Dirichlet-Neumann Preconditioning for Stabilised Unfitted Discretization of High Contrast Problems

B. Ayuso de Dios, K. Dunn, M. Sarkis, and S. Scacchi

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with an immersed simple closed smooth interface $\Gamma \in C^2$, such that $\overline{\Omega} = \overline{\Omega}^- \cup \overline{\Omega}^+$, and $\Gamma := \overline{\Omega}^- \cap \overline{\Omega}^+$ is far away from $\partial\Omega$ (i.e, either Ω^+ or Ω^- is a *floating subdomain*; i.e., one of them does not touch $\partial\Omega$). Given $f \in L^2(\Omega)$ we set $f^\pm = f_{|\Omega^\pm}$ and consider the problem of finding u_* such that

$$\begin{cases} -\nabla \cdot (\rho_\pm \nabla u_*^\pm) = f^\pm \quad \text{in } \Omega^\pm, & u_*^\pm \quad = 0 \quad \text{on } \partial\Omega^\pm \backslash \Gamma \\ \qquad [u_*] = 0 \quad \text{on } \Gamma, & [\rho \nabla u_*] \quad = 0 \quad \text{on } \Gamma, \end{cases} \tag{1}$$

where $u_*^\pm = u_*|_{\Omega^\pm}$ and $\mathbf{n}^\pm$ denote the unit normal outward to $\Omega^\pm$. The jump conditions on Γ enforce the continuity of the solution and its flux across the interface. The jump operators are defined by

$$[\rho \nabla u_*] = \rho_+ \nabla u_*^+ \cdot \mathbf{n}^+ + \rho_- \nabla u_*^- \cdot \mathbf{n}^- \quad \text{and} \quad [u_*] = u_*^+ - u_*^-. \tag{2}$$

We also assume that the diffusion coefficients $\rho_\pm > 0$ are constant and satisfy $\rho_- \leq \rho_+$. Note that $u_*^\pm \in H^2(\Omega^\pm)$, but $u_* \in H^{1+\epsilon}(\Omega)$ with $\epsilon > 0$. To approximate

Blanca Ayuso de Dios
Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Milan, Italy, e-mail: blanca.ayuso@unimib.it

Kyle Dunn
Cold Regions Research and Engineering Laboratory, ERDC - U.S. Army, Hanover, NH, e-mail: Kyle.G.Dunn@usace.army.mil

Marcus Sarkis
Mathematical Sciences Department, Worcester Polytechnic Institute, MA, e-mail: msarkis@wpi.edu

Simone Scacchi
Dipartimento di Matematica, Università degli Studi di Milano, Milan, Italy, e-mail: simone.scacchi@unimi.it

(1) we consider the stabilised unfitted FE approximation from [3].

A class of unfitted finite element methods were introduced in the seminal works of [1] and in recent years there has been a renewed interest in these type of approaches, giving rise to numerous novel methods; the immersed boundary method [2], XFEM [5], the finite cell method (FCM) [6], and CutFEM [4, 8]. The use of unfitted meshes is particularly relevant for interface problems. However, in spite of the upsurge in research for unfitted approaches, the design and analysis of robust solvers for the resulting linear and nonlinear systems still seem elusive. Simple preconditioning strategies are explored for finite cell discretizations in [10] and multigrid-type method are proposed in [9]. In the present contribution we focus on the construction of a simple Dirichlet-Neumann (DN) domain decomposition preconditioner for the CutFEM method introduced in [3] and demonstrate its robustness also in the hard inclusion case. Due to space restrictions, we focus on a very simple version and stick to the algebraic description of the solver. Details on the analysis as well as further tailored preconditioners will be found in [7].

2 Basic Notation and Unfitted Stabilized Discretization

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of uniform partitions of Ω into squares T of diameter h. We assume that for each T, $\Gamma \cap \partial T$, is either empty or occurs at exactly two different edges of ∂T[1]. We also define:

$$\mathcal{T}_h^{\pm} := \{T \in \mathcal{T}_h : \overline{T} \cap \overline{\Omega}^{\pm} \neq \emptyset\}, \quad \mathcal{T}_h^{\Gamma} := \{T \in \mathcal{T}_h : \overline{T} \cap \Gamma \neq \emptyset\}.$$

For $T \in \mathcal{T}_h^{\Gamma}$ we denote $T_\Gamma = \overline{T} \cap \Gamma$. We also introduce the discrete domains

$$\Omega_h^{\pm} := \mathrm{Int}\left(\bigcup_{T \in \mathcal{T}_h^{\pm}} \overline{T}\right) \qquad \Omega_h^{\Gamma} := \mathrm{Int}\left(\bigcup_{T \in \mathcal{T}_h^{\Gamma}} \overline{T}\right), \quad \text{and} \qquad \Omega_{h,0}^{\pm} = \Omega_h^{\pm} \backslash \overline{\Omega}_h^{\Gamma},$$

where $\mathrm{Int}(K)$ denotes the interior of the set K. Note that $\Omega_h^+ \cup \Omega_h^- = \Omega$ is an overlapping partition of Ω while a non-overlapping partition is given by $\Omega = \Omega_{h,0}^+ \cup \overline{\Omega}_h^{\Gamma} \cup \Omega_{h,0}^-$ (see Figure 1.) Finally we introduce the following subsets of edges of elements in $\mathcal{T}_h^{\Gamma}$:

$$\mathcal{E}_h^{\Gamma,\pm} := \{e = \mathrm{Int}(\partial T_1 \cap \partial T_2) : T_1 \neq T_2 \in \mathcal{T}_h^{\pm}, \text{ and } T_1 \in \mathcal{T}_h^{\Gamma} \text{ or/and } T_2 \in \mathcal{T}_h^{\Gamma}\}.$$

Note that $\mathcal{E}_h^{\Gamma,+}$ (resp. $\mathcal{E}_h^{\Gamma,-}$) does not contain any edges on $\partial\Omega_h^+$ (resp. $\partial\Omega_h^-$).
• *Finite Element Spaces:* We consider FE spaces of piecewise bilinear polynomials whose support is contained in $\Omega_h^{\pm}$, $\Omega_{h,0}^{\pm}$ and Ω_h^{Γ}, respectively:

[1] This assumption is only needed in the stability and error analysis of the method.

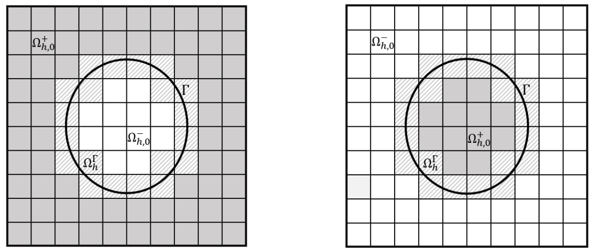

Fig. 1: Domain configurations: Soft inclusion (leftmost), Hard inclusion (center) cases

$$V^{\pm} = \{v \in C(\Omega_h^{\pm}) : v|_T \in \mathbb{Q}^1(T), \forall T \in \mathcal{T}_h^{\pm}, \text{ and } v|_{\partial\Omega_h^{\pm}\cap\partial\Omega} \equiv 0\},$$
$$V_0^{\pm} = \{v \in V^{\pm} \ : \ v|_T \equiv 0 \quad \forall\, T \in \Omega_h^{\Gamma}\}, \qquad W^{\pm} = \{v \text{ restricted to } \Omega_h^{\Gamma}, \quad v \in V^{\pm}\} \ .$$

With a small abuse of notation, we set $V_h = V^+ \times V^-$ where it is understood

$$u_h \in V_h = V^+ \times V^- \qquad u_h = (u^+, u^-) \text{ with } u^+ \in V^+ \ , \quad u^- \in V^- \ .$$

That is, the FE space V_h is defined by *a copy of two FE piecewise functions*: one from V^+ defined on Ω_h^+ and another from V^- defined over Ω_h^-.
• *The stabilised unfitted Nitsche approximation:* the method reads: find $u_h = (u^+, u^-) \in V_h = V^+ \times V^-$, such that:

$$a_h(u_h, v_h) = (f^+, v^+)_{\Omega^+} + (f^-, v^-)_{\Omega^-}, \qquad \text{for all } v_h = (v^+, v^-) \in V^+ \times V^- \ , \tag{3}$$

where $(\cdot,\cdot)_{\Omega^\pm}$ denotes the $L^2(\Omega^\pm)$ inner product and $a_h : V_h \times V_h \longrightarrow \mathbb{R}$ is given as:

$$\begin{aligned} a_h(u_h, v_h) = & \int_{\Omega^-} \rho_- \nabla u^- \cdot \nabla v^- dx + \int_{\Omega^+} \rho_+ \nabla u^+ \cdot \nabla v^+ dx \\ & + \int_\Gamma \left(\{\rho \nabla v_h\}_w \cdot \mathbf{n}^- \, [u_h] + \{\rho \nabla u_h\}_w \cdot \mathbf{n}^- \, [v_h]\right) ds + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\gamma_\Gamma}{h_T} \{\rho\}_H \int_{T_\Gamma} [u_h] \, [v_h] \, ds \\ & + \sum_{e \in \mathcal{E}_h^{\Gamma,-}} \gamma_- |e| \int_e \rho_- [\nabla u^-] \, [\nabla v^-] \, ds + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} \gamma_+ |e| \int_e \rho_+ [\nabla u^+] \, [\nabla v^+] \, ds, \end{aligned} \tag{4}$$

where γ^Γ, γ^-, and γ^+ are positive (moderate) constants and $|e|$ is the diameter of the edge e. Here, $[\cdot]$ refers to the jump operator as in (2) while $\{\cdot\}_H$ and $\{\cdot\}_\omega$ denote the harmonic and weighted averages defined by:

$$\{\rho\}_H = \frac{2\rho^+\rho^-}{\rho^+ + \rho^-} \ , \qquad \{\rho \nabla v_h\}_\omega := (\omega_- \, \rho^- \nabla v^- + \omega_+ \, \rho^+ \nabla v^+), \quad \omega_\mp = \frac{\rho^\pm}{\rho^+ + \rho^-} \ .$$

Continuity and coercivity of $a_h(\cdot,\cdot)$ in (4) can be shown with respect to the norm:

$$\|v_h\|^2_{V_h} := |v^+|^2_{V^+} + |v^-|^2_{V^-} + \sum_{T\in\mathcal{T}_h^\Gamma} \frac{\gamma_\Gamma}{h_T}\{\rho\}_H \int_{T_\Gamma} [v_h]^2\, ds \quad \forall\, v_h \in V_h\ , \quad \text{with}$$

$$|v^\pm|^2_{V^\pm} := \int_{\Omega^\pm} \rho_\pm |\nabla v^\pm|^2\, dx + \sum_{e\in\mathcal{E}_h^{\Gamma,\pm}} \gamma_\pm |e| \int_e \rho_\pm [\nabla v^\pm]^2\, ds, \quad \forall\, v^\pm \in V^\pm\ . \tag{5}$$

We remark that the semi-norm $|\cdot|_{V^+}$ is a norm if Ω^+ is non floating. We will denote by $(\cdot,\cdot)_{V^+}$ to its originating inner product. Optimal and robust error estimates are proved in [3].

3 Dirchlet-Neumann preconditioner

We describe now a preconditioner for the linear system resulting from (3) based on the non-overlapping decomposition $\Omega^+_{h,0} \cup \overline{\Omega}^\Gamma_h \cup \Omega^-_{h,0}$. Associated with such a decomposition, and owing to the *fat interface* we consider the somehow asymmetric splitting of the space $V_h = (V_0^+, W^+) \times V^-$, we first introduce some notation. We denote by $\mathcal{R}_\pm : V_h \longrightarrow V^\pm$ the restriction operators to $\Omega_h^\pm$ such that $\mathcal{R}_\pm u_h = u^\pm$. The corresponding prolongation operators $\mathcal{R}_\pm^T : V^\pm \longrightarrow V_h$ are defined as the extension to V_h by zero, i.e., $\mathcal{R}_+^T u^+ = (u^+, 0)$ and $\mathcal{R}_-^T u^- = (0, u^-)$. Similarly, we introduce the restriction and prolongation operators

$$\begin{array}{lll} \mathcal{R}_{W^\pm} : V_h \longrightarrow W^\pm & \mathcal{R}_{0^\pm} : V_h \longrightarrow V_0^\pm & \mathcal{R}_W : V_h \longrightarrow W_h \\ \mathcal{R}^T_{W^\pm} : W^\pm \longrightarrow V_h & \mathcal{R}^T_{0^\pm} : V_0^\pm \longrightarrow V_h & \mathcal{R}^T_W : W_h \longrightarrow V_h \end{array}$$

We define the bilinear forms $a_0^+ : V_0^+ \times V_0^+ \longrightarrow \mathbb{R}$ and $a^- : V^- \times V^- \longrightarrow \mathbb{R}$

$$\begin{array}{lll} a_0^+ : V_0^+ \times V_0^+ \longrightarrow \mathbb{R} & a_0^+(u_0^+, v_0^+) := a_h(\mathcal{R}^T_{0^+} u_0^+, \mathcal{R}^T_{0^+} v_0^+) & \forall\, u_0^+, v_0^+ \in V_0^+ \\ a^- : V^- \times V^- \longrightarrow \mathbb{R} & a^-(u^-, v^-) := a_h(\mathcal{R}^T_- u^-, \mathcal{R}^T_- v^-) & \forall\, u^-, v^- \in V^- \end{array}$$

We now introduce the *local solvers*. Let $u^+_{f,0} \in V_0^+$ and $u^-_f \in V^-$ be the local solutions with support in $\Omega^+_{h,0}$ and Ω^-_h, respectively, defined by:

$$a_0^+(u^+_{f,0}, v_0^+) = (f^+, v_0^+)_{\Omega^+} \ \ \forall\, v_0^+ \in V_0^+ \qquad a^-(u^-_f, v^-) = (f^-, v^-)_{\Omega^-} \quad v^- \in V^-\ .$$

We set $\mathcal{P}_h u_h = \mathcal{R}^T_{0^+} u^+_{f,0} + \mathcal{R}^T_- u^-_f$ and note that $u_h - \mathcal{P}_h u_h$ lies in the orthogonal complement of $\mathcal{R}^T_{0^+} V_0^+ + \mathcal{R}^T_- V^-$ in V_h with respect to the inner product $a_h(\cdot,\cdot)$. This suggests the splitting $u_h = \mathcal{P}_h u_h + \mathcal{H}_h u_h$, with $\mathcal{H}_h u_h = (\mathcal{H}_+ u_h, \mathcal{H}_- u_h) \in V_h$ a suitable *discrete harmonic extension* of $(u_h^+)|_{\Omega_h^\Gamma}$ that we briefly sketch next. Recall that W^+ is the restriction of the space V^+ to Ω_h^Γ. Given $\eta^+ \in W^+$, we define $\mathcal{H}_\pm : W^+ \longrightarrow V^\pm$ to be the discrete harmonic extension of η^+ such that

$$a_h(\mathcal{R}_+^T \mathcal{H}_+ \eta^+, \mathcal{R}^T_{0^+} v_0^+) = 0 \quad \forall\, v_0^+ \in V_0^+ \qquad \text{and} \qquad \mathcal{R}_{W^+} \mathcal{R}_+^T \mathcal{H}_+ \eta^+ = \eta^+$$

and

$$a_h\big((\mathcal{R}_+\mathcal{R}_{W^+}^T\eta^+, \mathcal{H}_-\eta^+), \mathcal{R}_-^T v^-\big) = 0 \qquad \forall\, v^+ \in V^- .$$

Finally, we set $\mathcal{H}_h\eta^+ = (\mathcal{H}_+\eta^+, \mathcal{H}_-\eta^+)$ and introduce the Schur complement operator $\mathcal{S} : W^+ \longrightarrow W^+$:

$$< \mathcal{S}\eta, w > := a_h(\mathcal{H}_h\eta^+, \mathcal{H}_h w^+) \qquad \forall\, \eta^+, w^+ \in W^+ \ . \tag{6}$$

From the definition of $\mathcal{P}_h u_h$ it follows

$$a_h(\mathcal{H}_h u_h, \mathcal{H}_h v_h) = (f, v_h)_\Omega - a_h(\mathcal{P}_h u_h, v_h) \qquad \forall\, v_h \in V_h \ . \tag{7}$$

We focus now on constructing preconditioners $\mathcal{B}^{-1}$ for the operator $\mathcal{S}$ and hence for the system (7). The basic guide to ensure robustness will be to use, when possible, the local Schur complement corresponding to the largest coefficient, ρ_+:

$$< \mathcal{S}_+\eta, w > := (\mathcal{H}_+\eta, \mathcal{H}_+ w^+)_{V^+} \qquad \forall\, \eta, w \in W^+ \ , \tag{8}$$

where $(\cdot,\cdot)_{V+}$ is the originating inner-product for the norm $|\cdot|_{V^+}$ in (5). We need to distinguish two cases:

- Ω^+ is not floating subdomain and we set $\mathcal{B}^{-1} = \mathcal{S}_+^{-1}$.
- Ω^+ is a floating subdomain; since $\mathcal{S}_+$ is not invertible, we define $\mathcal{B}^{-1}$ as a suitable regularisation of $\mathcal{S}_+$. We propose one level and two level methods.

4 Algebraic formulation of the DN preconditioner

After choosing standard Lagrangian basis for $V^\pm$, problem (3) reduces to a linear algebraic system $\mathbb{A}\mathbb{U} = \mathbb{F}$. We consider the block structure of $\mathbb{A}$ that results from splitting the degrees of freedom (dofs) of the discrete space V_h into three sets:

- dofs associated with V_0^+ (in the interior of Ω^+) are indicated by I^+;
- dofs related to W^+, indicated by W^+;
- dofs associated with V^-(dofs related to V_0^- and W^-), indicated by V^-.

$$\begin{bmatrix} \mathbb{A}_{I^+I^+} & \mathbb{A}_{I^+W^+} & 0 \\ \mathbb{A}_{W^+I^+} & \boxed{\mathbb{A}^+_{W^+W^+} + \mathbb{A}^-_{W^+W^+}} & \mathbb{A}_{W^+V^-} \\ 0 & \mathbb{A}_{V^-W^+} & \mathbb{A}_{V^-V^-} \end{bmatrix} \begin{bmatrix} \mathbb{U}_{I^+} \\ \mathbb{U}_{W^+} \\ \mathbb{U}_{V^-} \end{bmatrix} = \begin{bmatrix} \mathbb{F}_{I^+} \\ \mathbb{F}_{W^+} \\ \mathbb{F}_{V^-} \end{bmatrix}.$$

Here, we have highlighted that the stiffness block with dofs from W^+ in the *fat interface* has contributions from Ω_h^+ and Ω_h^-. Performing static condensation of the interior variables I^+ and V^- we obtain the Schur complement system

$$\mathbb{S}\mathbb{U}_{W^+} = \mathbb{G}_{W^+}, \qquad \mathbb{S} = \mathbb{S}_+ + \mathbb{S}_- \ ,$$

where $\mathbb{G}_{W^+} = \mathbb{F}_{W^+} - \mathbb{A}_{W^+I^+}\mathbb{A}^{-1}_{I^+I^+}\mathbb{F}_{I^+} - \mathbb{A}_{W^+V^-}\mathbb{A}^{-1}_{V^-V^-}\mathbb{F}_{V^-}$, and $\mathbb{S}$ is given by

$$\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_- \qquad \text{with} \qquad \begin{cases} \mathbb{S}_+ = \mathbb{A}^+_{W^+W^+} - \mathbb{A}_{W^+I^+}\mathbb{A}^{-1}_{I^+I^+}\mathbb{A}_{I^+W^+} \\ \mathbb{S}_- = \mathbb{A}^-_{W^+W^+} - \mathbb{A}_{W^+V^-}\mathbb{A}^{-1}_{V^-V^-}\mathbb{A}_{V^-W^+}, \end{cases}$$

Soft inclusion: Ω_h^+ in Non-Floating Subdomain Case: In this case we set $\mathcal{B}^{-1} = \mathcal{S}_+^{-1}$ since the operator is invertible. At the algebraic level we arrive at $\mathbb{S}_+^{-1}\mathbb{S}\mathbb{U}_{W^+} = \mathbb{S}_+^{-1}\mathbb{G}_{W^+}$. The action of the DN preconditioner $\mathbb{S}_+^{-1}$ on a generic residual vector $\mathbb{R}_{W^+}$ consists of solving the linear system

$$\begin{bmatrix} \mathbb{A}_{I^+I^+} & \mathbb{A}_{I^+W^+} \\ \mathbb{A}_{W^+I^+} & \mathbb{A}^+_{W^+W^+} \end{bmatrix} \begin{bmatrix} \mathbb{V}_{I^+} \\ \mathbb{V}_{W^+} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbb{R}_{W^+} \end{bmatrix}.$$

and letting $\mathbb{V}_{W^+} := \mathbb{S}_+^{-1}\mathbb{R}_{W^+}$.

Hard inclusion: Ω_h^+ is the Floating Subdomain: Since $\mathbb{S}_+$ is not invertible we consider two different strategies: a regularisation and the use of a one dimensional coarse solver to account for the kernel of $\mathbb{S}_+$.

• *One-Level DN:* The action of the preconditioner amounts to solving

$$\left(\begin{bmatrix} \mathbb{A}_{I^+I^+} & \mathbb{A}_{I^+W^+} \\ \mathbb{A}_{W^+I^+} & \mathbb{A}^+_{W^+W^+} \end{bmatrix} + \frac{\{\rho\}_H}{D_+^2} \begin{bmatrix} \mathbb{M}^+_{I^+I^+} & \mathbb{M}^+_{I^+W^+} \\ \mathbb{M}^+_{W^+I^+} & \mathbb{M}^+_{W^+W^+} \end{bmatrix} \right) \begin{bmatrix} \mathbb{V}_{I^+} \\ \mathbb{V}_{W^+} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbb{R}_{W^+} \end{bmatrix},$$

and setting $\mathbb{S}^{-1}_{+,one}\mathbb{R}_{W^+} = \mathbb{V}_{W^+}$. Here, $\mathbb{M}^+$ stands for the mass matrix associated with V^+ (i.e., defined over Ω_h^+), and $D_+ :=\mathrm{diam}(\Omega_h^+)$ and is used to regularise $\mathbb{S}_+$.

• *Two Level DN preconditioner:* The idea is to first solve in the space orthogonal to the (one-dimensional) kernel of $\mathbb{S}_+$ and then correct with a coarse solver that accounts for the contribution in $\ker(\mathbb{S}_+)$. Hence, the practical implementation of the two level solver $\mathbb{S}^{-1}_{+,two}$ amounts to first solving

$$\begin{bmatrix} \mathbb{A}_{I^+I^+} & \mathbb{A}_{I^+W^+} \\ \mathbb{A}_{W^+I^+} & \mathbb{A}^+_{W^+W^+} \end{bmatrix} \begin{bmatrix} \mathbb{V}_{I^+} \\ \mathbb{V}_{W^+} \end{bmatrix} + \begin{bmatrix} \mathbb{M}^+_{I^+I^+} & \mathbb{M}^+_{I^+W^+} \\ \mathbb{M}^+_{W^+I^+} & \mathbb{M}^+_{W^+W^+} \end{bmatrix} \begin{bmatrix} \mathbf{1}_{I^+} \\ \mathbf{1}_{W^+} \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ \mathbb{R}_{W^+} \end{bmatrix},$$

with the constraint

$$\begin{bmatrix} \mathbf{1}_{I^+} \\ \mathbf{1}_{W^+} \end{bmatrix}^T \begin{bmatrix} \mathbb{M}^+_{I^+I^+} & \mathbb{M}^+_{I^+W^+} \\ \mathbb{M}^+_{W^+I^+} & \mathbb{M}^+_{W^+W^+} \end{bmatrix} \begin{bmatrix} \mathbb{V}_{I^+} \\ \mathbb{V}_{W^+} \end{bmatrix} = 0$$

and then define $\mathbb{S}_+^{\dagger}\mathbb{R}_{W^+} = \mathbb{V}_{W^+}$. Here $\mathbf{1}^+_{I^+}$ and $\mathbf{1}^+_{W^+}$ are vectors of ones in V_0^+ and W^+, respectively. The matrix representation of the two level preconditioner (with coarse space) is defined via $\mathbb{S}^{-1}_{+,two} = \mathbb{S}_+^{\dagger} + \mathbf{1}_{W^+}(\mathbf{1}_{W^+}, \mathbb{S}\mathbf{1}_{W^+})^{-1}\mathbf{1}^T_{W^+}$. Note that $\mathbb{S}\mathbf{1}_{W^+} = \mathbb{S}_-\mathbf{1}_{W^+}$.

5 Numerical Results

We consider the domain $\Omega = (0,1)^2$ and study the performance of the Dirichlet-Neumann (DN) preconditioner for the CutFEM approximation (3) to (1) with $\Omega^{\mp}$ a disk of radius 0.15 and $\Omega^{\pm} = (0,1)^2 \setminus \overline{\Omega^{\mp}}$ and always $\rho_- \leq \rho_+$. We use CG and PCG as a solver with zero initial guess and tolerance 10^{-6} for the relative residual. In the tables we report the estimated (via Lanzcos algorithm) condition numbers (denoted by κ_2) and the number of iterations (denoted by **it**) required by CG and PCG for convergence. Table 1 reports the results in the case where Ω^+ is

ρ_-	**full CG**		**schur NO precond.**		**schur DN preconditioned**	
	κ_2	it	κ_2	it	κ_2	it
1	3.32e+3	(218)	388.40	(75)	1.95	(14)
10^{-2}	2.06e+4	(575)	362.15	(91)	1.01	(15)
10^{-4}	2.01e+6	(2828)	361.71	(93)	1.00	(4)
10^{-6}	2.01e+8	(5418)	361.70	(93)	1.00	(3)

Table 1: Robustness with respect to ρ: Ω^- is the floating subdomain. Here, $\rho_+ = 1$ and $h = 1/64$.

non-floating, therefore using $\mathcal{S}_+^{-1}$ as a preconditioner. $\mathcal{S}_+^{-1}$ performs robustly when the ratio ρ_+/ρ_- increases. In the case where Ω^+ is the floating subdomain, we use one level and two level DN preconditioners. The results regarding optimality and robustness of these preconditioners are reported in Table 2 and 3, respectively. Notice that both preconditioners perform optimally and show robustness with respect to the jumping coefficient. In particular, the one-level DN preconditioner seems to be enough effective for the considered setting.

$1/h$	**full CG**		**schur NO precond.**		**DN Two-Level**		**DN one-level**	
	κ_2	it	κ_2	it	κ_2	it	κ_2	it
8	6.38e+3	252	4.09e+2	79	6.76	11	3.51	14
16	1.77e+4	520	8.60e+3	224	6.39	15	2.11	14
32	5.83e+4	863	1.09e+4	423	6.29	16	2.09	14
64	2.14e+4	1625	1.86e+4	551	6.34	16	2.08	14
128	8.19e+5	3163	3.79e+4	832	6.37	16	2.13	14
256	3.20e+6	6140	7.43e+4	1148	6.39	16	2.19	14

Table 2: Optimality with respect to h: floating circle Ω^+ embedded in $[0,1]^2$. $\rho_+ = \rho_- = 1$.

ρ_+	full CG		schur NO precond.		DN Two-Level		DN one-level	
	κ_2	it	κ_2	it	κ_2	it	κ_2	it
1	2.14e+5	1625	1.86e+4	539	6.37	16	2.13	14
10^2	2.00e+7	12906	1.81e+6	765	6.33	6	1.83	5
10^4	2.00e+9	>100000	1.81e+8	897	6.33	4	1.83	4
10^6	5.70e+10	>100000	1.81e+10	1026	6.33	3	1.83	3
10^8	4.20e+12	>100000	1.83e+12	1326	6.33	3	1.83	3

Table 3: Robustness with respect to ρ. Floating Ω^+ with jumping coefficients. Here, $\rho_- = 1$, $1/h = 64$.

References

1. Barrett, J.W., Elliott, C.M.: Fitted and unfitted finite-element methods for elliptic equations with smooth interfaces. IMA J. Numer. Anal. **7**(3), 283–300 (1987). DOI:10.1093/imanum/7.3.283
2. Boffi, D., Gastaldi, L.: A finite element approach for the immersed boundary method. Comput. & Structures **81**(8-11), 491–501 (2003). DOI:10.1016/S0045-7949(02)00404-2. In honour of Klaus-Jürgen Bathe
3. Burman, E., Guzmán, J., Sánchez, M.A., Sarkis, M.: Robust flux error estimation of an unfitted Nitsche method for high-contrast interface problems. IMA J. Numer. Anal. **38**(2), 646–668 (2018)
4. Burman, E., Hansbo, P.: Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method. Appl. Numer. Math. **62**(4), 328–341 (2012). DOI:10.1016/j.apnum.2011.01.008
5. Chessa, J., Smolinski, P., Belytschko, T.: The extended finite element method (XFEM) for solidification problems. Internat. J. Numer. Methods Engrg. **53**(8), 1959–1977 (2002). DOI:10.1002/nme.386
6. Dauge, M., Düster, A., Rank, E.: Theoretical and numerical investigation of the finite cell method. J. Sci. Comput. **65**(3), 1039–1064 (2015). DOI:10.1007/s10915-015-9997-3
7. Ayuso de Dios, B., Dunn, K., Sarkis, M., Scacchi, S.: Simple preconditioners for cutfem methods (2019). (work in preparation)
8. Hansbo, A., Hansbo, P.: An unfitted finite element method, based on Nitsche's method, for elliptic interface problems. Comput. Methods Appl. Mech. Engrg. **191**(47-48), 5537–5552 (2002). DOI:10.1016/S0045-7825(02)00524-8
9. Ludescher, T., Groβ, S., Reusken, A.: A multigrid method for unfitted finite element discretizations of elliptic interface problems. Techical report IGPM 481, RWTH Aachen (2018)
10. de Prenter, F., Verhoosel, C.V., van Zwieten, G.J., van Brummelen, E.H.: Condition number analysis and preconditioning of the finite cell method. Comput. Methods Appl. Mech. Engrg. **316**, 297–327 (2017). DOI:10.1016/j.cma.2016.07.006