme without dispersion correction, and this at already few points per wavelength resolution. We then also show that this dispersion correction has a very good effect on a two-grid method, by studying numerically the contraction factor of a two grid scheme in the important regime where the coarse and fine mesh are far from resolving the problem. We then conclude by discussing further research directions.

2 Problem Setting

We consider the 1D Helmholtz equation[1] with source term $f \in L^2(-1,1)$,

$$-\partial_{x^2}^2 u(x) - k(x)^2 u(x) = f(x), \quad x \in (-1,1), \quad u(-1) = 0, \quad u(1) = 0, \tag{1}$$

where $k(x)$ is the wave number, which we assume to be piecewise constant, $k(x) := k_1$ if $x \le 0$, and $k(x) := k_2$ if $x > 0$. We discretize Problem (1) with a standard 3-point centered finite difference scheme on a uniform mesh[2] with n interior mesh points and meshsize $h = 1/(n+1)$. Assuming that $x = 0$ is always a grid point, and denoting by n_1 the number of interior mesh points in $(-1,0)$ and n_2 the number of interior mesh points in $(0,1)$, the continuous problem is thus approximated by a linear system $A\mathbf{u} = \mathbf{f}$ with

$$A = \frac{1}{h^2}\text{tridiag}\,(-1,2,-1) - \text{diag}(k_1^2 I_{n_1}, k_0^2, k_2^2 I_{n_2}), \quad \mathbf{f} = \left(f(x_j)\right)_{j=1}^n, \tag{2}$$

[1] We only choose wave number configurations such that this problem is well posed

[2] We use for simplicity the same mesh size in both regions

where $k_0 = k(0) = k_1$, but we could also have chosen k_2 here.

3 Dispersion Correction for Piecewise Constant Wave Number

We use a modified wave number like in [13, p. 26 Eq. (3.15)] in the regions where the wave number is constant,

$$\widehat{k}_h(x) := \begin{cases} \sqrt{2h^{-2}(1-\cos(k_1 h))} & \text{if } x < 0, \\ \widehat{k}_0 & \text{at } x = 0, \\ \sqrt{2h^{-2}(1-\cos(k_2 h))} & \text{if } x > 0. \end{cases}$$

This modified wave number was obtained in [13] by making the exact solution of the homogeneous Helmholtz equation on $\mathbb{R}$ satisfy the finite difference scheme. Similarly we determine $\widehat{k}_0$ such that it satisfies the equation

$$h^{-2}(2u_e(0) - u_e(-h) - u_e(h)) - \widehat{k}_0^2 u_e(0) = 0, \tag{3}$$

where u_e is the exact solution of the Helmholtz equation with discontinuous k on $\mathbb{R}$ for the transmission problem of an incoming wave, given by

$$u_e(x) := \begin{cases} Ae^{ik_1 x} + Be^{-ik_1 x} & \text{if } x < 0 \\ Ce^{ik_2 x} & \text{if } x \geq 0 \end{cases}, \text{ with } A = \frac{k_1 + k_2}{2k_1},\ B = \frac{k_1 - k_2}{2k_1},\ C = 1. \tag{4}$$

The matrix associated to the new FD scheme with dispersion correction is then given by (2) with k replaced by $\widehat{k}_h$.

We show in Figure 1 the great influence this dispersion correction has on the numerical quality of the solution. We used $k_1 = 3.2\pi$ and $k_2 = k_1/2$ (top) and $k_2 = k_1/4$ (bottom) and solved (2) with and without dispersion correction using a mesh size $h = \frac{1}{16}$ which implies 10 points per wavelength for $x < 0$ (left) and $h = \frac{1}{8}$ (right), which implies 5 points per wavelength for $x < 0$. As a source term, we used a linear combination of the first sine functions $\sin(\omega \frac{\pi(x+1)}{2})$, $\omega = 1, 2, \ldots, 16$ with random coefficients, and we denote by exact a numerical solution without dispersion correction using a four times finer grid. We clearly see that dispersion correction is also possible in the case of a non-constant wave number, and we next study the influence of such a correction on a two-grid method.

4 Influence of Dispersion Correction on Multigrid

A two-grid algorithm for a general linear system $A\mathbf{u} = \mathbf{f}$ is given by performing for $n = 0, 1, \ldots$

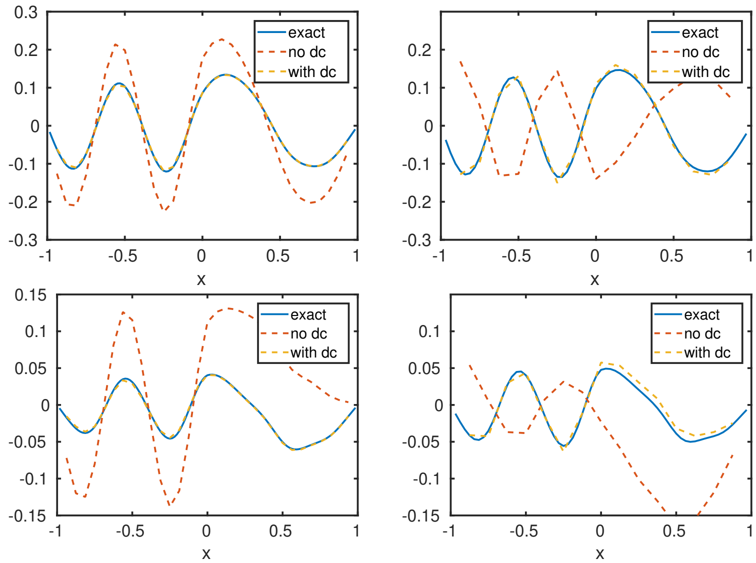

Fig. 1: Four numerical examples showing the impact of dispersion correction in the case of piecewise constant wave speed: contrast 2 (top row) and 4 (bottom row) and 10 points per wavelength (left column) and 5 points per wavelength (right column)

$$\begin{aligned}
&\tilde{\mathbf{u}}^n := S^{\nu_1}(\mathbf{u}^n, \mathbf{f}); \quad \% \text{ pre} - \text{smoothing} \\
&\mathbf{r}_c^n := R(\mathbf{f} - A\tilde{\mathbf{u}}^n); \\
&\mathbf{e}_c^{n+1} = A_c^{-1}\mathbf{r}_c^n; \\
&\tilde{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}^n + P\mathbf{e}^{n+1}; \\
&\mathbf{u}^{n+1} = S^{\nu_2}(\tilde{\mathbf{u}}^{n+1}, \mathbf{f}); \ \% \text{ post} - \text{smoothing}
\end{aligned} \tag{5}$$

where R denotes a restriction operator, P a prolongation operator, S^ν represents ν iterations of a smoother, and A_c is a coarse matrix. We define the fine grid Ω^h with meshsize $h = 1/(n+1)$ by

$$\Omega^h := \left\{x_j = jh : \ j = 0, \cdots, n+1\right\}.$$

The coarse grid is defined from Ω^h with mesh width $H = 2h$ by coarsening,

$$\Omega^H := \left\{x_j = jH : \ j = 0, \cdots, N+1\right\},$$

where $n = 2N + 1$. The prolongation operator maps grid functions $\mathbf{u}^H$ defined on a coarse grid Ω^H to a function $I_H^h \mathbf{u}^H$ defined on the fine grid Ω^h using linear interpolation. Its matrix representation P can be found in [13, p.18, Eq.(3.1)]. For the restriction operator, we use the *full weighting restriction operator* whose matrix representation is $R = P^T/2$ (see [13, p.20, Eq. (3.4)]).

For the smoother, we use a damped Kacmarz smoother whose iteration matrix is given by

$$S := I_N - \omega A_h^* A_h.$$

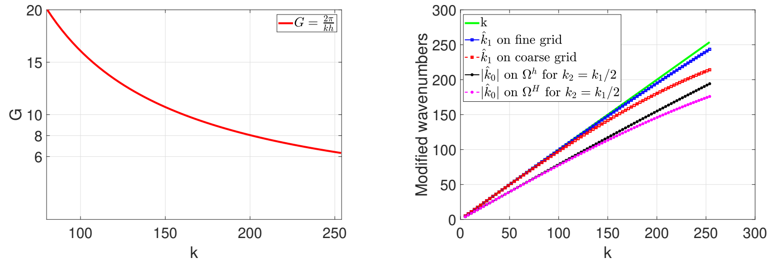

Fig. 2: Left: number of grid points per wavelength. Right: modified wavenumbers

Necessary conditions for the two-grid algorithm to converge can be found in [7, p.12 Theorem 4.1], and having $\|S^\nu\|_2 \leq C_S$, where $C_S > 0$ does not depend on ν, is needed. Since $S^* = S$, one has $\|S\|_2 = \max_{\lambda \in \sigma(S)} |\lambda|$ and one can thus chose ω to ensure that $\sigma(S) \subset [0,1]$. This can be achieved with $\omega = \rho(A_h)^{-2}$ and it is worth noting that this gives $\|S^\nu\|_2 \leq 1$.

The two-grid operator with ν_1 pre- and ν_2 post-smoothing steps then reads

$$T(\nu_1, \nu_2) = S^{\nu_1} \left(I_N - P A_H^{-1} R A_h \right) S^{\nu_2}, \tag{6}$$

where now both the fine matrix A_h and the coarse matrix A_H are defined by (2) when no dispersion correction is used and with k replaced by $\hat{k}_h$ (respectively $\hat{k}_H$) when we use dispersion correction.

Note that $\rho(T(\nu_1, \nu_2)) = \rho(T(\nu_1 + \nu_2, 0))$ and thus we are going to present our numerical results using $\nu = \nu_1 + \nu_2$ smoothing steps. We use a fine grid with $n = 255$ grid points which gives $h = 1/256$, and $N = 127$ coarse grid points. We compute the spectral radius of the two-grid operator (6) for the sequence of wave numbers

$$k_{1,j} = \sqrt{\frac{2}{h^2} \left(\sin\left(\frac{(j-1)\pi h}{2} \right)^2 + \sin\left(\frac{j \pi h}{2} \right)^2 \right)}, \; j = 1, \cdots, \widetilde{N},$$

placing $k_{1,j}^2$ exactly between two eigenvalues of the discrete Laplace operator[3]. The integer $\widetilde{N}$ is chosen so that we have a number of grid points per wavelength G satisfying $G = 2\pi/(k_{1,j}h) \geq 2\pi$ since, otherwise, the discrete dispersion relation at the coarse level is empty. The value of G satisfies $G \geq 20$ for $k \leq 80$. Figure 2 gives G for $k \geq 80$ and the modified wavenumbers as functions of k when $k_2 = k_1/2$ (similar results can be obtained for $k_2 = k_1/4$). We present in Figure 3 the spectral radius of the two grid operator without dispersion correction (left), and with dispersion correction (right), for a contrast of two in the wave number (top), and four (bottom), as a function of an increasing wave number, using various numbers of smoothing steps. These results show that without dispersion correction, $\rho(T(\nu, 0))$

[3] This choice allows us to systematically test sequences of wavenumbers for similarly conditioned Helmholtz problems as long as the contrast is not too large.

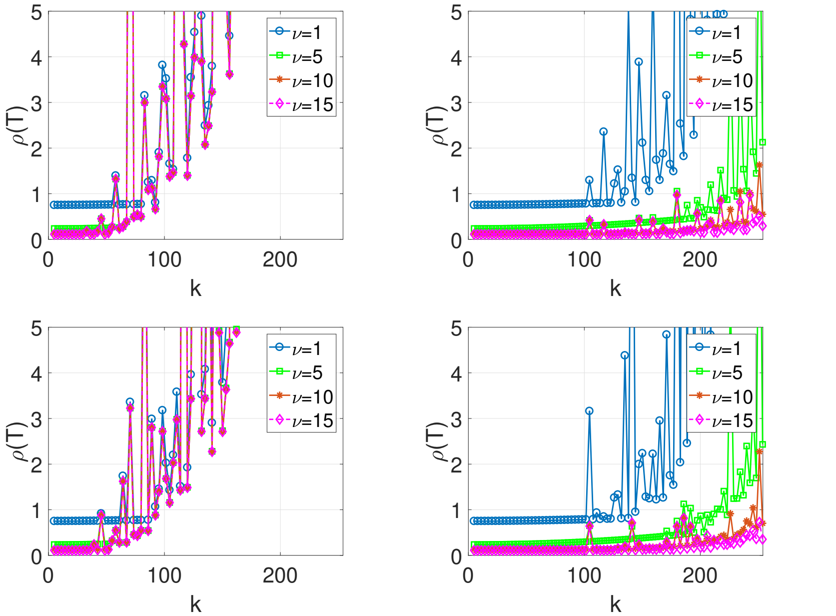

Fig. 3: Spectral radius of the two-grid operator. Left column: no dispersion correction. Right column: with dispersion correction. Top row: $k_2 = k_1/2$. Bottom row: $k_2 = k_1/4$.

is decreasing as the number of smoothing steps ν increases but the minimal ν to get $\rho(T(\nu, 0)) < 1$ is becoming too large for this method to be used in practice. In contrast, the two-grid scheme with dispersion correction is a convergent iterative method for a relatively small number of smoothing steps already.

In our approach, we computed a modified wave number $\hat{k}_h(0)$ at the interface using (3), which requires the computation of an exact solution of a transmission problem for the Helmholtz equation with piecewise constant wave number. Since it might be difficult to compute an exact solution of such a transmission problem in higher dimensions, we now test how important this dispersion correction at the interface is. The idea of this test is that since without dispersion correction we had $k(0) = k_1$, one could choose in the dispersion correction for the modified wave number $\hat{k}_h$ such that $\hat{k}_h(0) = \hat{k}_1 = \sqrt{2h^{-2}(1 - \cos(k_1 h))}$, i.e. just use the same dispersion correction at the interface as in the left region. We show in Figure 4 the spectral radius of the two-grid operator as a function of the number of smoothing steps for these two possible choices of $\hat{k}_h(0)$ for two different wave number contrasts and $k_1 = \max_j(k_{1,j}) = 253.73$. These results show that the modified wave number with $\hat{k}_h(0) = \hat{k}_1$ also yields a convergent two-grid method for a large enough number of smoothing steps, but the specific dispersion correction from the transmission problem in (3) needs a smaller number of smoothing steps to ensure that $\rho(T(\nu, 0)) < 1$ and also has a much smaller contraction factor.

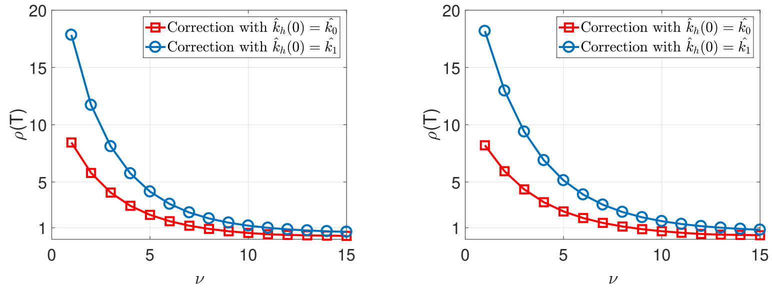

Fig. 4: Comparison of $\rho(T)$ for $k = \max_j(k_{1,j})$ using a shifted wave number $\hat{k}_h(x)$ such that $\hat{k}_h(0) = \hat{k}_1$ and $\hat{k}_h(0) = \hat{k}_0$ satisfying (3). Left: $k_2 = k_1/2$. Right: $k_2 = k_1/4$.

5 Conclusions and Outlook

We have introduced a new technique for dispersion correction for discretized Helmholtz problems in 1D for the case of piecewise constant wave numbers at the interface between regions where the wave number has a jump. The idea is to use a discrete wave number stemming from a transmission problem. We showed numerically that this dispersion correction leads to much more accurate numerical solutions, and also leads to much more efficient multigrid techniques when applied on each level of the grid hierarchy. Dispersion correction is more difficult in higher dimensions, but modifying the wave number in addition to specialized stencils has led to very good results in [8]. We are currently working on a 2D variant of the ideas presented here.

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