Non-overlapping Spectral Additive Schwarz Methods

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1 Discrete Problem

For a given domain $\Omega \subset \mathbb{R}^d$, we impose homogeneous Dirichlet data on $\partial\Omega$. Let us introduce the Sobolev space $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$.
The continuous variational formulation is given by: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v) \qquad \text{for all} \qquad v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u, v) := \int_\Omega \rho(x)\nabla u \cdot \nabla v dx \qquad f(v) := \int_\Omega f v dx,$$

where we assume $\rho(x) \geq \rho_{\min} > 0$ almost everywhere in Ω.

2 Discretization

We begin by discretizing Problem (1) in an algebraic framework. Let us consider a conforming triangulation $\mathcal{T}_h$ of Ω where $\overline{\Omega} = \bigcup_{\tau \in \mathcal{T}_h} \bar{\tau}$ and basis functions $\{\phi_k\}_{1 \leq k \leq n}$ for the finite element space $V_h(\Omega)$. We use the convention that an element $\tau \in \mathcal{T}_h$, the domain Ω, and the subdomains Ω_i are treated as open sets.

The finite element space $V_h(\Omega)$ is defined as:

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$$V_h(\Omega) := \{v \in H_0^1(\Omega); v|_\tau \in P_1(\tau) \ \ \forall \tau \in \mathcal{T}_h\} = \text{Span}\{\phi_k : 1 \le k \le n\}.$$

The FEM matrix form associated with (1) can be written as

$$Au = b, \tag{2}$$

where

$$(A)_{kl} := a(\phi_k, \phi_l) = \sum_{\tau \in \mathcal{T}_h} a_\tau(\phi_k|_\tau, \phi_l|_\tau) \quad \text{for all } \ 1 \le k, l \le n,$$

and

$$(b)_k := f(\phi_k) = \sum_{\tau \in \mathcal{T}_h} f_\tau(\phi_k|_\tau) \qquad \text{for all } \ 1 \le k \le n.$$

2.1 Finite Element Spaces

We decompose Ω into N non-overlapping polygonal subdomains Ω_i which satisfy

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \ne j.$$

We require that each subdomain be a union of shape-regular triangular elements with nodes on the boundaries of neighboring subdomains matching across the interface. We define the interface of each subdomain Γ_i and the interior of each subdomain I_i, global interface $\Gamma \subset \Omega$ and global interior I as:

$$\Gamma_i := \partial\Omega_i \backslash \partial\Omega \quad \text{and} \quad \Gamma := \bigcup_{i=1}^{N} \Gamma_i \quad \text{and} \quad I = \Omega/\Gamma = \bigcup_{i=1}^{N} I_i.$$

For any finite element subset $D \subset \Omega$ let the set of degrees of freedom in D be the hat functions

$$\text{dof}(D) := \{1 \le k \le n; \phi_k|D \ne 0|D\},$$

where $0|D : D \to \mathbb{R}$ is identically zero. The finite element space on D is defined as

$$V_h(D) := \{u|D; u \in V_h(\Omega)\} = \text{span}\{\phi_k|D; k \in \text{dof}(D)\}.$$

2.2 Decomposition of $V_h(\Omega)$

Let us consider a family of local spaces $\{V_i, 1 \le i \le N\}$, where

$$V_i = V_h(\Omega_i) \cap H_0^1(\Omega_i),$$

and we define the extrapolation operators $R_i^T : V_i \rightarrow V_h(\Omega)$ where R_i^T is the extension by zero outside of Ω_i.
The coarse space V_0 is defined as the space of piecewise linear and continuous functions on Γ:

$$V_0 = V_h(\Gamma) := \{v|_\Gamma; \forall v \in V_h(\Omega)\}.$$

In Section 3, we will present different choices of the extension operator $R_0^T : V_0 \rightarrow V_h(\Omega)$. The space $V_h(\Omega)$ admits the following direct sum decomposition:

$$V_h(\Omega) = R_0^T V_0 \oplus R_1^T V_1 \oplus \cdots \oplus R_N^T V_N$$

2.3 Additive Schwarz Methods

Local solvers: For $1 \leq i \leq N$, let us introduce the exact local bilinear form

$$a_i(u, v) := a(R_i^T u, R_i^T v) \qquad u, v \in V_i,$$

and let us define $\tilde{T}_i : V_h(\Omega) \rightarrow V_i$ by

$$a_i(\tilde{T}_i u, v) = a(u, R_i^T v) \qquad v \in V_i \quad 1 \leq i \leq N, \tag{3}$$

and let $T_i : V_h(\Omega) \rightarrow V_h(\Omega)$ be given by $T_i := R_i^T \tilde{T}_i$.
Global solver: For $i = 0$ first we consider the exact global solver

$$a_0(u, v) := a(R_0^T u, R_0^T v) \qquad u, v \in V_0$$

and let us define $\tilde{T}_0 : V_h(\Omega) \rightarrow V_0$ by

$$a_0(\tilde{T}_0 u, v) = a(u, R_0^T v) \qquad v \in V_0 \tag{4}$$

and let $T_0 : V_h(\Omega) \rightarrow V_h(\Omega)$ be given by $T_0 := R_0^T \tilde{T}_0$. Note that we also will consider inexact solvers $\hat{a}_0(\cdot, \cdot)$ later in this paper. We replace (2) by the linear system

$$T_A u = g_h \quad \text{where} \quad T_A := T_0 + T_1 + \cdots + T_N, \quad g_h = \sum_{i=0}^{N} g_i$$

where g_i are obtained from (3) and (4); see [8].

3 Schur complement system

The linear system (2) can be written as

$$\begin{bmatrix} A_{\Gamma\Gamma} & A_{\Gamma I} \\ A_{I\Gamma} & A_{II} \end{bmatrix} \begin{bmatrix} u_\Gamma \\ u_I \end{bmatrix} = \sum_{i=1}^{N} R^{(i)^T} \begin{bmatrix} A_{\Gamma\Gamma}^{(i)} & A_{\Gamma I}^{(i)} \\ A_{I\Gamma}^{(i)} & A_{II}^{(i)} \end{bmatrix} R^{(i)} \begin{bmatrix} u_\Gamma \\ u_I \end{bmatrix} = \sum_{i=1}^{N} R^{(i)^T} \begin{bmatrix} b_\Gamma^{(i)} \\ b_I^{(i)} \end{bmatrix}.$$

In this equation the extrapolation operators $R^{(i)^T} : V_h(\Omega_i) \longrightarrow V_h(\Omega)$ is the extension by zero at nodes outside of $\bar{\Omega}_i$. Thus we have,

$$A = \sum_{i=1}^{N} R^{(i)^T} A^{(i)} R^{(i)} = \sum_{i=1}^{N} R^{(i)^T} \begin{bmatrix} A_{\Gamma\Gamma}^{(i)} & A_{\Gamma I}^{(i)} \\ A_{I\Gamma}^{(i)} & A_{II}^{(i)} \end{bmatrix} R^{(i)} \qquad \text{and} \qquad b = \begin{bmatrix} b_\Gamma \\ b_I \end{bmatrix} = \sum_{i=1}^{N} R^{(i)^T} \begin{bmatrix} b_\Gamma^{(i)} \\ b_I^{(i)} \end{bmatrix},$$

where $A^{(i)}$ is the matrix corresponding to the bilinear form of

$$a^{(i)}(u_i, v_i) = \sum_{\tau \in \mathcal{T}_h|_{\Omega_i}} a_\tau(u_i|_\tau, v_i|_\tau) \qquad u_i, v_i \in V_h(\Omega_i).$$

Moreover, if we label the interface nodes first and then label the interior nodes, we can decompose the Boolean matrices $R^{(i)^T}$ as:

$$R^{(i)^T} = \begin{bmatrix} R_{\Gamma_i\Gamma}^T & 0 \\ 0 & R_{I_iI}^T \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_\Gamma^{(i)} \\ u_I^{(i)} \end{bmatrix} = R^{(i)} \begin{bmatrix} u_\Gamma \\ u_I \end{bmatrix},$$

where $R_{\Gamma_i\Gamma}^T : V_h(\Gamma_i) \longrightarrow V_h(\Omega)$ and $R_{I_iI}^T : V_i \longrightarrow V_h(I)$ are zero extension operators. We now rewrite (2) in terms of Schur complement system (see [8])

$$\begin{bmatrix} S & 0 \\ A_{I\Gamma} & A_{II} \end{bmatrix} \begin{bmatrix} u_\Gamma \\ u_I \end{bmatrix} = \sum_{i=1}^{N} R^{(i)^T} \begin{bmatrix} S_{\Gamma\Gamma}^{(i)} & 0 \\ A_{I\Gamma}^{(i)} & A_{II}^{(i)} \end{bmatrix} \begin{bmatrix} u_\Gamma^{(i)} \\ u_I^{(i)} \end{bmatrix} = \sum_{i=1}^{N} R^{(i)^T} \begin{bmatrix} b_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} b_I^{(i)} \\ b_I^{(i)} \end{bmatrix} = \begin{bmatrix} \tilde{b}_\Gamma \\ b_I \end{bmatrix},$$

where

$$S_{\Gamma\Gamma}^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} A_{I\Gamma}^{(i)},$$

$$\tilde{b}_{\Gamma_i} = \sum_{i=1}^{N} R_{\Gamma_i\Gamma}^T (b_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} b_I^{(i)}) \quad \text{and} \quad S = \sum_{i=1}^{N} R_{\Gamma_i\Gamma}^T S_{\Gamma\Gamma}^{(i)} R_{\Gamma_i\Gamma} \quad \text{and} \quad S u_\Gamma = \tilde{b}_\Gamma.$$

We note that the best extension R_0^T is the a-discrete harmonic extension from Γ to I due to the orthogonality of the coarse and local problems. In this case

$$a(R_0^T v_\Gamma, R_0^T u_\Gamma) = v_\Gamma^T S u_\Gamma.$$

The motivation is to replace S by a good preconditioner S_0 of S; see [1],[3],[2],[6].

4 New Method: Spectral Schwarz methods with exact solver

In our spectral method, we define a new R_0^T extension operator. To do that, the first goal is to represent the best k_i-dimensional subspace of V_i to approximate the a-discrete harmonic extension operator inside the subdomains. We fix a threshold

$\delta < 1$, and choose the smallest k_i eigenvalues in each subdomain smaller than δ. First solve the following generalized eigenproblem in each subdomain separately:

$$S^{(i)}\xi_j^{(i)} \equiv (A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)}(A_{II}^{(i)})^{-1}A_{I\Gamma}^{(i)})\xi_j^{(i)} = \lambda_j^{(i)} A_{\Gamma\Gamma}^{(i)}\xi_j^{(i)} \tag{5}$$

These eigenvalue problems are based on Neumann matrix associated to nonoverlapping subdomains, therefore, differ from those in GenEO [7] and AGDSW [4].

We choose the smallest k_i eigenvalues and corresponding eigenvectors:
For $j = 1 : k_i$, let $Q_j^{(i)} = \xi_j^{(i)} \qquad P_j^{(i)} = -(A_{II}^{(i)})^{-1}A_{I\Gamma}^{(i)}\xi_j^{(i)}$
And $Q^{(i)} = [Q_1^{(i)}, Q_2^{(i)}, \cdots, Q_{k_i}^{(i)}]$ and $P^{(i)} = [P_1^{(i)}, P_2^{(i)}, \cdots, P_{k_i}^{(i)}]$.
Thus we have three identities, where the left-hand sides involve operators on Γ_i only:

1. $-A_{\Gamma\Gamma}^{(i)}Q^{(i)}D^{(i)} = A_{\Gamma I}^{(i)}P^{(i)}$
2. $-D^{(i)}Q^{(i)^T}A_{\Gamma\Gamma}^{(i)} = P^{(i)^T}A_{I\Gamma}^{(i)}$
3. $D^{(i)}Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}Q^{(i)} = Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}Q^{(i)}D^{(i)} = P^{(i)^T}A_{II}^{(i)}P^{(i)}$, where

$$D^{(i)} = \text{diagonal}(1-\lambda_1, 1-\lambda_2, \cdots, 1-\lambda_{k_i}) = I - \Lambda^{(i)}$$

Also the $Q^{(i)}$ consist of the generalized eigenvectors from (5), and we can normalize the eigenvectors so that $Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}Q^{(i)} = I^{(i)}$ and $Q^{(i)^T}S^{(i)}Q^{(i)} = \Lambda^{(i)}$ however in the implementation we do not assume normalized eigenvectors, so we keep $Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}Q^{(i)}$. Define the global extension $R_0^T : V_0 \to V_h(\Omega)$ as:

$$\begin{aligned} R_0^T u_\Gamma &= \begin{bmatrix} u_\Gamma \\ -\sum_{i=1}^{N} R_{I_i I}^T P^{(i)}(P^{(i)^T}A_{II}^{(i)}P^{(i)})^{-1}P^{(i)^T}A_{I\Gamma}^{(i)}R_{\Gamma_i\Gamma}u_\Gamma \end{bmatrix} \\ &= \begin{bmatrix} u_\Gamma \\ \sum_{i=1}^{N} R_{I_i I}^T P^{(i)}(Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}Q^{(i)})^{-1}Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}R_{\Gamma_i\Gamma}u_\Gamma \end{bmatrix}. \end{aligned}$$

And for $u, v \in V_0$, we define the exact coarse solver as:

$$\begin{aligned} a_0(u,v) = a(R_0^T u, R_0^T v) &= v^T \sum_{i=1}^{N} R_{\Gamma_i\Gamma}^T\big(A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)}P^{(i)}(P^{(i)^T}A_{II}^{(i)}P^{(i)})^{-1}P^{(i)^T}A_{I\Gamma}^{(i)}\big)R_{\Gamma_i\Gamma}u \\ &= v^T \sum_{i=1}^{N} R_{\Gamma_i\Gamma}^T\big(A_{\Gamma\Gamma}^{(i)} - A_{\Gamma\Gamma}^{(i)}Q^{(i)}D^{(i)}(Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}Q^{(i)})^{-1}Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}\big)R_{\Gamma_i\Gamma}u. \end{aligned}$$

On each subdomain, we have the following lemmas and theorem:

Lemma 1 *([5]) Let $\Pi_S^{(i)}u$ be the projection of $u \in V_h(\Gamma_i)$ onto Span of $Q^{(i)}$, that is, $\Pi_S^{(i)}u \triangleq Q^{(i)}(Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}Q^{(i)})^{-1}Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}u$. Define the local bilinear form $a_0^{(i)}(u,v) = v^T(A_{\Gamma\Gamma}^{(i)} - A_{\Gamma\Gamma}^{(i)}Q^{(i)}D^{(i)}(Q^{(i)^T}A_{\Gamma\Gamma}^{(i)}Q^{(i)})^{-1}Q^{(i)^T}A_{\Gamma\Gamma}^{(i)})u$ where $u, v \in V_h(\Gamma_i)$. Then:*

$$a_0^{(i)}(u,v) = (\Pi_S^{(i)}v)^T S^{(i)}(\Pi_S^{(i)}u) + (v - \Pi_S^{(i)}v)^T A_{\Gamma\Gamma}^{(i)}(u - \Pi_S^{(i)}u).$$

Lemma 2 *([5]) Let $u \in V_0$ then*

$$a_0(u,u) = \sum_{i=1}^{N} a_0^{(i)}(R_\Gamma^{(i)}u, R_\Gamma^{(i)}u) \le \sum_{i=1}^{N} \frac{1}{\delta} u^T R_\Gamma^{(i)^T} S^{(i)} R_\Gamma^{(i)} u = \frac{1}{\delta} u^T S u$$

From Lemma 1 and Lemma 2 and the classical Schwarz Theory [8] we have:

Theorem 1 *([5]) For any $u \in V_h(\Omega)$ the following holds:*

$$(2+\frac{3}{\delta})^{-1} a(u,u) \le a(T_A u, u) \le 2a(u,u) \Longrightarrow k(T_A) \le 2(2+\frac{3}{\delta})$$

5 Complexity of Spectral Schwarz Method and with inexact coarse solver

The solution $u_\Gamma = \tilde{T}_0 u$ of $a_0(u_\Gamma, v_0) = a(u, R_0^T v_0) = (R_0^T v_0)^T b$ is of the form:

$$\sum_{i=1}^{N} R_{\Gamma_i\Gamma}^T \big(A_{\Pi\Gamma}^{(i)} - A_{\Pi\Gamma}^{(i)} Q^{(i)} D^{(i)} (Q^{(i)^T} A_{\Pi\Gamma}^{(i)} Q^{(i)})^{-1} Q^{(i)^T} A_{\Pi\Gamma}^{(i)}\big) R_{\Gamma_i\Gamma} u_\Gamma$$
$$= \sum_{i=1}^{N} R_{\Gamma_i\Gamma}^T \big(b_\Gamma^{(i)} + A_{\Pi\Gamma}^{(i)} Q^{(i)} (Q^{(i)} A_{\Pi\Gamma}^{(i)} Q^{(i)})^{-1} P^{(i)^T} b_I^{(i)}\big).$$

Denote $A_{\Gamma\Gamma} = \sum_{i=1}^{N} R_{\Gamma_i\Gamma}^T A_{\Gamma\Gamma}^{(i)} R_{\Gamma_i\Gamma}$, $U = \sum_{i=1}^{N} R_{\Gamma_i\Gamma}^T A_{\Gamma\Gamma}^{(i)} Q^{(i)} R_{\lambda_i}$,
$D = \sum_{i=1}^{N} R_{\lambda_i}^T D^{(i)} R_{\lambda_i}$, $C = \sum_{i=1}^{N} R_{\lambda_i}^T (Q^{(i)^T} A_{\Gamma\Gamma}^{(i)} Q^{(i)})^{-1} R_{\lambda_i}$ and $P = \sum_{i=1}^{N} R_{I_i I}^T P^{(i)} R_{\lambda_i}$.
where R_{λ_i} is a restriction chosen $[u_{i1}, \cdots, u_{ik_i}]^T$ from $\bar{u} = [u_{11}, \cdots, u_{1k_1}, \cdots, u_{Nk_1}, \cdots, u_{Nk_N}]^T$.
Here k_i is the number of eigenfunctions chosen from the i-th subdomain, and $\bar{u}$ has dimension k equals to the number of all eigenvectors chosen from all N subdomains. Then we can rewrite the coarse mesh problem as:

$$(A_{\Gamma\Gamma} - UDCU^T) u_\Gamma = b_\Gamma + UCP^T b_I,$$

and we use Woodbury identity for implementation:

$$(A_{\Gamma\Gamma} - UDCU^T)^{-1} = A_{\Gamma\Gamma}^{-1} + A_{\Gamma\Gamma}^{-1} U (C^{-1}D^{-1} - U^T A_{\Gamma\Gamma}^{-1} U)^{-1} U^T A_{\Gamma\Gamma}^{-1}.$$

Then the complexity of the method is associated with $A_{\Gamma\Gamma}^{-1}$, C^{-1} and the $k \times k$ matrix $(C^{-1}D^{-1} - U^T A_{\Gamma\Gamma}^{-1} U)^{-1}$, where k = the number of all eigenfunctions.

We can make $A_{\Gamma\Gamma}$ and C block diagonal or diagonal matrices if we replace the exact $A_{\Gamma\Gamma}^{(i)}$ on the right-hand side of the generalized eigenproblems by $\hat{A}_{\Gamma\Gamma}^{(i)}$, where the $\hat{A}_{\Gamma\Gamma}^{(i)}$ are block diagonal or diagonal versions of the $A_{\Gamma\Gamma}^{(i)}$. Note that for the block diagonal case we eliminate the connections across different faces, edges and corners of the subdomains. These inexact cases can be analyzed and given in Theorem 2.

We introduce the local generalized eigenproblems:

$$S^{(i)}\hat{\xi}_i^{(i)} \equiv (A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)}(A_{II}^{(i)})^{-1}A_{I\Gamma}^{(i)})\hat{\xi}_j^{(i)} = \hat{\lambda}_j^{(i)}\hat{A}_{\Gamma\Gamma}^{(i)}\hat{\xi}_j^{(i)}.$$

And for $u, v \in V_0(\Omega)$, we define the inexact coarse solver as:

$$\hat{a}_0(u,v) = v^T \sum_{i=1}^{N} R_\Gamma^{(i)^T}\big(\hat{A}_{\Gamma\Gamma}^{(i)} - \hat{A}_{\Gamma\Gamma}^{(i)}\hat{Q}^{(i)}\hat{D}^{(i)}(\hat{Q}^{(i)^T}\hat{A}_{\Gamma\Gamma}^{(i)}\hat{Q}^{(i)})^{-1}\hat{Q}^{(i)^T}\hat{A}_{\Gamma\Gamma}^{(i)}\big)R_\Gamma^{(i)}u.$$

Where $\hat{Q}^{(i)}$ are the generalized eigenvectors and $\hat{D}^{(i)} = \text{diagonal}(1-\hat{\lambda}_1, \cdots, 1-\hat{\lambda}_{k_i})$. Then, a condition number estimate for the inexact case is given by the following theorem:

Theorem 2 *([5])*

For any $u \in V_h(\Omega)$ the following holds:

$$(2{+}7\max\{1, \frac{1}{\delta}\})^{-1}a(u,u) \le a(\hat{T}_A u, u) \le 4a(u,u) \Longrightarrow k(\hat{T}_A) \le 4(2{+}7\max\{1, \frac{1}{\delta}\})$$

6 Numerical Experiments

We present results for problem (1) for $f \equiv 1$ of our Adaptive Spectral Schwarz method with highly heterogeneous coefficients in the format of stripes crossing the interface of the subdomains (see Figure 1). We divide the square domain into $H \times H$ congruent square subdomains and in each subdomain we have two horizontal stripes and two vertical stripes. The coefficient on the stripe (in grey) is $\rho(x) = 10^{-6}$ and $\rho(x) = 1$ elsewhere. Experiments show (not presented here) that the Additive Average Schwarz method can lead to a large condition number that depends on ρ_{max}/ρ_{min}. In contrast, when we use Adaptive Spectral Schwarz method with a threshold $\delta = \frac{1}{4}\frac{h}{H}$, we have a well conditioned problem with a low number of iterations; in Table 1 we see the robustness of the adaptive spectral Schwarz method with exact solver and Table 2 with inexact solver using diagonal of $A_{\Gamma\Gamma}^{(i)}$.

Length of subdomain	Iterations	Condition number	Number of eigenvectors	Complexity of problem
H=1/4	11	6.4719	84	84×84
H=1/8	12	6.4719	420	420×420
H=1/16	12	6.4719	1860	1860×1860

Table 2: Adaptive Spectral Schwarz method with diagonal inexact solver and the number of eigenvectors. We fix $H/h = 8$ and the number of iterations required to reduce the residual by 10^{-6}. The condition number is estimated by the Arnoldi matrix in the CG method.

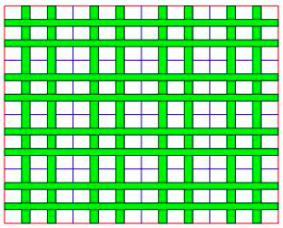

Fig. 1: In the stripe mesh, coefficient $\rho(x) = 10^{-6}$ in each stripe, and $\rho(x) = 1$ in other area.

Length of subdomain	CG Iterations	Condition number	Number of eigenvectors
$H = 1/4$	10	4.7684	84
$H = 1/8$	11	4.7684	420
$H = 1/16$	11	4.7684	1860

Table 1: Adaptive Spectral Schwarz method with exact solver and the number of eigenvectors. We fix $H/h = 8$ and the number of iterations required to reduce the residual by 10^{-6}. The condition number is estimated by the Arnoldi matrix in the CG method.

7 Conclusion

We introduced new two-dimensional and three-dimensional adaptive Schwarz methods derived from the additive average Schwarz method which are robust with respect to the jumps of coefficients with $O(H/h)$ condition number estimates. A unique feature of our methods is that our coarse space is based on generalized eigenvectors obtained in each nonoverlapping subdomain separately. One of the new methods has good parallelization properties since the global coarse matrix is sparse.

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