Asymptotic Analysis for the Coupling Between Subdomains in Discrete Fracture Matrix Models

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1 Introduction

We study the behavior of solutions of PDE models on domains containing a heterogeneous layer of aperture tending to zero. We consider general second order differential operators on the outer domains and elliptic operators inside the layer. Our study is motivated by the modeling of flow through fractured porous media, when one represents the fractures as entities of co-dimension one with respect to the surrounding rock matrix. These models are called Discrete Fracture Matrix (DFM) models [2, 4, 1]. A recent study on DFM models and their discretization can be found in [3]. Our focus lies on the derivation of coupling conditions, which have to be satisfied by the traces of the solutions for the matrix domain on each side of the matrix-fracture interfaces. We emphasize that we are not only concerned with the derivation of coupling conditions that have to be fulfilled in the limit of vanishing aperture, but in particular with the derivation of coupling conditions that have to be fulfilled up to a certain order of the aperture, which in turn occurs as a model parameter. In our work flow, we first derive exact coupling conditions by means of Fourier analysis. Reduced order coupling conditions are then obtained by truncation of the exact conditions at the desired order. Our approach is very systematic and allowed us to reproduce various coupling conditions from the literature as well as assess the error of the reduced models.

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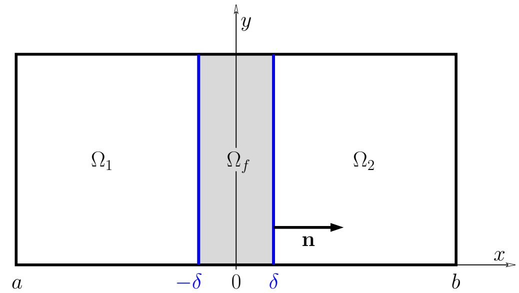

Fig. 1 Illustration of the domain under consideration. In our study, we restrict ourselves to a simple geometry, where $\Omega_1 = (a, -\delta) \times \mathbb{R}$, $\Omega_2 = (\delta, b) \times \mathbb{R}$ and $\Omega_f = (-\delta, \delta) \times \mathbb{R}$, with $a, b \in \overline{\mathbb{R}}$. $\mathbf{n}$ denotes the unit normal in x-direction.

2 Model problem

We consider the following problem on a threefold domain as illustrated in Figure 1:

$$\mathcal{L}_j(u_j, \mathbf{q}_j) = h_j \quad \text{in} \quad \Omega_j,\ j = 1, 2, f, \tag{1}$$

$$\mathbf{q}_j = \mathcal{G}_j u_j \quad \text{in} \quad \Omega_j,\ j = 1, 2, f, \tag{2}$$

$$u_j = u_f \quad \text{on} \quad \partial\Omega_j \cap \partial\Omega_f,\ j = 1, 2, \tag{3}$$

$$\mathbf{q}_j \cdot \mathbf{n} = \mathbf{q}_f \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega_j \cap \partial\Omega_f,\ j = 1, 2, \tag{4}$$

where $\mathcal{L}_j, \mathcal{G}_j$ are differential operators, together with some suitable boundary conditions. Only inside the fracture domain Ω_f, we will restrict our study to the class of general elliptic models, i.e. we assume that

$$\mathcal{L}_f(u_f, \mathbf{q}_f) = -\mathrm{div}\mathbf{q}_f + \frac{\mathbf{b}}{2} \cdot \nabla u_f + (\eta - \mathrm{div}\frac{\mathbf{b}}{2})u_f \quad \text{and} \quad \mathcal{G}_f u_f = (\mathbf{A}\nabla - \frac{\mathbf{b}}{2})u_f \tag{5}$$

with $\eta \in \mathbb{R}_{\geq 0}$, $\mathbf{b} \in \mathbb{R}^2$ and coercive $\mathbf{A} \in \mathbb{R}^{2\times 2}$. For simplicity, we also assume a trivial source term inside the fracture, i.e. $h_f = 0$.

3 Derivation of the reduced models by Fourier analysis

From (1),(2),(5) the Fourier coefficients $\hat{u}_f(x, k)$ of $u_f(x, y)$ have to fulfill for all $k \in \mathbb{R}$

$$-a_{11}\partial_{xx}\hat{u}_f + \Big(b_1 - (a_{12} + a_{21})ik\Big)\partial_x\hat{u}_f + (a_{22}k^2 + b_2 ik + \eta)\hat{u}_f = 0 \quad \text{in} \quad \Omega_f. \tag{6}$$

The roots of the characteristic polynomial associated with (6) are $\lambda_{1,2} = r \pm s$, where

$$r = -\frac{1}{2a_{11}}((a_{12} + a_{21})ik - b_1) \quad \text{and} \quad s = \Big(r^2 + \frac{1}{a_{11}}(a_{22}k^2 + b_2 ik + \eta)\Big)^{\frac{1}{2}}.$$

The ansatz for the solution of (6),

$$\hat{u}_f(x,k) = A_f(k)e^{\lambda_1 x} + B_f(k)e^{\lambda_2 x},$$

together with (3) and (4) immediately yields for the Fourier coefficients $\hat{u}_j(x,k)$ of $u_j(x,y)$ and $\hat{\mathbf{q}}_j(x,k)$ of $\mathbf{q}_j(x,y)$, $j = 1,2$, on the interfaces,

$$\hat{u}_1(-\delta,k) = A_f(k)e^{-\delta\lambda_1} + B_f(k)e^{-\delta\lambda_2}, \tag{7}$$

$$\hat{u}_2(\delta,k) = A_f(k)e^{\delta\lambda_1} + B_f(k)e^{\delta\lambda_2}, \tag{8}$$

$$\hat{\mathbf{q}}_1(-\delta,k)\cdot\mathbf{n} = a_{11}\lambda_1 A_f(k)e^{-\delta\lambda_1} + a_{11}\lambda_2 B_f(k)e^{-\delta\lambda_2} + (a_{12}ik - \frac{b_1}{2})\hat{u}_1(-\delta,k), \tag{9}$$

$$\hat{\mathbf{q}}_2(\delta,k)\cdot\mathbf{n} = a_{11}\lambda_1 A_f(k)e^{\delta\lambda_1} + a_{11}\lambda_2 B_f(k)e^{\delta\lambda_2} + (a_{12}ik - \frac{b_1}{2})\hat{u}_2(\delta,k). \tag{10}$$

Equations (7) and (8) are now solved for A_f and B_f, which can then be substituted into the remaining two equations (9) and (10). After some calculations, this leads to the exact coupling conditions

$$\sinh(2s\delta)\hat{\mathbf{q}}_1(-\delta)\cdot\mathbf{n} + (a_{11}s\cosh(2s\delta) + \rho\sinh(2s\delta))\hat{u}_1(-\delta) = a_{11}se^{-2\delta r}\hat{u}_2(\delta), \tag{11}$$

$$-\sinh(2s\delta)\hat{\mathbf{q}}_2(\delta)\cdot\mathbf{n} + (a_{11}s\cosh(2s\delta) - \rho\sinh(2s\delta))\hat{u}_2(\delta) = a_{11}se^{2\delta r}\hat{u}_1(-\delta), \tag{12}$$

where $\rho = \frac{a_{21}-a_{12}}{2}ik$. For the remaining part of the paper, we will drop the arguments indicating the evaluation at $x = -\delta$ for the functions living in Ω_1 and at $x = \delta$ for those living in Ω_2. Taking the sum (11) + (12) yields an expression related to the normal velocity jump across the fracture, whereas the difference (11) − (12) gives an expression related to the pressure jump accross the fracture,

$$\sinh(2s\delta)(\hat{\mathbf{q}}_2 - \hat{\mathbf{q}}_1)\cdot\mathbf{n} = a_{11}s\Big(\cosh(2s\delta)(\hat{u}_1+\hat{u}_2) - (e^{2\delta r}\hat{u}_1 + e^{-2\delta r}\hat{u}_2)\Big) + \rho\sinh(2s\delta)(\hat{u}_1 - \hat{u}_2), \tag{13}$$

$$a_{11}s\Big(\cosh(2s\delta)(\hat{u}_2-\hat{u}_1) + (e^{-2\delta r}\hat{u}_2 - e^{2\delta r}\hat{u}_1)\Big) = \sinh(2s\delta)(\hat{\mathbf{q}}_1+\hat{\mathbf{q}}_2)\cdot\mathbf{n} + \rho\sinh(2s\delta)(\hat{u}_1+\hat{u}_2). \tag{14}$$

We now expand (13), (14) into a series in δ and truncate at a given order. We then obtain the following reduced order coupling conditions at $x = \pm\delta$:

1. Truncation after the leading-order term, which we call coupling conditions of type zero (CC0 coupling conditions):

$$(\hat{\mathbf{q}}_2 - \hat{\mathbf{q}}_1)\cdot\mathbf{n} = 0 \qquad \text{and} \qquad \hat{u}_2 - \hat{u}_1 = 0.$$

2. Truncation after the next-to-leading-order term, which we call CC1 coupling conditions:

$$(\hat{\mathbf{q}}_2 - \hat{\mathbf{q}}_1) \cdot \mathbf{n} = \delta\Big(a_{22}k^2 + b_2 ik + \eta\Big)(\hat{u}_1 + \hat{u}_2) + \Big(-a_{21}ik + \frac{b_1}{2}\Big)(\hat{u}_2 - \hat{u}_1),$$
$$\delta(\hat{\mathbf{q}}_2 + \hat{\mathbf{q}}_1) \cdot \mathbf{n} = a_{11}(\hat{u}_2 - \hat{u}_1) + \delta\Big(a_{12}ik - \frac{b_1}{2}\Big)(\hat{u}_1 + \hat{u}_2).$$

Of course, we could derive higher order coupling conditions by using higher order expansions.

We now want to get back to the physical unknowns u_j and $\mathbf{q}_j$, $j = 1, 2$. To do so, we perform an inverse Fourier transform by formally applying the rules,

$$\hat{u}_j \mapsto u_j, \quad \hat{\mathbf{q}}_j \mapsto \mathbf{q}_j, \quad k^2 \mapsto -\partial_{yy}, \quad ik \mapsto \partial_y.$$

We therefore obtain as reduced order approximations of the exact coupling conditions between the subdomains Ω_1 and Ω_2

1. CC0 coupling conditions:

$$\mathbf{q}_2 \cdot \mathbf{n} - \mathbf{q}_1 \cdot \mathbf{n} = 0 \qquad \text{and} \qquad u_2 - u_1 = 0. \tag{15}$$

2. CC1 coupling conditions:

$$(\mathbf{q}_2 - \mathbf{q}_1) \cdot \mathbf{n} = \delta\Big(-a_{22}\partial_{yy} + b_2\partial_y + \eta\Big)(u_1 + u_2) + \Big(-a_{21}\partial_y + \frac{b_1}{2}\Big)(u_2 - u_1), \tag{16}$$

$$\delta(\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{n} = a_{11}(u_2 - u_1) + \delta\Big(a_{12}\partial_y - \frac{b_1}{2}\Big)(u_1 + u_2). \tag{17}$$

4 Comparison to the literature

DFM models are a tool for the simulation of flow through fractured porous media, where the governing equations are mass conservation and Darcy's law. The approach illustrated above covers more general problems, and in order to compare our coupling conditions to existing ones from the literature, we now let

$$\mathbf{b} := 0, \quad \eta := 0, \quad \text{and} \quad \mathbf{A} := \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

As outlined in [4], one typically derives the reduced order coupling conditions by integrating the equations over the fracture width,

$$0 = \int_{-\delta}^{\delta} \mathrm{div}\mathbf{q}_f \, \mathrm{d}x = \mathbf{q}_f \cdot \mathbf{n}(\delta) - \mathbf{q}_f \cdot \mathbf{n}(-\delta) + \partial_y \int_{-\delta}^{\delta} \mathbf{q}_f \, \mathrm{d}x$$

$$= \mathbf{q}_2 \cdot \mathbf{n} - \mathbf{q}_1 \cdot \mathbf{n} + 2\delta a_{22} \partial_y^2 U_f \,, \tag{18}$$

$$\int_{-\delta}^{\delta} \mathbf{q}_f \cdot \mathbf{n} \mathrm{d}x = a_{11}(u_f(\delta) - u_f(-\delta)) = a_{11}(u_2 - u_1), \tag{19}$$

and then uses some ad-hoc approximations

$$\int_{-\delta}^{\delta} \mathbf{q}_f \cdot \mathbf{n} \mathrm{d}x \approx 2\delta \frac{\mathbf{q}_f \cdot \mathbf{n}(\delta) + \mathbf{q}_f \cdot \mathbf{n}(-\delta)}{2} = \delta(\mathbf{q}_1 \cdot \mathbf{n} + \mathbf{q}_2 \cdot \mathbf{n}), \tag{20}$$

$$u_2 + u_1 \approx 2U_f \,, \tag{21}$$

where $U_f := \frac{1}{2\delta} \int_{-\delta}^{\delta} u_f \, \mathrm{d}x$. Combining these equations leads to the coupling conditions

$$\delta a_{22} \partial_y^2 (u_1 + u_2) + \mathbf{q}_2 \cdot \mathbf{n} - \mathbf{q}_1 \cdot \mathbf{n} = 0, \tag{22}$$

$$\delta(\mathbf{q}_2 \cdot \mathbf{n} + \mathbf{q}_1 \cdot \mathbf{n}) = a_{11}(u_2 - u_1). \tag{23}$$

Note that, by means of (21), condition (22) is equivalent to the tangential mass conservation inside the fracture together with mass exchange between the fracture and rock matrix.

Theorem 1 *The coupling conditions* (22)*,* (23) *coincide with the coupling conditions* (16)*,*(17) *for the diffusion equation with diagonal matrix A. Furthermore, the exact solution obeys formally* (22)*,* (23) *with an error of order three, for* $\delta \to 0$*.*

Proof For the diffusion equation with diagonal matrix A, the terms in the coupling conditions (16), (17), which are related to the advection and reaction constants and the terms related to the off-diagonal entries in the diffusion matrix vanish. By direct comparison, we observe that the resulting equations coincide with the coupling conditions (22), (23), which shows the first statement of the theorem. Furthermore, for the diffusion equation with diagonal matrix A, the coupling conditions (13), after dividing by $\sinh(2s\delta)$, and (14), after dividing by $\cosh(2s\delta)$, yield

$$\mathbf{q}_1 \cdot \mathbf{n} - \mathbf{q}_2 \cdot \mathbf{n} = \Big(\delta a_{22} \partial_{yy} + \frac{1}{3} \frac{\delta^3 a_{22}^2}{a_{11}} \partial_{yy}^2 + \frac{2}{15} \frac{\delta^5 a_{22}^3}{a_{11}^2} \partial_{yy}^3 + \cdots \Big)(u_1 + u_2), \tag{24}$$

$$u_2 - u_1 = \Big(\frac{\delta}{a_{11}} + \frac{1}{3} \frac{\delta^3 a_{22}}{a_{11}^2} \partial_{yy} + \frac{2}{15} \frac{\delta^5 a_{22}^2}{a_{11}^3} \partial_{yy}^2 + \cdots \Big)(\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{n}. \tag{25}$$

Hence, by substitution of the exact solution into the approximate coupling conditions (22), (23), we formally obtain residuals of order three, for $\delta \to 0$, which confirms the second statement of the theorem. □

From (24),(25), we observe that the asymptotic behavior of the exact coupling conditions depends only on the asymptotic behavior of the ratio $\frac{\delta}{a_{11}}$ and of the product δa_{22}. We call these two characteristic quantities the fracture resistivity and conductivity, respectively. In [5], a rigorous asymptotic analysis for the Laplace equation is conducted, with the focus on the solution in the limit $\delta = 0$. In this context, coupling conditions (at $x = \pm 0$) are derived, for the cases $\frac{\delta}{a_{11}} \to \gamma \in \mathbb{R}$, $\frac{\delta}{a_{11}} \to \infty$, $\frac{\delta}{a_{11}} \to 0$, provided $a_{11} \to 0$, which turn out to correspond to the coupling conditions, which we derive by means of truncating (24),(25) at order δ^0 (with $a_{11} = a_{22}$ for isotropic diffusion).

1. Case $\frac{\delta}{a_{11}} \to \gamma \in \mathbb{R}$ (note that this implies $\delta a_{11} \to 0$):
$$\mathbf{q}_1 \cdot \mathbf{n} - \mathbf{q}_2 \cdot \mathbf{n} = 0 \quad \text{and} \quad u_2 - u_1 = \gamma(\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{n}.$$
2. Case $\frac{\delta}{a_{11}} \to \infty$ (note that this implies $\delta a_{11} \to 0$):
$$\mathbf{q}_1 \cdot \mathbf{n} - \mathbf{q}_2 \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{q}_1 \cdot \mathbf{n} + \mathbf{q}_2 \cdot \mathbf{n} = 0.$$
3. Case $\frac{\delta}{a_{11}} \to 0$ and $\delta a_{11} \to 0$ corresponds to (15).

We can now complete this study by considering the cases $\delta a_{11} \to \gamma \in \mathbb{R}$ or $\delta a_{11} \to \infty$ (which both imply $\frac{\delta}{a_{11}} \to 0$). We obtain

4. Case $\delta a_{11} \to \gamma \in \mathbb{R}$:
$$\mathbf{q}_1 \cdot \mathbf{n} - \mathbf{q}_2 \cdot \mathbf{n} = \gamma \partial_{yy}(u_1 + u_2) \quad \text{and} \quad u_2 - u_1 = 0.$$
5. Case $\delta a_{11} \to \infty$:
$$\partial_{yy}(u_1 + u_2) = 0 \quad \text{and} \quad u_2 - u_1 = 0.$$

5 Numerical results

We present here a series of test cases for isotropic diffusion in all of the three domains $\Omega_1 = (-10, -\delta) \times (-10, 10)$, $\Omega_2 = (\delta, 10) \times (-10, 10)$ and $\Omega_f = (-\delta, \delta) \times (-10, 10)$. The diffusion coefficients are ν in Ω_f and 1 in the domains Ω_1, Ω_2. This means that we consider the model solved on the full domain, which consists of the Laplace equation $\Delta u_j = 0$ in Ω_j, $j = 1, 2, f$, together with the coupling conditions

$$\begin{aligned} u_1(-\delta) = u_f(-\delta) \quad &\text{and} \quad u_2(\delta) = u_f(\delta), \\ \partial_x u_1(-\delta) = \nu \partial_x u_f(-\delta) \quad &\text{and} \quad \partial_x u_2(\delta) = \nu \partial_x u_f(\delta), \end{aligned}$$

and compare the solution to those obtained by the reduced models, which consist of the Laplace equation $\Delta u_j = 0$ in Ω_j, $j = 1, 2$, together with either leading order (CC0) coupling conditions,

$$u_1(-\delta) = u_2(\delta) \quad \text{and} \quad \partial_x u_1(-\delta) = \partial_x u_2(\delta),$$

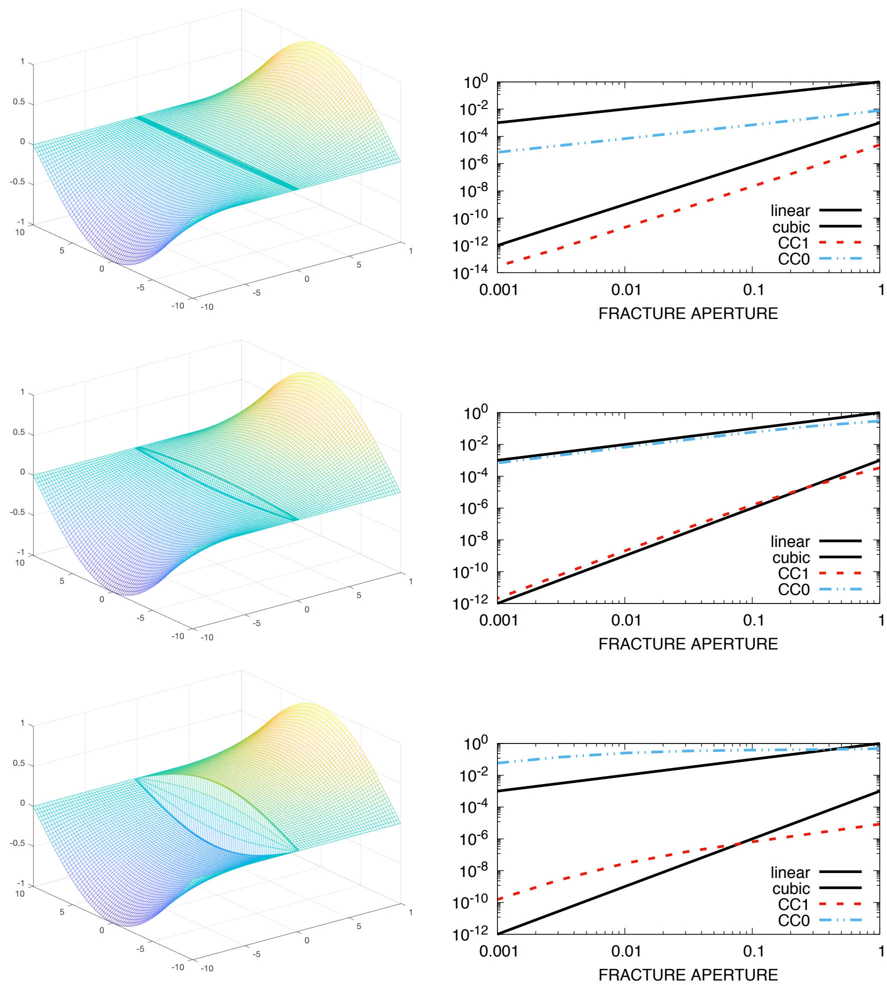

Fig. 2: The reference solution and the L^∞-error for the solutions of the reduced models for $\nu = 10$, $\nu = 0.1$ and $\nu = 0.001$ (from top to bottom). The error is plotted for CC0 and for CC1 coupling conditions.

or coupling conditions containing next-to-leading-order corrections (CC1),

$$\partial_x u_1(-\delta) - \partial_x u_2(\delta) = \delta \nu \partial_{yy}(u_1(-\delta) + u_2(\delta)),$$
$$u_2(\delta) - u_1(-\delta) = \delta \nu^{-1}(\partial_x u_2(\delta) + \partial_x u_1(-\delta)),$$

which have been shown to have an error of $O(\delta^3)$ compared to the exact solution, for diffusion problems with diagonal matrix A. We use homogeneous Dirichlet boundary conditions at $y = \pm 10$ and non-homogeneous Dirichlet boundary conditions with values $\pm \cos(\pi y/20)$ at $x = \pm 10$. From Figure 2, we observe an increase of the pressure jump across the fracture, when increasing the fracture resistivity, as encoded in the coupling conditions. From the error plots, we see that the theoretical order of convergence is reproduced, although we note that, in the case of $\nu = 0.001$, we need to decrease the fracture width quite severely to enter the regime of theoretical order of convergence.

6 Conclusion

We presented a rigorous derivation of coupling conditions for DFM models of very general type, i.e. advection-diffusion-reaction in the fracture and even more general second order PDEs in the surrounding matrix domains. The derivation of coupling conditions relies on a Fourier transform of the physical unknowns in direction tangential to the fracture and, subsequently, on the elimination of the fracture unknowns' Fourier coefficients by performing a continuous Schur complement. Reduced order coupling conditions are then obtained by straightforward truncation of an expansion. We compared the coupling conditions to a commonly used family of (diffusion) models from the literature and obtained correspondence for the coupling conditions truncated after the next-to-leading-order terms. We further derived coupling conditions for the fracture resistivity tending to a constant, to infinity and to zero, and found correspondence to the literature, which contains results for the special case of the Laplace equation only.

References

1. P. Angot, F. Boyer, and F. Hubert. Asymptotic and numerical modelling of flows in fractured porous media. *ESAIM: Mathematical Modelling and Numerical Analysis*, 43(2):239–275, mar 2009.
2. E. Flauraud, F. Nataf, I. Faille, and R. Masson. Domain decomposition for an asymptotic geological fault modeling. *Comptes Rendus Mécanique*, 331(12):849–855, dec 2003.
3. J. Hennicker. *Hybrid dimensional modeling of multi-phase Darcy flows in fractured porous media*. PhD thesis, Université Côte d'Azur, 2017.
4. V. Martin, J. Jaffré, and J. E. Roberts. Modeling fractures and barriers as interfaces for flow in porous media. *SIAM Journal on Scientific Computing*, 26(5):1667–1691, 2005.
5. E. Sánchez-Palencia. Problèmes de perturbations liés aux phénomènes de conduction à travers des couches minces de grande résistivité. *J. Math. Pures et Appl.*, 53(9):251–269, 1974.