Localization of Nonlinearities and Recycling in Dual Domain Decomposition

Andreas S. Seibold, Michael C. Leistner, and Daniel J. Rixen

1 Introduction

Newton-Krylov domain decomposition methods are well suited for solving nonlinear structural mechanics problems in parallel, especially due to their scalability properties. A Newton-Raphson method in combination with a dual domain decomposition technique, such as a FETI method, takes advantage of the quadratic convergence behaviour of the Newton-Raphson algorithm and the scalabality and high parallelizability of FETI methods. In order to reduce expensive communication between computing cores and thus Newton-iterations, a localization step for nonlinearities was proposed for FETI2, FETI-DP and BDDC solvers [12, 8]. Further methodologies on nonlinear preconditioning of a global Newton method for cases with high local nonlinearities can be found in literature as well [2]. To further improve the efficiency of FETI2-solvers methods have been developed, such as adaptive multipreconditioning [15], derived from simultaneous FETI [7], and reuse techniques of Krylov subspaces [6]. These reuse techniques are rather memory-intensive. More efficient recycling strategies based on Ritz-vectors were therefore developed [9]. In this contribution, we combine those recycling methods with localization of nonlinearities and apply them to static and dynamic structural mechanics problems. We start with the introduction of the model problems and the solution strategy in Sec. 2.1. Then we introduce the localization technique in Sec. 2.2, the adaptive multipreconditioning in Sec. 3 and the used recycling methods in Sec. 4. Finally, we present our numerical results in Sec. 5.

Andreas S. Seibold, e-mail: andreas.seibold@tum.de
Michael C. Leistner, e-mail: m.leistner@tum.de
Daniel J. Rixen, e-mail: rixen@tum.de
Technical University of Munich, Department of Mechanical Engineering, Boltzmannstr. 15 D - 85748 Garching

2 Localized nonlinearities in dual domain decomposition

2.1 Modelproblem and nonlinear solution strategy

We consider a static structural mechanics problem with nonlinear material behavior, discretized with Finite Elements and decomposed into N_s substructures s of the form

$$\mathbf{f}_{int}^{(s)}(\mathbf{u}^{(s)}) + \mathbf{B}^{(s)^T}\boldsymbol{\lambda} - \mathbf{f}_{ext}^{(s)} = \mathbf{0}, \qquad \sum_{s=1}^{N_s} \mathbf{B}^{(s)}\mathbf{u}^{(s)} = \mathbf{0}, \tag{1}$$

where $\mathbf{u}$ describes the displacements of the elastic structure and the primary solution of the problem. The substructures are coupled with Lagrange-multipliers $\boldsymbol{\lambda}$ imposed on the boundary of each substructure by a signed Boolean matrix $\mathbf{B}$ [5]. Accelerations $\ddot{\mathbf{u}}$ and velocities $\dot{\mathbf{u}}$ are added with the related mass $\mathbf{M}$ and damping $\mathbf{D}$ for a structural mechanics problem, which results in the dynamic nonlinear system of equations

$$\mathbf{M}^{(s)}\ddot{\mathbf{u}}^{(s)} + \mathbf{D}^{(s)}\dot{\mathbf{u}}^{(s)} + \mathbf{f}_{int}^{(s)}(\mathbf{u}^{(s)}) + \mathbf{B}^{(s)^T}\boldsymbol{\lambda} - \mathbf{f}_{ext}^{(s)} = \mathbf{0}, \qquad \sum_{s=1}^{N_s} \mathbf{B}^{(s)}\ddot{\mathbf{u}}^{(s)} = \mathbf{0}.$$

This dynamic system can now be integrated by a suitable time-integration scheme and handled as subsequently described for the static system. For our experiments we use a generalized-α scheme [3]. These systems are solved by a Newton-Raphson scheme and the resulting linearized system by a FETI-method [5] at each time or load step. By linearizing the system of equations (1) at Newton-iteration n and resolving it for the incremental displacements, we get the tangent interface problem

$$\begin{bmatrix} \mathbf{F}_n & -\mathbf{G}_n \\ \mathbf{G}_n^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\lambda} \\ \delta\boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_n \end{bmatrix} - \begin{bmatrix} \mathbf{d}_n \\ \mathbf{G}_n^T\boldsymbol{\lambda}_n \end{bmatrix} \qquad \mathbf{F}_n = \sum_{s=1}^{N_s} \mathbf{B}^{(s)}\mathbf{K}_{T,n}^{(s)^+}\mathbf{B}^{(s)^T}$$

$$\mathbf{G}_n = \left[\mathbf{B}^{(1)}\mathbf{R}_n^{(1)} \ \dots \ \mathbf{B}^{(N_s)}\mathbf{R}_n^{(N_s)}\right] \qquad \mathbf{e}_n = \left[\mathbf{R}_n^{(1)^T}\mathbf{f}_{ext}^{(1)} \ \dots \ \mathbf{R}_n^{(N_s)^T}\mathbf{f}_{ext}^{(N_s)}\right]^T$$

$$\mathbf{d}_n = -\sum_{s=1}^{N_s} \mathbf{B}^{(s)}\mathbf{K}_{T,n}^{(s)^+}\left(\mathbf{f}_{int}(\mathbf{u}_n^{(s)}) - \mathbf{f}_{ext} + \mathbf{B}^{(s)^T}\boldsymbol{\lambda}_n\right) + \mathbf{B}^{(s)}\mathbf{u}_n^{(s)}$$

and the local linear solves for the incremental displacements

$$\delta\mathbf{u}^{(s)} = -\mathbf{K}_{T,n}^{(s)^+}\mathbf{B}^{(s)^T}\delta\boldsymbol{\lambda} - \mathbf{K}_{T,n}^{(s)^+}\left(\mathbf{f}_{int}^{(s)}(\mathbf{u}_n^{(s)}) - \mathbf{f}_{ext}^{(s)} + \mathbf{B}^{(s)^T}\boldsymbol{\lambda}_n\right)\Big) + \mathbf{R}_n^{(s)}\delta\boldsymbol{\alpha}$$

Here, $\mathbf{K}_{T,n}^{(s)}$ is the tangent stiffness, $\mathbf{R}_n^{(s)}$ its null space and $\mathbf{F}_n$ the tangent interface operator with the superscript + denoting a pseudoinverse. The null space $\mathbf{R}_n^{(s)}$ and its corresponding additional unknowns $\delta\boldsymbol{\alpha}$ can be seen as rigid body modes of floating substructures and are needed for solvability [5]. This isn't needed in structural

dynamics due to the additional mass matrix. The interface problem is then solved by a preconditioned conjugate gradient.

2.2 Localization of nonlinearities

In order to reduce Newton-iterations and hence iterations of the conjugate gradient method, one can solve local nonlinear problems as some kind of preconditioning step for the global linear solution step within the Newton algorithm [12, 11, 8]. In case of a FETI2-solver this is achieved by solving local nonlinear Neumann-problems while keeping the Lagrange-multipliers constant, whereas for FETI-DP and BDDC far more options are available [8]. Thus, the displacements

$$\delta\mathbf{u}^{(s)} = -\mathbf{K}^{(s)^+}_{T,ng,nl}\left(\mathbf{f}^{(s)}_{int}(\mathbf{u}^{(s)}_{ng,nl}) - \mathbf{f}^{(s)}_{ext} + \mathbf{B}^{(s)^T}\boldsymbol{\lambda}_{ng-1}\right), \quad \mathbf{u}^{(s)}_{ng,nl+1} = \mathbf{u}^{(s)}_{ng,nl} + \delta\mathbf{u}^{(s)}$$

are calculated within local Newton iterations nl after using the displacements of the previous global Newton-iteration $ng-1$ as an initialization. To ensure local solvability, the Lagrange-multipliers have to be initialized with the natural coarse grid [12, 11]

$$\mathbf{G}_0^T\boldsymbol{\lambda}_0 = \mathbf{e}_0 \qquad \boldsymbol{\lambda}_0 = \mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{G}_0\right)^{-1}\mathbf{e}_0 \tag{2}$$

3 Adaptive Multipreconditioning

A preconditioner $\mathbf{H} = \sum_{s=1}^{N_s}\mathbf{B}^{(s)}\mathbf{S}^{(s)}\mathbf{B}^{(s)}$ is commonly used for an efficient solution of the interface problem, here a Dirichlet-preconditioner with the Schur-complement $\mathbf{S}$ [13]. Due to the summation of the local preconditioners, some local information gets lost. Hence, multipreconditioning, also known as *simultaneous FETI* (S-FETI) [7], has been proposed using separated preconditioners $\mathbf{H}^{(s)}$ leading to independent search directions $\mathbf{z}_i^{(s)} = \mathbf{H}^{(s)}\mathbf{r}_i$ in each FETI-iteration i for the residual $\mathbf{r}$. To avoid large search spaces, a τ-criterion has been introduced to modify the S-FETI to an *adaptive multipreconditioned FETI* (AMP-FETI) method [16, 9]. The τ-criterion controls which substructures are chosen for multipreconditioning. To this end the expression

$$\Theta_i^{(s)} = \frac{\boldsymbol{\gamma}_i^T\mathbf{W}_i^T\mathbf{F}^{(s)}\mathbf{W}_i\boldsymbol{\gamma}_i}{\mathbf{r}_{i+1}^T\mathbf{H}^{(s)}\mathbf{r}_{i+1}}, \qquad \mathbf{W}_i = \mathbf{P}\mathbf{Z}_i$$

is used with $\boldsymbol{\gamma}_i$ being step-lengths from the CG iteration i and the natural coarse-grid projector $\mathbf{P}$. Only the substructures that fulfill the criterion $\Theta_i^{(s)} < \tau$ are chosen. The parameter τ can be set by the user and $\tau = 0.1$ leads to robust behavior in most cases and has been used in this paper [15, 1]. The search space is constructed with such $J = (j_1, j_2, \dots)$ chosen substructures as

$$\mathbf{Z}_i = \left[\sum_{k \notin J} \mathbf{z}_i^{(k)} \mid \mathbf{z}_i^{(j_1)} \mid \mathbf{z}_i^{(j_2)} \mid \ldots \right]$$

4 Recycling methods for dual solutions

In order to further increase the FETI-solver's efficiency and render it scalable, we introduce a deflation or coarse space $\mathbf{C}$ for the search directions $\mathbf{W}_i$, which leads to a two-level FETI (FETI2) solver [4]. A coarse-problem is solved during the initialization and iterations of the FETI-solver. The remaining search space has to be $\mathbf{F}$-conjugate, which is ensured by the projector

$$\mathbf{P}_C = \mathbf{I} - \mathbf{C}(\mathbf{C}^T \mathbf{F} \mathbf{C})^{-1} \mathbf{C}^T \mathbf{F}.$$

In the so-called *total reuse of Krylov subspaces* (TRKS), proposed in [6], all the previous solutions are reused to build the coarse grid

$$\mathbf{C}_{ng} = \left[\mathbf{C}_{ng-1} \ \mathbf{w}_{ng,i=1} \ \ldots \ \mathbf{w}_{ng,i=i_{end}} . \right]$$

In order to improve memory-efficiency, people have investigated the convergence behavior of a preconditioned conjugate gradient algorithm. This is mainly governed by the eigenspectrum of the preconditioned operator $\mathbf{HF}$. High, well-separated eigenvalues might slow down convergence according to studies in [14]. These high eigenvalues are usually captured during the first few iterations of the FETI-solver. Hence, by first solving the eigenvalue-problem

$$\mathbf{S}^{(s)} \mathbf{y}^{(s)} = \Phi^{(s)} \mathbf{B}^{(s)^T} \mathbf{H} \mathbf{B}^{(s)} \mathbf{y}^{(s)}, \tag{3}$$

called *generalized eigenvalues in the overlaps* (GenEO), the high eigenmodes are precomputed separately [16]. Here $\Phi^{(s)}$ are the eigenvalues and $\mathbf{y}^{(s)}$ the corresponding eigenvectors. To reduce the high initial cost of the Schur-complements, a local Ritz Ansatz has been applied in [9], approximating the GenEO eigenvectors and resulting in a smaller eigenproblem. The Ritz space of substructure s is then constructed as

$$\mathbf{V}^{(s)} = \mathbf{S}^{(s)^{-1}} \mathbf{B}^{(s)^T} \mathbf{V}_W^{(s)}, \qquad \mathbf{V}_W^{(s)} = \left[\mathbf{W}_0 \boldsymbol{\gamma}_0 \ \ldots \ \mathbf{W}_{n^s-1} \boldsymbol{\gamma}_{n^s-1} \right], \quad n^s \leq i_{end},$$

where the solution space of the first n^s iterations is considered and n_s limits the Ritz space size. With such a Ritz space follows the approximation of (3)

$$\mathbf{V}^{(s)^T} \mathbf{S}^{(s)} \mathbf{V}^{(s)} \mathbf{q}^{(s)} = \Phi^{(s)} \mathbf{V}^{(s)^T} \mathbf{B}^{(s)^T} \mathbf{H} \mathbf{B}^{(s)} \mathbf{V}^{(s)} \mathbf{q}^{(s)},$$

which can be rewritten as

$$\mathbf{V}_W^{(s)^T} \mathbf{F}^{(s)} \mathbf{V}_W^{(s)} \mathbf{q}^{(s)} = \Phi^{(s)} \mathbf{V}_W^{(s)^T} \mathbf{F}^{(s)} \mathbf{H} \mathbf{F}^{(s)} \mathbf{V}_W^{(s)} \mathbf{q}^{(s)}, \quad \mathbf{F}^{(s)} = \mathbf{B}^{(s)} \mathbf{S}^{(s)^{-1}} \mathbf{B}^{(s)^T}.$$

The resulting coarse space with the first k^s local Ritz vectors is

$$\mathbf{C}^{(s)} = \left[\mathbf{HF}^{(s)}\mathbf{V}_W^{(s)}\mathbf{q}_1^{(s)} \ \ldots \ \mathbf{HF}^{(s)}\mathbf{V}_W^{(s)}\mathbf{q}_{k^s}^{(s)},\right]$$

where k^s has to fulfill $k^s \leq n^s$. This method is subsequently called *local Ritz* (LRitz) approach. It may be even reasonable to build the coarse space directly out of Ritz spaces only, without solving an eigenproblem [10]

$$\mathbf{C} = \left[\mathbf{HF}^{(1)}\mathbf{V}_W^{(1)} \ \ldots \ \mathbf{HF}^{(N_s)}\mathbf{V}_W^{(N_s)}.\right]$$

This method is referred to as *local Ritz direct* (LRitzDir) below.

5 Numerical results

5.1 Recycling methods applied to static mechanical problems

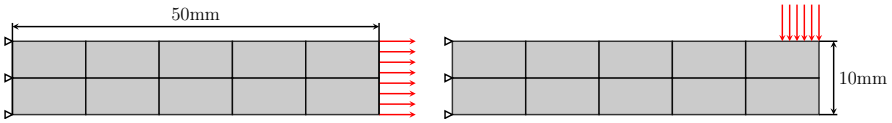

Fig. 1: Left clamped cantilever beam partitioned to 10 rectangular substructures under pull load (left) and bending load (right). Mooney-Rivlin-material (Invariant-parameters: $A_{10} = 0.4N/mm^2$, $A_{01} = 0.1N/mm^2$, $K = 1 \cdot 10^2 N/mm^2$); pull-load: $5N$, bending-load: $1.5 \cdot 10^{-3} N$

Table 1: FETI-iterations cumulated over Newton-iterations and loadsteps and normalized to the NLF-method without recycling. (NLF: classic nonlinear FETI, LoNo: FETI with localized nonlinearities) Load case: pull, 10 loadsteps; Absolute cumulated number of iterations for NLF None: 224

NL-Method	NLF	NLF	LoNo	NLF	NLF	LoNo	LoNo	LoNo
Recycling	None	plReuse	None	LRitzDir	LRitz	plReuse	LRitz	LRitzDir
rel. Iter	1	0.75	0.7366	0.6964	0.5804	0.5268	0.4955	0.4420

We apply the methods introduced above to a homogeneous, nonlinear cantilever beam (Mooney-Rivlin material model and geometrical nonlinearity without damping) under static pull and static bending load and rectangular substructuring, as shown in Fig. 1. The cumulated numbers of FETI-iterations are normalized to the classic nonlinear FETI method (NLF) without recycling in Table 1 since that is the reference we want to compare the performance gain to. The TRKS approach is renamed *plain Reuse* (plReuse) as we no longer have Krylov-subspaces due to multipreconditioning [9]. The coarse spaces are limited to a fixed global size to get compareable results. The combination of localizations and LRitzDir resulted in a reduction of global iterations by 55%. Hence, localizations combine well with recycling methods. The LRitzDir method in particular performs better with localizations than in combination

with the classic nonlinear FETI. The LRitz approach suffers from a slower build up

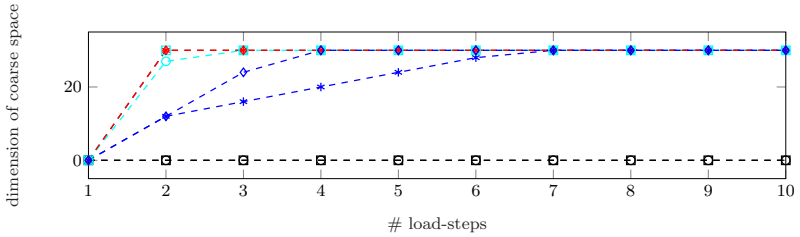

Fig. 2: Coarse grid dimension over load steps in static pull case (Colours are the same as in Fig. 3).

Table 2: Over Newton-iterations and load steps cumulated numbers of FETI-iterations normalized to the NLF-method without recycling. Load case: bending; Absolute cumulated number of iterations for NLF None: 519

NL-Method	LoNo	NLF	LoNo	NLF	LoNo	NLF	NLF	LoNo
Recycling	None	None	plReuse	LRitzDir	LRitz	LRitz	plReuse	LRitzDir
rel. Iter	1.0617	1	0.9422	0.8112	0.7380	0.4566	0.3988	0.3738

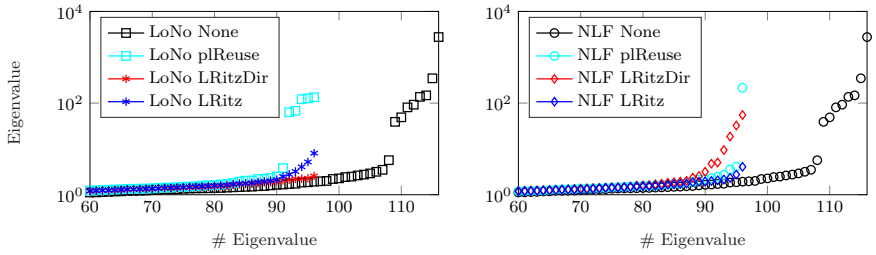

Fig. 3: Eigenvalue spectrum of interface operator $\mathbf{HP}_C^T\mathbf{F}$ sorted from lowest to highest in last load step 40 with localizations (left) and classical nonlinear FETI (right). Loadcase: bending

Table 3: Global Newton iterations of first 3 (of 40) loadsteps. Loadcase: bending

NL-Method	LoNo	LoNo	NLF	NLF	LoNo	LoNo	NLF	NLF
Recycling	None	plReuse	None	LRitzDir	LRitz	LRitzDir	LRitz	plReuse
LoadStep1	14	19	4	3	10	13	3	3
LoadStep2	4	4	4	3	2	2	2	2
LoadStep3	4	4	4	3	2	2	2	2

of the coarse grid in Fig. 2. Due to the small chosen limit of the Ritz-space by $n_s = 4$ and a reduction of global Newton-iterations, the solver is unable to capture the high modes fast enough. In the bending case, the localizations lead to worse performance than the NLF method without recycling due to instabilities of local rotational modes, mentioned as non-physical nonlinearities in literature [12, 11]. The combined local-

ization and LRitzDir outperforms the NLF though. In Fig. 3, it is able to capture all the bad modes better than in the NLF. This doesn't apply for the LRitz method. The coarse grid is filled up within the first load step of LoNo-method due to many global Newton iterations, but with unfavorable modes. In NLF, it takes more load steps, but apparently better modes are chosen here, which accelerates the solution process. In the case of LRitzDir, the higher number of global Newton iterations in the first load step is well compensated by fewer Newton iterations compared to NLF in later load steps. Anyway, the high number of load steps has been chosen to obtain a stable convergence of the algorithm with localizations. Fewer load steps would have been needed for the classic nonlinear method. Moreover, one has to bear in mind the cost of more local solves for the localization method.

5.2 Recycling methods applied to dynamic mechanical problems

We also apply the localization and recycling methods to a dynamic mechanical bending problem, meshed with Gmsh 3.0.6 and partitioned with its Metis partitioner. Here with the plain reuse technique more iterations are needed than with Ritz

Table 4: Number of FETI-iterations cumulated over time steps and global Newton-iterations and normalized to the NLF-method without recycling. Load case: dynamic bending beam. Absolute number of cumulated iterations for NLF None: 1473

NL-Method	LoNo	LoNo	NLF	NLF	LoNo	NLF	LoNo	NLF
Recycling	None	plReuse	None	plReuse	LRitz	LRitz	LRitzDir	LRitzDir
rel. Iter	1.0930	1.0088	1	0.8771	0.8629	0.7916	0.7461	0.7264

approximations due to persistent high modes. The application of localizations leads to slightly more iterations, even with recycling methods. The influence of nonlinear material is rather low due to time stepping and localization won't be able to reduce global iterations significantly.

6 Conclusions

In this work, we applied recent recycling methods and adaptive multipreconditioning for a FETI2-method together with nonlinear localization to static and dynamic structural mechanics problems. We were able to reduce global iterations by up to 62% with this combination, even for homogeneous material properties in the static bending case. This is counterbalanced by very low load step-sizes though, as otherwise the localized method would not converge due to instabilities in rotational rigid body modes. However, the static case under pull load shows quite promising results and localization combines well with recycling techniques. Hence, if the

stability issues could be fixed, these methods would be a reasonable technique to reduce communication, but at a cost of additional local solves. We were unable to test these methods in parallel due to our current implementation limitations. So it still has to be evaluated, whether the increased local solves are compensated by the reduced global iterations. Moreover, we applied these methods to dynamic structural mechanics problems, where we don't encounter the stability issues due to the present mass-matrix. Localizations didn't provide any reduction of iterations either due to limited nonlinear influences caused by time stepping. Hence, it might be different for a model with local, highly nonlinear phenomena, such as cracks and damaging, which will be supported by our implementation in the future.

References

1. Bovet, C., Parret-Fréaud, A., Spillane, N., Gosselet, P.: Adaptive multipreconditioned feti: scalability results and robustness assessment. Computers and Structures **193**, 1–20 (2017). DOI:10.1016/j.compstruc.2017.07.010
2. Cai, X.C., Li, X.: Inexact Newton methods with restricted additive Schwarz based nonlinear elimination for problems with high local nonlinearity. SIAM J. Sci. Comput. **33**(2), 746–762 (2011). DOI:10.1137/080736272
3. Chung, J., Hulbert, G.M.: A time integration algorithm for structural dynamics with improved numerical dissipation: the generalized-α method. Trans. ASME J. Appl. Mech. **60**(2), 371–375 (1993). DOI:10.1115/1.2900803
4. Farhat, C., Pierson, K., Lesoinne, M.: The second generation feti methods and their application to the parallel solution of large-scale linear and geometrically non-linear structural analysis problems. Computer methods in applied mechanics and engineering **184**, 333–374 (2000)
5. Farhat, C., Roux, F.X.: A method of finite element tearing and interconnecting and its parallel solution algorithm. Internat. J. Numer. Methods Engrg. **32**(6), 1205–1227 (1991). DOI:10.1002/nme.1620320604
6. Gosselet, P., Rey, C., Pebrel, J.: Total and selective reuse of Krylov subspaces for the resolution of sequences of nonlinear structural problems. Internat. J. Numer. Methods Engrg. **94**(1), 60–83 (2013). DOI:10.1002/nme.4441
7. Gosselet, P., Rixen, D., Roux, F.X., Spillane, N.: Simultaneous FETI and block FETI: robust domain decomposition with multiple search directions. Internat. J. Numer. Methods Engrg. **104**(10), 905–927 (2015). DOI:10.1002/nme.4946
8. Klawonn, A., Lanser, M., Rheinbach, O., Uran, M.: Nonlinear FETI-DP and BDDC methods: a unified framework and parallel results. SIAM J. Sci. Comput. **39**(6), C417–C451 (2017). DOI:10.1137/16M1102495
9. Leistner, M.C., Gosselet, P., Rixen, D.J.: Recycling of solution spaces in multipreconditioned FETI methods applied to structural dynamics. Internat. J. Numer. Methods Engrg. **116**(2), 141–160 (2018). DOI:10.1002/nme.5918
10. Leistner, M.C., Gosselet, P., Rixen, D.J.: A simplification of the ritzgeneo recycling strategy for adaptive multi-preconditioned feti applied to multiple right-hand sides. PAMM Preprint pp. 1–2 (2018). DOI:hal-01815402
11. Negrello, C., Gosselet, P., Rey, C., Pebrel, J.: Substructured formulations of nonlinear structure problems—influence of the interface condition. Internat. J. Numer. Methods Engrg. **107**(13), 1083–1105 (2016). DOI:10.1002/nme.5195
12. Pebrel, J., Rey C. Gosselet, P.: A nonlinear dual domain decomposition method: application to structural problems with damage. Internat. J. Multiscale Comput. Engrg. **6**(3), 251–262 (2008). DOI:10.1615/IntJMultCompEng.v6.i3.50

13. Rixen, D.J., Farhat, C.: A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. Internat. J. Numer. Methods Engrg. **44**(4), 489–516 (1999). DOI:10.1002/(SICI)1097-0207(19990210)44:4<489::AID-NME514>3.3.CO;2-Q
14. Roux, F.X.: Spectral analysis of the interface operators associated with the preconditioned saddle-point principle domain decomposition method. In: Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991), pp. 73–90. SIAM, Philadelphia, PA (1992)
15. Spillane, N.: An adaptive multipreconditioned conjugate gradient algorithm. SIAM Journal on Scientific Computing **38**(3), 1896–1918 (2016). DOI:10.1137/15M1028534
16. Spillane, N., Rixen, D.J.: Automatic spectral coarse spaces for robust finite element tearing and interconnecting and balanced domain decomposition algorithms. Internat. J. Numer. Methods Engrg. **95**(11), 953–990 (2013). DOI:10.1002/nme.4534