Fictitious Domain Method for an Inverse Problem in Volcanoes

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1 General framework and problem setting

Problems in volcanology often involve elasticity models in presence of cracks (see e.g. [5]). Most of the time the force exerted on the crack is unknown, and the position and shape of the crack are also frequently unknown or partially known (see e.g. [2]). The model may be approximated *via* boundary element methods.These methods are quite convenient to take into account the crack since the problem is then reformulated into an external problem where the crack is the only object to be meshed. However these methods do not allow to take the heterogeneity and/or the anisotropy of the medium into account. Another drawback is that, when it comes to identifying the shape and/or location of the crack, the variation of the latter implies a remeshing and assembling of all the matrices of the problem.

Using a domain decomposition technique then appears as the natural solution to these problems. In [1], a first step was made with the development of a direct solver implementing a domain decomposition method. The present work represents a step further with the use of such a solver, which has been improved since the publication of [1], to solve inverse problems in the field of earth sciences. To our knowledge, this is the first work using these kind of techniques in this field of application. The next step of our project will be the shape optimization problem to identify the shape and location of the crack.

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Let Ω be a bounded open set in $\mathbb{R}^d$, $d = 2, 3$ with smooth boundary $\partial\Omega := \Gamma_D \cup \overline{\Gamma_N}$ where Γ_D and Γ_N are of nonzero measure and $\Gamma_D \cap \Gamma_N = \emptyset$. We assume that Ω is occupied by an elastic solid and we denote by $\mathbf{u}$ the displacement field of the solid and the density of external forces by $\mathbf{f} \in \mathbf{L}^2(\Omega)$. The Cauchy stress $\sigma(\mathbf{u})$ and strain $\varepsilon(\mathbf{u})$ are given by

$$\sigma(\mathbf{u}) = \lambda\big(\mathrm{Tr}\varepsilon(\mathbf{u})\big)\mathbf{I}_{\mathbb{R}^d} + 2\mu\varepsilon(\mathbf{u}) \quad \text{and} \quad \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathsf{T}}),$$

where (λ, μ) are the Lamé coefficients, $\mathbf{I}_{\mathbb{R}^d}$ denotes the identity tensor, and $\mathrm{Tr}(\cdot)$ represents the matrix trace. Consider a crack $\Gamma_C \subset \Omega$ represented by a line ($d = 2$) or a surface ($d = 3$) parametrized by an injective mapping. Around the crack, Ω is split into Ω^- and Ω^+. The deformation field of the solid is supposed to satisfy the following elastostatic system:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{H}^1(\Omega \setminus \Gamma_C) \text{ such that :} & \\ -\mathrm{div}\, \sigma(\mathbf{u}) = \mathbf{f} & \text{in } \Omega \setminus \Gamma_C, \\ \mathbf{u} = 0 & \text{in } \Gamma_D, \\ \sigma(\mathbf{u}) \cdot \mathbf{n} = 0 & \text{on } \Gamma_N, \\ \sigma(\mathbf{u}) \cdot \mathbf{n}^{\pm} = p\mathbf{n}^{\pm} & \text{on } \Gamma_C. \end{cases} \tag{1}$$

where $\mathbf{n}$ is the outward unit normal to its boundaries. Typically in such a situation, Γ_N is the ground surface and free to move. Practically, the displacement field can be observed on Γ_N, whereas the pressure p exerted on the crack is unknown most of the time.

Consider the following function defined on $\mathbf{L}^2(\Gamma_C)$:

$$J(p) := \frac{1}{2}\int_{\Gamma_N} (\mathbf{u} - \mathbf{u}_d)\mathrm{C}^{-1}(\mathbf{u} - \mathbf{u}_d)^{\mathsf{T}}\, d\Gamma_N + \frac{\alpha}{2}\|p\|^2_{\mathbf{L}^2(\Gamma_C)}, \tag{2}$$

where $u_d \in \mathbf{L}^2(\Gamma_N)$ is the measured displacement field and u is the solution of (1) associated with p. Moreover, the matrix C is the covariance operator of the measurements uncertainties, and is assumed to be positive definite (see e.g. [4]), and finally $\alpha > 0$ is a regularization parameter. The aim of this work is to study the following problem, of optimal control type:

$$\min_{p \in \mathbf{L}^2(\Gamma_C)} J(p). \tag{3}$$

The paper is organized as follows : the next section will be devoted to the presentation of the domain decomposition method and its discretization. Section 3 gives the optimality conditions for the problem (3) and establishes their discrete version. A special focus will be made on the adaptation of the problem to a domain decomposition formulation. Finally we present a relevant numerical test in section 4 and discuss the next steps of our project.

2 Domain decomposition : the direct solver

To solve the direct problem (1), we use a domain decomposition method. More precisely, following [1], the domain Ω is split into two subdomains such that each point of the domain lies on one side of the crack or on the crack. Moreover, the global unknown solution $\mathbf{u}$ is decoupled in two sub–solutions for each side of the crack. For this purpose, we are using an artificial extension of the considered crack Γ_C (e.g. Γ_0 in Figure 1). Therefore, instead of the crack problem (1), we have to solve two

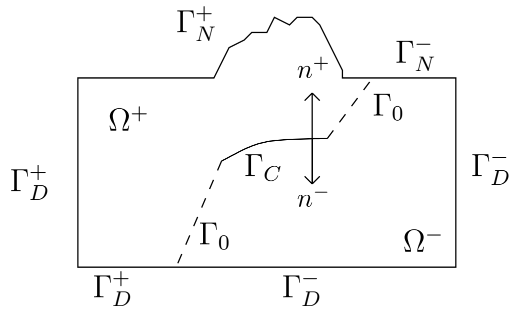

Fig. 1: Splitting the volcanic cracked domain

Neumann–type boundary problems such that for each problem we impose a pressure on Γ_C, which is more convenient from both theoretical and numerical points of view. More precisely we solve the following system:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{H}^1(\Omega) \text{ such that :} & \\ -\text{div}\, \sigma(\mathbf{u}^\pm) = \mathbf{f}^\pm & \text{in } \Omega^\pm, \\ \mathbf{u}^\pm = 0 & \text{on } \Gamma_D \cap \partial\Omega^\pm, \\ (\sigma(\mathbf{u}) \cdot \mathbf{n})^\pm = 0 & \text{on } \Gamma_N \cap \partial\Omega^\pm, \\ (\sigma(\mathbf{u}) \cdot \mathbf{n})^\pm = p\mathbf{n}^\pm & \text{on } \Gamma_C, \\ [\mathbf{u}] = 0 & \text{on } \Gamma_0, \\ [\sigma(\mathbf{u})] \cdot \mathbf{n}^+ = 0 & \text{on } \Gamma_0, \end{cases} \tag{4}$$

where $\mathbf{u}^+ = \mathbf{u}_{|\Omega^+}$ and $\mathbf{u}^- = \mathbf{u}_{|\Omega^-}$, and $[\mathbf{v}]$ denotes the jump of v across Γ_0. The two last conditions in (4) enforce the continuity of displacement and stress across Γ_0. Notice that the boundary conditions on Γ_0 ensure the construction of a global displacement field in $\mathbf{H}^1(\Omega \setminus \Gamma_C)$ solving the original problem (1).

Let us define the following Hilbert spaces

$$\mathbf{V}^\pm = \{v \in \mathbf{H}^1(\Omega^\pm) \mid v = 0 \text{ on } \Gamma_D \cap \partial\Omega^\pm\}, \qquad \mathbf{W} = (\mathbf{H}^{\frac{1}{2}}(\Gamma_0)).$$

and their dual spaces $\mathbf{V}'^\pm$ and $\mathbf{W}'$, endowed with their usual norms. Prescribing the continuity of displacement across the Γ_0 *via* a Lagrangian formulation, the mixed weak formulation of Problem (4) reads as follows:

$$\begin{cases} \text{Find } \mathbf{u}^{\pm} \in \mathbf{V}^{\pm} \text{ and } \boldsymbol{\lambda} \in \mathbf{W}' \text{ such that :} \\ a(\mathbf{u}^{\pm}, \mathbf{v}^{\pm}) \pm b(\boldsymbol{\lambda}, \mathbf{v}^{\pm}) = l^{\pm}(\mathbf{v}^{\pm}) & \forall \mathbf{v}^{\pm} \in \mathbf{V}^{\pm}, \\ b(\boldsymbol{\mu}, [\mathbf{u}]) = 0 & \forall \boldsymbol{\mu} \in \mathbf{W}', \end{cases} \tag{5}$$

with

$$a(\mathbf{u}^{\pm}, \mathbf{v}^{\pm}) = \int_{\Omega^{\pm}} \sigma(\mathbf{u}^{\pm}) : \varepsilon(\mathbf{v}^{\pm}) \, \mathrm{d}\Omega^{\pm}$$

bilinear, symmetric, coercive and

$$l^{\pm}(\mathbf{v}^{\pm}) = \int_{\Omega^{\pm}} f \cdot \mathbf{v}^{\pm} \, \mathrm{d}\Omega^{\pm} + \int_{\Gamma_C} (p\mathbf{n})^{\pm} \cdot \mathbf{v}^{\pm} \, \mathrm{d}\Gamma_C$$

linear and continuous. Moreover, b is defined as the duality pairing between $\mathbf{W}'$ and $\mathbf{W}$: $b(\boldsymbol{\lambda}, \mathbf{v}^{\pm}) = \langle \boldsymbol{\lambda}, \mathbf{v}^{\pm} \rangle_{\mathbf{W}', \mathbf{W}}$. Therefore, it is straightforward to prove the existence and uniqueness of a solution to Problem (5) (see e.g. [1] ans references within).

Denoting then $\mathbf{f}^{\pm}$ and $\mathbf{p}$ the approximations of f and p in $\mathbf{V}^{\pm}$ and $\widehat{\mathbf{W}}_h$, setting $p_n = \mathbf{p}\mathbf{n}^+ = -\mathbf{p}\mathbf{n}^-$ and

$$\mathrm{K} = \begin{pmatrix} A^+ & 0 & B^{+T} \\ 0 & A^- & -B^{-T} \\ B^+ & -B^- & 0 \end{pmatrix}, \qquad \mathbf{X} = \begin{pmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \\ \boldsymbol{\lambda} \end{pmatrix},$$

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \\ 0 \end{pmatrix} = \begin{pmatrix} M_\Omega^+ \cdot \mathbf{f}^+ \\ M_\Omega^- \cdot \mathbf{f}^- \\ 0 \end{pmatrix} + \begin{pmatrix} +M_c^+ \cdot p_n \\ -M_c^- \cdot p_n \\ 0 \end{pmatrix} := L_\Omega.\mathbf{f} + L_c.\mathbf{p},$$

the discretized form of system (5) has the linear algebraic formulation

$$\mathrm{K}\mathbf{X} = \mathbf{F}. \tag{6}$$

The system (6) can be solved by a Uzawa Conjugate gradient/domain decomposition method [1]. The method can be classically stabilized and the convergence of the numerical scheme can be proved as $h \to 0$.

In what follows, we will focus on the adaptation of a crack inverse problem to this domain decomposition formulation and its application to a realistic problem.

3 The crack inverse problem

First, we have the following result.

Proposition 1 *For any* $\alpha > 0$*, the problem* (3) *admits a unique solution* p^* *in* $\mathbf{L}^2(\Gamma_C)$.

Proof The proof is classical: applying the same method as in [3], one easily shows that J is strictly convex and coercive on $\mathbf{L}^2(\Gamma_C)$. □

The objective function J being strictly convex, first order optimality conditions can be computed to implement an suitable optimization method (in our case the conjugate gradient). Let us introduce the adjoint system

$$\begin{cases} -\mathrm{div}\,\sigma(\boldsymbol{\phi}) = 0 & \text{in } \Omega, \\ \boldsymbol{\phi} = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma(\boldsymbol{\phi}) \cdot \mathbf{n} = \mathrm{C}^{-1}(\mathbf{u} - \mathbf{u}_d) & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{7}$$

where $\mathbf{u}$ is a solution of system (1). Is is easy to prove that this adjoint system admits a unique solution $\boldsymbol{\phi} \in H^1(\Omega)$. We have the following

Proposition 2 *Let p^* be the solution of problem* (3) *and* $(\mathbf{u}^*, \boldsymbol{\phi}^*)$ *be the associated solutions of* (1) *and* (7). *Then, the following optimality condition holds:*

$$\alpha p^* + (\boldsymbol{\phi}^* \cdot \mathbf{n}^{\pm}) = 0. \tag{8}$$

This result can be proved using a classical sensitivity analysis technique. The important point here is that it gives a way to compute the gradient of the function J: for a given $p \in L^2(\Gamma_{\mathrm{C}})$, compute $(\mathbf{u}, \boldsymbol{\phi})$ which solve (1) and (7). Then, the Gâteaux derivative of J is given in $L^2(\Gamma_{\mathrm{C}})$ by

$$J'(p) = \alpha p + (\boldsymbol{\phi} \cdot \mathbf{n}^{\pm}), \tag{9}$$

For a given pressure $p \in L^2(\Gamma_{\mathrm{C}})$, the computation of the gradient $J'(p)$ then requires to solve two systems.

Since we transformed our direct problem into system (4), we now need to adapt the inverse problem to this formulation. The cost function J defined by (2) then rewrites into

$$J(p) := \frac{1}{2} \int_{\Gamma_{\mathrm{N}}^{\pm}} (\mathbf{u}^{\pm} - \mathbf{u}_d) \mathrm{C}^{-1} (\mathbf{u}^{\pm} - \mathbf{u}_d)^{\mathsf{T}} \, \mathrm{d}\Gamma_{\mathrm{N}}^{\pm} + \frac{\alpha}{2} \|p\|^2_{\mathbf{L}^2(\Gamma_{\mathrm{C}})}. \tag{10}$$

Notice that the observed data $\mathbf{u}_d$ can be interpolated on two sub–domains $\Omega^{\pm}$ to obtain $\mathbf{u}_d^{\pm}$ corresponding to $\mathbf{u}^{\pm}$.

In view of (6), denoting R the reduction matrix $R : \mathbf{X} \longrightarrow \mathbf{U}$ and

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \end{pmatrix}, \quad \mathbf{U}_d = \begin{pmatrix} \mathbf{u}_d^+ \\ \mathbf{u}_d^- \end{pmatrix},$$

the discrete cost function is defined as

$$J_d(\mathbf{p}) = \frac{1}{2} (R\mathbf{X} - \mathbf{U}_d)^{\mathsf{T}} \mathrm{C}^{-1} M_N (R\mathbf{X} - \mathbf{U}_d) + \frac{\alpha}{2} (\mathbf{p}^{\mathsf{T}} M_F \mathbf{p}), \tag{11}$$

where $\mathbf{X}$ is the solution of (6), M_N and M_F are the mass matrices on Γ_N and Γ_C, respectively. This finite dimensional problem then boils down to finding the saddle point of the following Lagrangian

$$\mathcal{L}(\mathbf{X}, \mathbf{p}, \mathbf{\Phi}) = J_d(\mathbf{p}) - \langle \mathrm{K}\mathbf{X} - (L_\Omega \mathbf{f} + L_c \mathbf{p}), \mathbf{\Phi} \rangle.$$

Computing the KKT conditions for this problem allows to compute the gradient of J_d : for a given vector $\mathbf{p}$, let $\mathbf{X}$ be the solution of (6) and $\mathbf{\Phi}$ be the solution of the adjoint problem

$$\mathrm{K}^{\mathsf{T}}\mathbf{\Phi} = \mathrm{C}^{-1} M_N (R\mathbf{X} - \mathbf{U}_d). \tag{12}$$

Then, we have

$$\nabla J_d(\mathbf{p}) = \alpha M_F \mathbf{p} + L_c^{\mathsf{T}} \mathbf{\Phi}, \tag{13}$$

The system (12) and the gradient (13) are the discrete counterparts of (7) and (9). As (6), the adjoint system (12) is solved by a Uzawa conjugate gradient/domain decomposition method.

Computational aspects: the problem studied here is actually of quadratic type. Hence it is natural to use a suitable minimization technique, namely a conjugate gradient algorithm. It is important to notice that, using the underlying quadratic form, one can determine the optimal step size. Therefore no line search algorithm is necessary, which consequently reduces the computational cost.

4 Numerical experiments

Aiming at practical applications, we applied the technique to a realistic volcano, the Piton de la Fournaise, Île de la Réunion, France. The mesh was built from a digital elevation model (DEM), provided by the french institute IGN (Institut Géographique National, French National Geographic Institute). Both the boundary and volume mesh for the whole domain were generated by Gmsh software (Figure 2, left). The crack geometry is assumed to be quadrangular and intersecting the surface. It is constructed following [2] (see Figure 2, left). The crack mesh does not match the volume mesh. Moreover, it can be easily extended in order to split the domain. We assume that the crack is submitted to an initial pressure $\mathbf{p}^0$. The inverse problem will consist in determining the unknown pressure from the surface displacements (Figure 2, right). The convergence curves in Figure 3, highlight the efficiency of adapted optimization algorithm. The conjugate gradient minimization performs efficiently, even for fine meshes.

5 Conclusion

We have studied a conjugate gradient type method for an interface pressure inverse problem using a Uzawa conjugate gradient domain decomposition method (from [1]) as inner solver. Further study is underway to derive a single-loop conjugate gradient domain decomposition method by (directly) considering the constrained minimization problem (1)-(2) and using sensitivity and adjoint systems techniques.

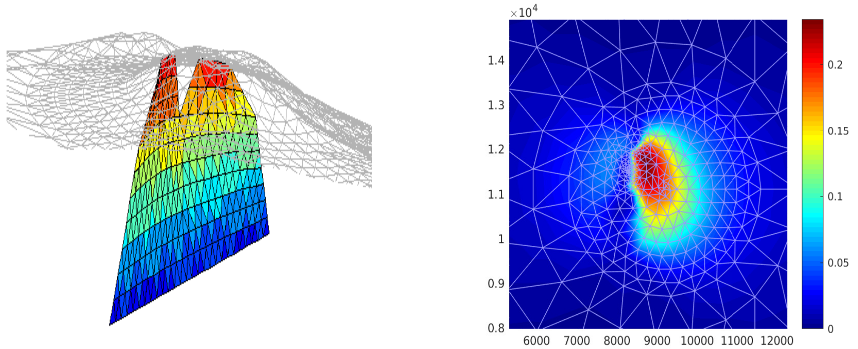

Fig. 2: Triangular surface mesh [2] representing the crack (left), amplitude of the displacement of a realistic volcano (right).

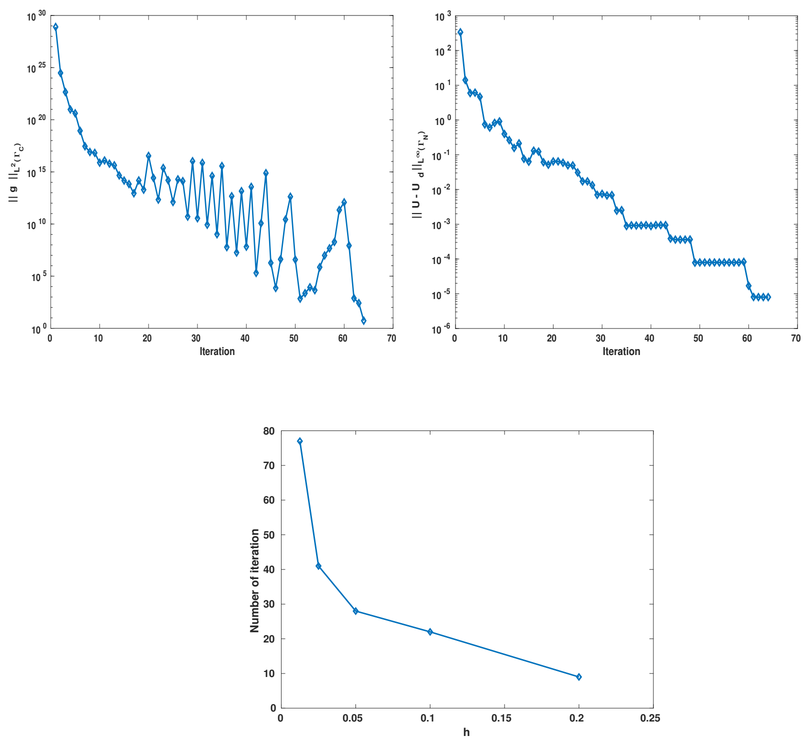

Fig. 3: Decay of the norm of gradient (left), the error of displacement on the ground in each iteration (right) and number of iteration to the converged **p** after each refinement of the mesh (bottom)

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