BDDC Preconditioners for a Space-time Finite Element Discretization of Parabolic Problems

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1 Introduction

Continuous space-time finite element methods for parabolic problems have been recently studied, e.g., in [1, 9, 10, 13]. The main common features of these methods are very different from those of time-stepping methods. Time is considered to be just another spatial coordinate. The variational formulations are studied in the full space-time cylinder that is then decomposed into arbitrary admissible simplex elements. In this work, we follow the space-time finite element discretization scheme proposed in [10] for a model initial-boundary value problem, using continuous and piecewise linear finite elements in space and time simultaneously.

It is a challenging task to efficiently solve the large-scale linear system of algebraic equations arising from the space-time finite element discretization of parabolic problems. In this work, as a preliminary study, we use the balancing domain decomposition by constraints (BDDC [2, 11, 12]) preconditioned GMRES method to solve this system efficiently. We mention that robust preconditioning for space-time isogeometric analysis schemes for parabolic evolution problems has been reported in [3, 4].

The remainder of the paper is organized as follows: Sect. 2 deals with the space-time finite element discretization for a parabolic model problem. In Sect. 3, we discuss BDDC preconditioners that are used to solve the linear system of algebraic equations. Numerical results are shown and discussed in Sect. 4. Finally, some conclusions are drawn in Sect. 5.

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2 The space-time finite element discretization

The following parabolic initial-boundary value problem is considered as our model problem: Find $u : \overline{Q} \to \mathbb{R}$ such that

$$\partial_t u - \Delta_x u = f \text{ in } Q, \quad u = 0 \text{ on } \Sigma,\ u = u_0 \text{ on } \Sigma_0, \tag{1}$$

where $Q := \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^2$ is a sufficiently smooth and bounded spatial computational domain, $\Sigma := \partial\Omega \times (0,T)$, $\Sigma_0 := \Omega \times \{0\}$, $\Sigma_T := \Omega \times \{T\}$.

Let us now introduce the following Sobolev spaces:

$$H_0^{1,0}(Q) = \{u \in L_2(Q) : \nabla_x u \in [L_2(Q)]^2, u = 0 \text{ on } \Sigma\},$$
$$H_{0,\bar{0}}^{1,1}(Q) = \{u \in L_2(Q) : \nabla_x u \in [L_2(Q)]^2, \partial_t u \in L_2(Q) \text{ and } u|_{\Sigma\cup\Sigma_T} = 0\},$$
$$H_{0,\underline{0}}^{1,1}(Q) = \{u \in L_2(Q) : \nabla_x u \in [L_2(Q)]^2, \partial_t u \in L_2(Q) \text{ and } u|_{\Sigma\cup\Sigma_0} = 0\}.$$

Using the classical approach [7, 8], the variational formulation for the parabolic model problem (1) reads as follows: Find $u \in H_0^{1,0}(Q)$ such that

$$a(u,v) = l(v), \quad \forall v \in H_{0,\bar{0}}^{1,1}(Q), \tag{2}$$

where

$$a(u,v) = -\int_Q u(x,t)\partial_t v(x,t)d(x,t) + \int_Q \nabla_x u(x,t)\cdot\nabla_x u(x,t)d(x,t),$$
$$l(v) = \int_Q f(x,t)v(x,t)d(x,t) + \int_\Omega u_0(x)v(x,0)dx.$$

Remark 1 (Parabolic solvability and regularity [7, 8]) If $f \in L_{2,1}(Q) := \{v : \int_0^T \|v(\cdot,t)\|_{L_2(\Omega)}dt < \infty\}$ and $u_0 \in L_2(\Omega)$, then there exists a unique generalized solution $u \in H_0^{1,0}(Q) \cap V_2^{1,0}(Q)$ of (2), where $V_2^{1,0}(Q) := \{u \in H^{1,0}(Q) : |u|_Q < \infty$ and $\lim_{\Delta t\to 0}\|u(\cdot,t+\Delta t) - u(\cdot,t)\|_{L_2(\Omega)} = 0$, uniformly on $[0,T]\}$, and $|u|_Q := \max_{0\le\tau\le T}\|u(\cdot,\tau)\|_{L_2(\Omega)} + \|\nabla_x u\|_{L_2(\Omega\times(0,T))}$. If $f \in L_2(Q)$ and $u_0 \in H_0^1(\Omega)$, then the generalized solution u belongs to $H_0^{\Delta,1}(Q) := \{v \in H_0^{1,1}(Q) : \Delta_x u \in L_2(Q)\}$ and continuously depends on t in the norm of the space $H_0^1(\Omega)$.

To derive the space-time finite element scheme, we mainly follow the approach proposed in [10]. Let $V_h = \text{span}\{\varphi_i\}$ be the span of continuous and piecewise linear basis functions φ_i on shape regular finite elements of an admissible triangulation $\mathcal{T}_h$. Then we define $V_{0h} = V_h \cap H_{0,\underline{0}}^{1,1}(Q) = \{v_h \in V_h : v_h|_{\Sigma\cup\Sigma_0} = 0\}$. For convenience, we consider homogeneous initial conditions, i.e., $u_0 = 0$ on Ω. Multiplying the PDE $\partial_t u - \Delta_x u = f$ on $K \in \mathcal{T}_h$ by an element-wise time-upwind test function $v_h + \theta_K h_K \partial_t v_h$, $v_h \in V_{0h}$, we get

$$\int_K (\partial_t u v_h + \theta_K h_K \partial_t u \partial_t v_h - \Delta_x u(v_h + \theta_K h_K \partial_t v_h))d(x,t) =$$
$$\int_K f(v_h + \theta_K h_K \partial v_h)d(x,t),$$

where h_K refers to the diameter of an element K in the space-time triangulation $\mathcal{T}_h$ of Q. Further, θ_K denotes a stabilization parameter [10]; see Remark 3. In the space-time finite element scheme [10], the time is considered as another spatial coordinate, and the partial derivative w.r.t. time is viewed as a convection term in the time direction. Therefore, as in the classical SUPG (streamline upwind Petrov-Galerkin) scheme, we use time-upwind test functions elementwise.

Integration by parts (the first part) with respect to the space and summation yields

$$\sum_{K\in\mathcal{T}_h} \int_K (\partial_t u v_h + \theta_K h_K \partial_t u \partial_t v_h + \nabla_x u \cdot \nabla_x v_h - \theta_K h_K \Delta_x u \partial_t v_h)d(x,t)$$
$$- \sum_{K\in\mathcal{T}_h} \int_{\partial K} n_x \cdot \nabla_x u v_h ds = \sum_{K\in\mathcal{T}_h} \int_K f(v_h + \theta_K h_K \partial_t v_h)d(x,t).$$

Since $n_x \cdot \nabla_x u$ is continuous across the inner boundary ∂K of K, $n_x = 0$ on $\Sigma_0 \cup \Sigma_T$, and $v_h = 0$ on Σ, the term $-\sum_{K\in\mathcal{T}_h} \int_{\partial K} n_x \cdot \nabla_x u v_h ds$ vanishes.

If the solution u of (2) belongs to $H^{\Delta,1}_{0,\underline{0}}(\mathcal{T}_h) := \{v \in H^{1,1}_{0,\underline{0}}(Q) : \Delta_x v|_K \in L_2(K), \forall K \in \mathcal{T}_h\}$, cf. Remark 1, then the consistency identity

$$a_h(u, v_h) = l_h(v_h), \quad v_h \in V_{0h}, \tag{3}$$

holds, where

$$a_h(u, v_h) := \sum_{K\in\mathcal{T}_h} \int_K (\partial_t u v_h + \theta_K h_K \partial_t u \partial_t v_h + \nabla_x u \cdot \nabla_x v_h - \theta_K h_K \Delta_x u \partial_t v_h)d(x,t),$$
$$l_h(v_h) := \sum_{K\in\mathcal{T}_h} \int_K f(v_h + \theta_K h_K \partial_t v_h)d(x,t).$$

With the restriction of the solution to the finite-dimensional subspace V_{0h}, the space-time finite element scheme reads as follows: Find $u_h \in V_{0h}$ such that

$$a_h(u_h, v_h) = l_h(v_h), \quad v_h \in V_{0h}. \tag{4}$$

Thus, we have the Galerkin orthogonality: $a_h(u - u_h, v_h) = 0, \forall v_h \in V_{0h}$.

Remark 2 Since we use continuous and piecewise linear trial functions, the integrand $-\theta_K h_K \Delta_x u_h \partial_t v_h$ vanishes element-wise, which simplifies the implementation.

Remark 3 On fully unstructured meshes, $\theta_k = O(h_k)$ [10]; on uniform meshes, $\theta_k = \theta = O(1)$ [9]. In this work, we have used $\theta = 0.5$ and $\theta = 2.5$ on uniform meshes for testing robustness of the BDDC preconditioners. The detailed results for $\theta = 2.5$ are presented in Table 1.

It was shown in [10] that the bilinear form $a_h(\cdot,\cdot)$ is V_{0h}-coercive: $a_h(v_h,v_h) \geq \mu_c \|v_h\|_h^2$, $\forall v_h \in V_{0h}$ with respect to the norm $\|v_h\|_h^2 = \sum_{K\in\mathcal{T}_h}(\|\nabla_x v_h\|^2_{L_2(K)} + \theta_K h_K \|\partial_t v_h\|^2_{L_2(K)}) + \frac{1}{2}\|v_h\|^2_{L_2(\Sigma_T)}$. Furthermore, the bilinear form is bounded on $V_{0h,*}\times V_{0h}$: $|a_h(u,v_h)| \leq \mu_b \|u\|_{0h,*}\|v_h\|_h$, $\forall u \in V_{0h,*}$, $\forall v_h \in V_{0h}$, where $V_{0h,*} = H^{\Delta,1}_{0,\underline{0}}(\mathcal{T}_h) + V_{0h}$ equipped with the norm $\|v\|^2_{0h,*} = \|v\|_h^2 + \sum_{K\in\mathcal{T}_h}(\theta_K h_K)^{-1}\|v\|^2_{L_2(K)} + \sum_{K\in\mathcal{T}_h}\theta_k h_k \|\Delta_x v\|^2_{L_2(K)}$. Let l and k be positive reals such that $l \geq k > 3/2$. We now define the broken Sobolev space $H^s(\mathcal{T}_h) := \{v \in L_2(Q) : v|_K \in H^s(K)\ \forall K \in \mathcal{T}_h\}$ equipped with the broken Sobolev semi-norm $|v|^2_{H^s(\mathcal{T}_h)} := \sum_{K\in\mathcal{T}_h}|v|^2_{H^s(K)}$. Using the Lagrangian interpolation operator Π_h mapping $H^{1,1}_{0,\underline{0}}(Q)\cap H^k(Q)$ to V_{0h}, we obtain $\|u-u_h\|_h \leq \|u-\Pi_h u\|_h + \|\Pi_h u - u_h\|_h$. The term $\|u-\Pi_h u\|_h$ can be bounded by means of the interpolation error estimate, and the term $\|\Pi_h u - u_h\|_h$ by using ellipticity, Galerkin orthogonality and boundedness of the bilinear form. The discretization error estimate $\|u-u_h\|_h \leq C(\sum_{K\in\mathcal{T}_h} h_K^{2(l-1)}|u|^2_{H^l(K)})^{1/2}$ holds for the solution u provided that u belongs to $H^{1,1}_{0,\underline{0}}(Q)\cap H^k(Q)\cap H^l(\mathcal{T}_h)$, and the finite element solution $u_h \in V_{0h}$, where $C > 0$, independent of mesh size; see [10].

3 Two-level BDDC preconditioners

After the space-time finite element discretization of the model problem (1), the linear system of algebraic equations reads as follows:

$$Kx = f, \tag{5}$$

with $K := \begin{bmatrix} K_{II} & K_{I\Gamma} \\ K_{\Gamma I} & K_{\Gamma\Gamma}\end{bmatrix}$, $x := \begin{bmatrix} x_I \\ x_\Gamma \end{bmatrix}$, $f := \begin{bmatrix} f_I \\ f_\Gamma \end{bmatrix}$, $K_{II} = \operatorname{diag}\left[K^1_{II},\ldots,K^N_{II}\right]$, where N denotes the number of polyhedral subdomains Q_i from a non-overlapping domain decomposition of Q. In system (5), we have decomposed the degrees of freedom into the ones associated with the internal (I) and interface (Γ) nodes, respectively. We aim to solve the Schur-complement system living on the interface:

$$Sx_\Gamma = g_\Gamma, \tag{6}$$

with $S := K_{\Gamma\Gamma} - K_{\Gamma I}K_{II}^{-1}K_{I\Gamma}$ and $g := f_\Gamma - K_{\Gamma I}K_{II}^{-1}f_I$.

The bilinear form $a_h(\cdot,\cdot)$ is coercive on the space-time finite element space V_{0h} like in the corresponding elliptic case. There are efficient domain decomposition preconditioners for such elliptic problems [14]. This motivated us to use such preconditioners for solving positive definite space-time finite element equations too. Following [12] (see also details in [5]), Dohrmann's (two-level) BDDC preconditioners P_{BDDC} for the interface Schur complement equation (6), originally proposed for symmetric and positive definite systems in [2, 11], can be written in the form

$$P^{-1}_{BDDC} = R^T_{D,\Gamma}(T_{sub} + T_0)R_{D,\Gamma}, \tag{7}$$

where the scaled operator $R_{D,\Gamma}$ is the direct sum of restriction operators $R^i_{D,\Gamma}$ mapping the global interface vector to its component on local interface $\Gamma_i := \partial Q_i \cap \Gamma$, with a proper scaling factor.

Here the coarse level correction operator T_0 is constructed as

$$T_0 = \Phi(\Phi^T S \Phi)^{-1} \Phi^T \tag{8}$$

with the coarse level basis function matrix $\Phi = \left[(\Phi^1)^T, \cdots, (\Phi^N)^T\right]^T$, where the basis function matrix Φ^i on each subdomain interface is obtained by solving the following augmented system:

$$\begin{bmatrix} S^i & (C^i)^T \\ C^i & 0 \end{bmatrix} \begin{bmatrix} \Phi^i \\ \Lambda^i \end{bmatrix} = \begin{bmatrix} 0 \\ R^i_\Pi x_\Gamma \end{bmatrix}. \tag{9}$$

with the given primal constraints C^i of the subdomain Q_i and the vector of Lagrange multipliers on each column of Λ^i. The number of columns of each Φ^i equals to the number of global coarse level degrees of freedom, typically living on the subdomain corners, and/or interface edges, and/or faces. Here the restriction operator R^i_Π maps the global interface vector in the continuous primal variable space on the coarse level to its component on Γ_i.

The subdomain correction operator T_{sub} is defined as

$$T_{sub} = \sum_{i=1}^{N} \left[(R^i_\Gamma)^T \quad 0\right] \begin{bmatrix} S^i & (C^i)^T \\ C^i & 0 \end{bmatrix}^{-1} \begin{bmatrix} R^i_\Gamma \\ 0 \end{bmatrix}, \tag{10}$$

with vanishing primal variables on all the coarse levels. Here the restriction operator R^i_Γ maps global interface vectors to their components on Γ_i.

4 Numerical experiments

We use $u(x, y, t) = \sin(\pi x)\sin(\pi y)\sin(\pi t)$ as exact solution of (1) in $Q = (0, 1)^3$; see the left plot in Fig. 1. We perform uniform mesh refinements of Q using tetrahedral elements. By using Metis [6], the domain is decomposed into $N = 2^k$, $k = 3, 4, ..., 9$, non-overlapping subdomains Q_i with their own tetrahedral elements; see the right plot in Fig. 1. The total number of degrees of freedom is $(2^k + 1)^3$, $k = 4, 5, 6, 7$. We run BDDC preconditioned GMRES iterations until the relative residual error reaches 10^{-9}. The experiments are performed on 64 nodes each with 8-core Intel Haswell processors (Xeon E5-2630v3, 2.4Ghz) and 128 GB of memory. Three variants of BDDC preconditioners are used with corner (C), corner/edge (CE), and corner/edge/face (CEF) constraints, respectively. The number of BDDC preconditioned GMRES iterations and the computational time measured in seconds [s] with respect to the number of subdomains (row-wise) and number of degrees of freedom (column-wise) are given in Table 1 for $\theta = 2.5$. Since the system is unsymmetric but

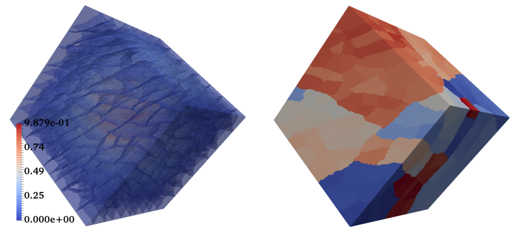

Fig. 1 Solution (left), space-time domain decomposition (right) with 129^3 degrees of freedom and 512 subdomains.

positive definite, the BDDC preconditioners do not show the same typical robustness and efficiency behavior when applied to the symmetric and positive definite system [14]. Nevertheless, we still observe certain scalability with respect to the number subdomains (up to 128), in particular, with corner/edge and corner/edge/face constraints. Increasing θ will improve the preformance of the BDDC preconditioners with respect to the number of GMRES iterations, computational time, and scalability with respect to the number of subdomains as well as number of degrees of freedom, whereas decreasing θ leads to a worse performance. For instance, in the case of $\theta = 0.5$, the last row of Table 1 reads as follows: 129^3 $\mathrm{OoM}/(-)$ $\mathrm{OoM}/(-)$ $173/(126.93s)$ $171/(109.94s)$ $185/(45.05s)$ $> 500/(-)$ $206/(33.13s)$. This behaviour is expected since larger θ makes the problem more elliptic. However, we note that θ also affects the norm $\|\cdot\|_h$ in which we measure the discretization error.

5 Conclusions

In this work, we have applied two-level BDDC preconditioned GMRES methods to the solution of finite element equations arising from the space-time discretization of a parabolic model problem. We have compared the performance of BDDC preconditioners with different coarse level constraints for such an unsymmetric, but positive definite system. The preconditioners show certain scalability provided that θ is sufficiently large. Future work will concentrate on improvement of coarse-level corrections in order to achieve robustness with respect to different choices of θ.

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Table 1: $\theta = 2.5$. BDDC performance using different coarse level constraints ($C/CE/CEF$), with respect to the number of subdomains (row-wise) and degrees of freedoms (column-wise).

	8	16	32	64	128	256	512
No preconditioner							
17^3	46	51	55	61	66	> 500	> 500
	(0.02*s*)	(0.02*s*)	(0.02*s*)	(0.03*s*)	(0.04*s*)	–	–
33^3	72	79	87	99	108	> 500	> 500
	(0.64*s*)	(0.32*s*)	(0.15*s*)	(0.12*s*)	(0.13*s*)	–	–
65^3	116	126	145	163	176	191	> 500
	(27.59*s*)	(9.17*s*)	(4.07*s*)	(1.92*s*)	(1.06*s*)	(2.02*s*)	–
129^3	OoM	OoM	240	271	304	> 500	382
	(–)	(–)	(145.64*s*)	(58.51*s*)	(24.3*s*)	(–)	(12.41*s*)
C (corner) preconditioner							
17^3	23	28	27	32	36	51	110
	(0.02*s*)	(0.02*s*)	(0.03*s*)	(0.03*s*)	(0.07*s*)	(0.72*s*)	(2.48*s*)
33^3	30	33	39	50	50	206	182
	(0.60*s*)	(0.26*s*)	(0.14*s*)	(0.10*s*)	(0.09*s*)	(3.86*s*)	(6.2*s*)
65^3	35	47	61	64	69	77	287
	(19.85*s*)	(7.42*s*)	(3.47*s*)	(1.44*s*)	(0.68*s*)	(1.05*s*)	(9.00*s*)
129^3	OoM	OoM	94	104	107	340	112
	(–)	(–)	(124.30*s*)	(46.45*s*)	(16.90*s*)	(33.45*s*)	(5.89*s*)
CE (corner+edge) preconditioner							
17^3	21	22	22	25	27	42	80
	(0.03*s*)	(0.02*s*)	(0.01*s*)	(0.03*s*)	(1.78*s*)	(0.78*s*)	(1.65*s*)
33^3	27	26	32	38	32	117	132
	(0.54*s*)	(0.23*s*)	(0.11*s*)	(0.12*s*)	(0.15*s*)	(2.52*s*)	(5.98*s*)
65^3	33	44	51	54	54	54	235
	(19.00*s*)	(7.27*s*)	(2.98*s*)	(1.33*s*)	(0.75*s*)	(1.32*s*)	(15.13*s*)
129^3	OoM	OoM	82	83	90	366	94
	(–)	(–)	(109.19*s*)	(38.59*s*)	(15.21*s*)	(37.00*s*)	(7.91*s*)
CEF (corner+edge+face) preconditioner							
17^3	21	21	22	22	22	38	74
	(0.02*s*)	(0.01*s*)	(0.02*s*)	(0.04*s*)	(0.13*s*)	(0.74*s*)	(1.74*s*)
33^3	27	26	30	35	31	115	123
	(0.68*s*)	(0.28*s*)	(0.14*s*)	(0.12*s*)	(0.25*s*)	(3.79*s*)	(7.08*s*)
65^3	34	44	49	52	53	51	226
	(26.64*s*)	(9.29*s*)	(3.86*s*)	(1.61*s*)	(1.11*s*)	(1.88*s*)	21.30*s*
129^3	OoM	OoM	82	83	88	369	92
	(–)	(–)	(145.39*s*)	(53.10*s*)	(23.04*s*)	(52.33)	(13.8*s*)

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