A Neumann-Neumann Method for Anisotropic TDNNS Finite Elements in 3D Linear Elasticity

Lukas Maly and Dalibor Lukas

1 Introduction

We are interested in solving a problem of linear elasticity in three dimensions. Let $\Omega \in \mathbb{R}^3$ be a bounded connected domain with the Lipschitz boundary $\partial\Omega$. An elastostatic problem is described by the equilibrium equation (2) and Hooke's law (1), which couples the strain and stress tensors for linear elastic materials. We seek for the displacement vector $\mathbf{u}\colon \Omega \to \mathbb{R}^3$ and the stress field $\underline{\boldsymbol{\sigma}}\colon \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$ subject to volume forces $\mathbf{f}$ and the boundary conditions (3) and (4) $\partial\Omega = \overline{\Gamma_D} \cap \overline{\Gamma_N}$. Therefore, we solve the problem

$$
\begin{aligned}
\underline{\mathbf{C}}^{-1}\,\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\varepsilon}}(\mathbf{u}) &= 0 && \text{in } \Omega, && (1)\\
-\,\mathbf{div}\,\underline{\boldsymbol{\sigma}} &= \mathbf{f} && \text{in } \Omega, && (2)\\
\mathbf{u} &= \mathbf{u}_D && \text{on } \Gamma_D, && (3)\\
\boldsymbol{\sigma}_n &= \mathbf{t}_N && \text{on } \Gamma_N, && (4)
\end{aligned}
$$

where $\mathbf{u}_D$ and $\mathbf{t}_N$ are the prescribed displacement and surface traction, respectively. The tensor $\underline{\boldsymbol{\varepsilon}}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right)$ is a symmetric strain tensor, and by $\underline{\mathbf{C}}^{-1}$ we denote the compliance tensor, which implements Hooke's law for a given Young modulus and Poisson ratio.

Let $\mathbf{n}$ be an outer unit normal vector. Then the normal component v_n and the tangential component $\mathbf{v}_\tau$ of a vector field $\mathbf{v}$ on the boundary are given by

Lukas Maly
IT4Innovations, VSBB – Technical University of Ostrava,
17. listopadu 2172/15, 708 00 Ostrava-Poruba, Czech Republic
e-mail: lukas.maly@vsb.cz

Lukas Maly, Dalibor Lukas
Department of Applied Mathematics, VSB – Technical University of Ostrava,
17. listopadu 2172/15, 708 00 Ostrava-Poruba, Czech Republic
e-mail: lukas.maly@vsb.cz,dalibor.lukas@vsb.cz

$$v_n = \mathbf{v} \cdot \mathbf{n}, \qquad \mathbf{v}_\tau = \mathbf{v} - v_n \mathbf{n},$$

where the dot symbol stands for the inner product between vector fields. The vector-valued normal component $\boldsymbol{\sigma}_n$ of a tensor $\underline{\boldsymbol{\sigma}}$ can be split into a normal-normal component σ_{nn} and a normal-tangential component $\boldsymbol{\sigma}_{n\tau}$ by

$$\boldsymbol{\sigma}_n = \underline{\boldsymbol{\sigma}}\mathbf{n}, \qquad \sigma_{nn} = \boldsymbol{\sigma}_n \cdot \mathbf{n}, \qquad \boldsymbol{\sigma}_{n\tau} = \boldsymbol{\sigma}_n - \sigma_{nn}\mathbf{n}.$$

2 TDNNS Formulation

We want to solve the stated problem on thin plate-type domains, therefore we use the tangential displacement normal-normal stress (TDNNS) formulation as introduced in [13] and published in a series of papers [11, 10, 9, 8]. The authors developed a new mixed method for the Hellinger-Reissner formulation of elasticity, where the displacement $\mathbf{u}$ is sought in the $\mathbf{H}(\mathbf{curl})$ Sobolev space, i.e., continuity of the tangential components of the displacements is preserved. Meanwhile the stresses live in a new Sobolev space $\underline{\mathbf{H}}(\operatorname{div}\mathbf{div})$, which can be approximated with a symmetric stress tensor preserving continuity of the normal part of their normal component.

The TDNNS elements are applicable for nearly incompressible materials and for structurally anisotropic discretization of slim domains. Here we assume that the Ω is a polyhedral Lipschitz domain (possibly a thin layer in one direction). Let $\mathcal{T}_h = \bigcup_{k=1}^m \{T_k\}$, $T_k = T^{\mathbf{x}} \times T^t : T^{\mathbf{x}} \in \mathcal{T}_h^{\mathbf{x}}, T^t \in \mathcal{T}_{h^t}^t$ be a tensor product triangulation of Ω. Since we want to incorporate anisotropic geometrical elements, we need to distinguish sizes of mesh elements in plane- (isotropy-) and thickness- (anisotropy-) directions. We denote them by h and h^t, respectively. Then for the displacements we use the second family of Nédélec space $\mathbf{V}_h$ with a continuous tangential component, and for the stresses, we use a normal-normal continuous space $\underline{\boldsymbol{\Sigma}}_h$. Correct definitions of the appropriate tensor product finite element spaces require more technical details, therefore we leave the spaces undefined here, and refer the interested reader to [10, Chapter 6] or [11, 6] for their correct definitions.

The discrete mixed TDNNS formulation of the original problem (1)–(4) reads as: find $\mathbf{u} \in \mathbf{V}_h$ and $\underline{\boldsymbol{\sigma}} \in \underline{\boldsymbol{\Sigma}}_h$ such that

$$\int_\Omega \left(\underline{\mathbf{C}}^{-1}\,\underline{\boldsymbol{\sigma}}\right) : \underline{\boldsymbol{\tau}}\,\mathrm{d}\mathbf{x} + \langle \mathbf{div}\,\underline{\boldsymbol{\tau}}, \mathbf{u}\rangle_V = \int_{\Gamma_D} u_{D,n}\tau_{nn}\,\mathrm{d}\mathbf{s} \qquad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}_h, \quad (5)$$

$$\langle \mathbf{div}\,\underline{\boldsymbol{\sigma}}, \mathbf{v}\rangle_V = -\int_\Omega \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_{N,\tau} \cdot \mathbf{v}_{\boldsymbol{\tau}}\,\mathrm{d}\mathbf{s} \quad \forall \mathbf{v} \in \mathbf{V}_h, \quad (6)$$

with duality pairing that can be evaluated by element-wise volume and boundary integrals

$$\langle \mathbf{div}\, \underline{\boldsymbol{\tau}}, \mathbf{v} \rangle_V = \sum_{T \in \mathcal{T}_h} \left[-\int_T \underline{\boldsymbol{\tau}} : \underline{\boldsymbol{\varepsilon}}\, \mathrm{d}\mathbf{x} + \int_{\partial T} \tau_{nn} v_n \, \mathrm{d}\mathbf{s} \right]. \tag{7}$$

We can identify the volume integral in (5) with matrix $\underline{\mathbf{A}}$, the duality product in (5) and (6) defined by (7) with matrix $\underline{\mathbf{B}}$, and the right hand sides in (5) and (6) with vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively. Similarly, the sought finite element solutions can be identified with vectors $\mathbf{S} \leftrightarrow \underline{\boldsymbol{\sigma}}$ and $\mathbf{U} \leftrightarrow \mathbf{u}$. Then we can write a linear system for the discrete mixed TDNNS formulation in the following form,

$$\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{B}}^\top \\ \underline{\mathbf{B}} & \underline{\mathbf{0}} \end{bmatrix} \begin{bmatrix} \mathbf{S} \\ \mathbf{U} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix}. \tag{8}$$

2.1 Hybridization

The system matrix in (8) is symmetric but indefinite, as is typical when mixed formulations are considered. So far, the required continuity of tangential displacement and normal-normal stress is enforced directly by the conforming choice of the solution spaces $\mathbf{V}_h$ and $\underline{\boldsymbol{\Sigma}}_h$. We can break the continuity of the stress space and re-enforce it via Langrangian multipliers. The Lagrangian multipliers will exist in the facet space $\overline{V}_n$ and will correspond to normal displacements on element interfaces. Therefore, we shall denote them as $\overline{u}_n$. To be equivalent with the normal-normal continuity condition for $\underline{\boldsymbol{\sigma}}$, together with the traction condition $\sigma_{nn}|_{\Gamma_N} = 0$, functions in $\overline{V}_n$ have to fulfill the following equation,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \tau_{nn} \overline{v}_n \, \mathrm{d}\mathbf{s} = 0 \qquad \forall v_n \in \overline{V}_n. \tag{9}$$

This leads to an enlarged system with discontinuous stress finite elements,

$$\begin{bmatrix} \underline{\widetilde{\mathbf{A}}} & \underline{\mathbf{B}}_1^\top & \underline{\mathbf{B}}_2^\top \\ \underline{\mathbf{B}}_1 & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{B}}_2 & \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{S}} \\ \mathbf{U} \\ \overline{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{F}}_1 \\ \mathbf{F}_2 \\ \mathbf{0} \end{bmatrix}, \tag{10}$$

where all coupling degrees of freedom are connected to displacement quantities. The matrix $\underline{\widetilde{\mathbf{A}}}$ is block-diagonal with each block corresponding to one element. Such a matrix can be inverted in optimal complexity and thus, we can eliminate all stress degrees of freedom from the system by static condensation,

$$\begin{bmatrix} \underline{\mathbf{B}}_1 \underline{\widetilde{\mathbf{A}}}^{-1} \underline{\widetilde{\mathbf{B}}}^\top & \underline{\mathbf{B}}_1 \underline{\widetilde{\mathbf{A}}}^{-1} \underline{\mathbf{B}}_2^\top \\ \underline{\mathbf{B}}_2 \underline{\widetilde{\mathbf{A}}}^{-1} \underline{\mathbf{B}}_1^\top & \underline{\mathbf{B}}_2 \underline{\widetilde{\mathbf{A}}}^{-1} \underline{\mathbf{B}}_2^\top \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \overline{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} \underline{\widetilde{\mathbf{B}}}_1 \underline{\widetilde{\mathbf{A}}}^{-1} \widetilde{\mathbf{F}}_1 - \mathbf{F}_2 \\ \underline{\mathbf{B}}_2 \underline{\widetilde{\mathbf{A}}}^{-1} \widetilde{\mathbf{F}}_1 \end{bmatrix}. \tag{11}$$

The system matrix in (11) is symmetric and positive definite. The Langrange functions are identified with the vector $\overline{\mathbf{U}}$. We will abbreviate the system using the notation

$$\underline{\mathbf{K}}\widehat{\mathbf{U}} = \widehat{\mathbf{F}}. \tag{12}$$

3 Domain Decomposition

Finite elements in linear elasticity have typically relative high number of degrees of freedom. This is even worse for mixed formulations. Although the stresses were hybridized from the system, there is still significant number of degrees of freedom for displacements. The lowest order anisotropic prismatic and hexahedral finite elements have 60 and 84 degrees of freedom per element, respectively.

This aspect clearly brings a limitation in the sense of problem size. For a large number of mesh elements, the corresponding linear system becomes too large to be solved using direct solvers. Therefore, we resort to an iterative solver and substructuring domain decomposition method as a preconditioner technique [2, 3, 12, 14]. We start our research of preconditioners of mixed TDNNS elements with one which is straightforward and relatively simple to implement, to get a preliminary overview, namely the Neumann-Neumann method described in Section 3.1.

We partition the original domain Ω into N non-overlapping subdomains $\Omega^{(i)}$:

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}^{(i)}, \qquad \Omega^{(i)} \cap \Omega^{(j)} = \emptyset \quad \text{for } i \neq j, \qquad \Gamma := \bigcup_{i=1}^{N} \partial\Omega^{(i)},$$

such that each subdomain is a union of elements of the global mesh. Using the index (i) we indicate an association to subdomain $\Omega^{(i)}$. By the union of individual subdomain boundaries without the global boundary of Ω we define the *interface* Γ.

The degrees of freedom can be subdivided into two groups; coupling, those being associated with the interface (shared with at least one of the other subdomains, or being on the Dirichlet or Neumann boundary), and interior, which are not coupling. In our setting, the coupling degrees of freedom are associated with an edge or face. All the coupling degrees of freedom in the system are denoted by the lower index C while the interior ones are denoted by the lower index I. The system (12) can then be reordered into the following form

$$\begin{bmatrix} \underline{\mathbf{K}}_{II} & \underline{\mathbf{K}}_{IC} \\ \underline{\mathbf{K}}_{CI} & \underline{\mathbf{K}}_{CC} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{U}}_I \\ \widehat{\mathbf{U}}_C \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{F}}_I \\ \widehat{\mathbf{F}}_C \end{bmatrix}. \tag{13}$$

The interior degrees of freedom are related only to the individual subdomains and thus can easily be eliminated from the system using the same idea we used in the hybridization of stresses. This procedure leads to a classical method referred to in the literature as primal domain decomposition, the Schur complement, particular-solution, and the three-step approach [7, 5].

The main idea of using the described domain decomposition procedure resides in preconditioning of the global Schur complement matrix in such a way that the resulting method can be effectively run in parallel. In future, we plan to apply some of the known preconditioner techniques used in [2, 3, 14].

3.1 Neumann-Neumann preconditioner

One of the basic preconditioners of the Schur complement system is the Neumann-Neumann method, which is derived from its local additive construction. Since the Schur complement can be assembled subdomain-wise using local Schur complements multiplied with the restriction operator

$$\underline{\mathbf{S}} = \sum_i \underline{\mathbf{R}}^{(i)\top} \underline{\mathbf{S}}^{(i)} \underline{\mathbf{R}}^{(i)}, \qquad \underline{\mathbf{S}}^{(i)} = \underline{\mathbf{K}}_{CC}^{(i)} - \underline{\mathbf{K}}_{CI}^{(i)} \left(\underline{\mathbf{K}}_{II}^{(i)}\right)^{-1} \underline{\mathbf{K}}_{IC}^{(i)}, \tag{14}$$

a simple idea for how to obtain an approximation of $\underline{\mathbf{S}}$ is to also assemble individual inverses subdomain-wise,

$$\underline{\mathbf{S}}^{-1} \approx \sum_i \underline{\mathbf{D}}^{(i)} \underline{\mathbf{R}}^{(i)\top} \left(\underline{\mathbf{S}}^{(i)}\right)^{-1} \underline{\mathbf{R}}^{(i)} \underline{\mathbf{D}}^{(i)} =: \underline{\mathbf{M}}^{-1}. \tag{15}$$

Matrix $\underline{\mathbf{D}}^{(i)}$ in the formula is a diagonal matrix, whose entry $\underline{\mathbf{D}}_{kk}^{(i)}$ is computed as a reciprocal of a number of subdomains that share the k-th degree of freedom.

It is important to note that the Schur complement $\underline{\mathbf{S}}^{(i)}$ on a subdomain has the same null space dimension as the stiffness matrix $\underline{\mathbf{K}}^{(i)}$. Therefore, local problems on floating subdomains have to be treated with care, since the matrices are singular. Then, the application of the preconditioner corresponds to solving a problem with pure Neumann boundary conditions. For more details on Neumann-Neumann preconditioning, see [1, 4]. In the numerical experiment presented in Section 4, none of the subdomains is floating due to the Dirichlet boundary condition and the two-dimensional decomposition.

4 Numerical Experiments

We present here a simple problem of linear elasticity in three dimensions. Our domain $\Omega := (0;1) \times (0;1) \times (0;L_z)$ is represented by a plate with varying thickness in the z-direction. The plate is rigidly fixed on the bottom and top side. As we described above, all volume forces are reflected on the right hand side in (12) and thus they do not play any role in the study of the system matrix properties. We set Young's modulus E to be 1, and Poisson's ration ν to be 0.285. A diagram of the model problem is depicted in Fig. 1 on the left. On the right, we present a simple two-dimensional $(N \times N)$ domain decomposition of the plate geometry.

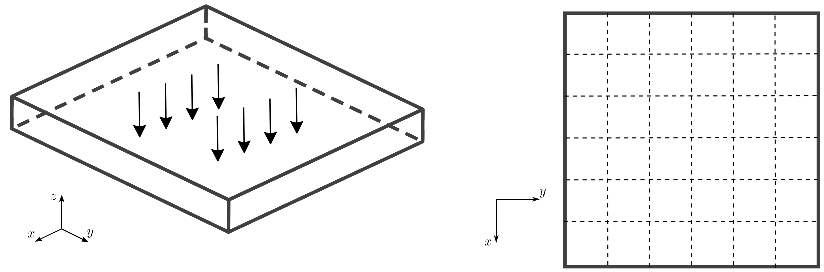

Fig. 1: A diagram of the model problem on plate geometry of thickness L_z and its two-dimensional decomposition into $N \times N$ subdomains in the xy-plane. Dashed lines on the right represent the interface.

We started the development of the scalable parallel algorithm with this simple $N \times N$ domain decomposition to study and fully understand the behavior of the presented systems within the TDNNS formulation of linear elasticity. We construct the Schur complement matrix and its Neumann-Neumann preconditioner as described above. In Table 1 and Table 2, we present condition numbers with respect to discretization size h. To discretize the domain Ω ,we use anisotropic hexahedral elements with only one element in the thickness direction, i.e. $h^t = L_z$. The number of subdomains varies from 2×2 to 8×8.

Presented numerical experiments were implemented in Matlab (version 8.5.0.19-7613 (R2015a)). The implementation uses sparse matrices. Condition numbers presented in Tables 1 and 2 were computed using the built-in *condest* function, and inverse matrices were assembled explicitly. Computations were performed on a classical portable laptop, the biggest problem consisted of 4,096 elements that in case of 2×2 decomposition translate into more than 20,000 inner, and more than 3,000 coupling degrees of freedom.

Table 1: Condition numbers of the Schur complement and preconditioned Schur complement for a plate with thickness $L_z = 0.25$.

$N \times N$	2×2		4×4		8×8	
H/h	$\kappa(\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{M}}^{-1}\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{M}}^{-1}\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{M}}^{-1}\underline{\mathbf{S}})$
1	$1.14 \cdot 10^4$	$6.16 \cdot 10^2$	$3.05 \cdot 10^3$	$1.61 \cdot 10^3$	$1.23 \cdot 10^3$	$1.56 \cdot 10^3$
2	$1.42 \cdot 10^3$	$8.72 \cdot 10^1$	$8.14 \cdot 10^2$	$2.54 \cdot 10^2$	$2.81 \cdot 10^3$	$4.78 \cdot 10^3$
4	$5.21 \cdot 10^2$	$8.50 \cdot 10^1$	$1.48 \cdot 10^3$	$1.00 \cdot 10^3$	$1.38 \cdot 10^4$	$3.03 \cdot 10^4$
8	$1.17 \cdot 10^3$	$5.63 \cdot 10^2$	$7.22 \cdot 10^3$	$8.31 \cdot 10^3$	$8.16 \cdot 10^4$	$2.72 \cdot 10^5$
16	$4.92 \cdot 10^3$	$3.81 \cdot 10^3$	$4.59 \cdot 10^4$	$8.22 \cdot 10^4$		
32	$3.23 \cdot 10^4$	$3.43 \cdot 10^4$				

As presented in Table 1, the preconditioner is not working very well when the thickness is 0.25. The conditioning stays more or less the same except in the case of 4 subdomains and a low H/h ratio. The situation significantly differs when the thickness is 0.01, as in Table 2. In this case, the conditioning is decreased by two orders for all decompositions regardless of the H/h ratio.

Table 2: Condition numbers of the Schur complement and preconditioned Schur complement for a plate with thickness $L_z = 0.01$.

$N \times N$	2×2		4×4		8×8	
H/h	$\kappa(\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{M}}^{-1}\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{M}}^{-1}\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{S}})$	$\kappa(\underline{\mathbf{M}}^{-1}\underline{\mathbf{S}})$
1	$2.19 \cdot 10^9$	$5.58 \cdot 10^7$	$5.36 \cdot 10^8$	$6.32 \cdot 10^7$	$1.36 \cdot 10^8$	$1.59 \cdot 10^7$
2	$3.19 \cdot 10^8$	$3.53 \cdot 10^6$	$8.14 \cdot 10^7$	$2.24 \cdot 10^6$	$2.10 \cdot 10^7$	$6.13 \cdot 10^5$
4	$6.76 \cdot 10^7$	$7.75 \cdot 10^5$	$1.80 \cdot 10^7$	$2.23 \cdot 10^5$	$4.99 \cdot 10^6$	$8.32 \cdot 10^4$
8	$1.78 \cdot 10^7$	$1.97 \cdot 10^5$	$4.60 \cdot 10^6$	$5.82 \cdot 10^4$	$1.57 \cdot 10^6$	$1.47 \cdot 10^4$
16	$4.60 \cdot 10^6$	$5.05 \cdot 10^4$	$1.26 \cdot 10^6$	$1.29 \cdot 10^4$		
32	$1.47 \cdot 10^6$	$1.29 \cdot 10^4$				

It is well known that the efficiency of the local Neumann-Neumann preconditioner deteriorates with a growing number of subdomains, therefore we expect the same trend for a continuation of Tables 1 and 2. In order to improve the presented preconditioner, one needs an additional coarse problem, e.g. by projecting (deflating) against certain modes (yet to be found) or by using sophisticated primal constraints (yet to be found) in a BDDC framework.

5 Conclusion and outlook

We have briefly introduced the TDNNS formulation for a problem of linear elasticity in 3-dimensions, which leads to large and ill-conditioned systems. Based on our experience, we apply a primal domain decomposition procedure to get an initial overview. We try to follow similar ideas as presented in [9], where the authors introduced FETI preconditioned methods for TDNNS elements in 2-dimensions. Our $N \times N$ domain decomposition of thin plate geometry demonstrates the limited efficiency of the Neumann-Neumann preconditioner, and moves us to further research. We aim to end up with a parallel and scalable method, therefore, next we plan to implement some of the modern methods that achieve a bound for the condition number of order $C(1 + \log(H/h))^2$, as discussed in [14].

Acknowledgements The research was supported by The Ministry of Education, Youth and Sports from the National Programme of Sustainability (NPS II) project 'IT4Innovations excellence in science – LQ1602' and by the IT4Innovations infrastructure which is supported from the Large Infrastructures for Research, Experimental Development and Innovations project 'IT4Innovations National Supercomputing Center – LM2015070'.

References

1. Bourgat, J.F., Glowinski, R., Le Tallec, P., Vidrascu, M.: Variational formulation and algorithm for trace operation in domain decomposition calculations. Research Report RR-0804, INRIA (1988). URL https://hal.inria.fr/inria-00075747
2. Bramble, J.H., Pasciak, J.E., Schatz, A.H.: The construction of preconditioners for elliptic problems by substructuring, I. Math. Comp. **47**(175), 103–134 (1986)
3. Bramble, J.H., Pasciak, J.E., Schatz, A.H.: The construction of preconditioners for elliptic problems by substructuring, II. Math. Comp. **49**, 1–16 (1987)
4. De Roeck, Y.H., Le Tallec, P.: Analysis and test of a local domain decomposition preconditioner. In: R. Glowinski, Y. Kuznetsov, G. Meurant, J. Périaux, O. Widlund (eds.) Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, pp. 112–128. SIAM, Philadelphia, PA (1991)
5. Lukas, D., Bouchala, J., Vodstrcil, P., Maly, L.: 2-dimensional primal domain decomposition theory in detail. Application of Mathematics **60**(3), 265–283 (2015)
6. Lukas, D., Schöberl, J., Maly, L.: Dispersion analysis of displacement-based and TDNNS mixed finite elements for thin-walled elastodynamics. – pp. 1–13 (2019). Submitted.
7. Maly, L.: Primal domain decomposition methods and boundary elements. Diploma thesis, VSB-Technical University of Ostrava (2013)
8. Meindlhumer, M., Pechstein, A.: 3d mixed finite elements for curved, flat piezoelectric structures. International Journal of Smart and Nano Materials pp. 1–19 (2018). DOI:10.1080/19475411.2018.1556186. URL https://doi.org/10.1080/19475411.2018.1556186
9. Pechstein, A., Pechstein, C.: A FETI method for a TDNNS discretization of plane elasticity. RICAM-Report **11**, 30 (2013). URL http://www.sfb013.uni-linz.ac.at/index.php?id=reportshttp://www.ricam.oeaw.ac.at/publications/list/http://www.numa.uni-linz.ac.at/Publications/List/
10. Pechstein, A., Schöberl, J.: Tangential-displacement and normal-normal-stress continuous mixed finite elements for elasticity. Mathematical Models and Methods in Applied Sciences **21**(08), 1761–1782 (2011). DOI:10.1142/S0218202511005568. URL http://www.worldscientific.com/doi/abs/10.1142/S0218202511005568
11. Pechstein, A., Schöberl, J.: Anisotropic mixed finite elements for elasticity. International Journal for Numerical Methods in Engineering **90**(2), 196–217 (2012). DOI:10.1002/nme.3319. URL http://dx.doi.org/10.1002/nme.3319
12. Saad, Y.: Iterative Methods for Sparse Linear Systems: Second Edition. Society for Industrial and Applied Mathematics (2003). URL https://books.google.cz/books?id=ZdLeBlqYeF8C
13. Sinwel, A.: A new family of mixed finite elements for elasticity. PhD thesis, Johannes Kepler University, Institute of Computational Mathematics (2008). URL http://www.numa.uni-linz.ac.at/Teaching/PhD/Finished/sinwel
14. Toselli, A., Widlund, O.: Domain Decomposition Methods - Algorithms and Theory, *Springer Series in Computational Mathematics*, vol. 34. Springer (2004)