Virtual Coarse Spaces for Irregular Subdomain Decompositions

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1 Introduction

Consider the model problem: Find $u \in H^1(\Omega)$ such that

$$-\nabla \cdot (\rho(x)\nabla u) = f(x), \quad x \in \Omega, \tag{1}$$

for a given polygonal domain $\Omega \subset \mathbb{R}^2$ and $\rho(x) > 0$, along with homogeneous boundary conditions. A standard approach to solve (1) is to discretize with Finite Element Methods (FEM) for which there is vast literature on the construction of Domain Decomposition (DD) algorithms; see, e.g., [11] for a complete study. As usual, we will decompose the domain Ω into N non-overlapping subdomains $\{\Omega_i\}_{i=1}^N$, each of which is the union of elements of the triangulation $\mathcal{T}_h$ of Ω. Each Ω_i will be simply connected and will have a connected boundary $\partial\Omega_i$. We then construct overlapping subdomains Ω_i' by adding layers of elements to Ω_i.

One of the simplest DD algorithms consists in splitting the finite dimensional space V_h (associated with the fine triangulation of the domain) as

$$V_h = R_0^T V_0 + \sum_{i=1}^{N} R_i^T V_i,$$

where $V_1, \ldots, V_N$ represent local spaces related to $\Omega_1', \ldots, \Omega_N'$, respectively, with corresponding extension operators $R_i^T : V_i \to V_h$, and V_0 is a coarse space which is related to V_h by the operator $R_0^T : V_0 \to V_h$. Originally, these methods arose in the presence of regular decompositions where usual Finite Element spaces can be defined. In the past few years, there has been some efforts to study how to define coarse spaces if irregular subdomains as the ones obtained by mesh partitioners

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are considered; see, e.g., [5, 14, 6], where a complete theory is developed for Jones subdomains and nodal elliptic problems. For Raviart-Thomas and Nédélec elements, see [9, 2]. These studies are based on energy minization, and require to obtain local discrete harmonic functions by solving Dirichlet problems on the subdomains. In a more general setting, adaptive coarse spaces can be defined as in [7, 10, 8].

On the other hand, Virtual Element Methods (VEM) [1, 12, 13] allow to handle general polygonal elements. In the case of triangular elements, VEM reduces to the usual FEM. Thus, VEM is a natural choice for constructing space of functions on irregular subdomains. As studied in [3, 4], considering a virtual space on an irregular decomposition allows us to avoid the computation of discrete harmonic functions, while we keep the typical bound for the condition number of the preconditioned system; see Theorem 1 below. In this setting, we can define general virtual functions for such irregular decompositions. However, virtual functions cannot be evaluated at interior nodes, and the operator R_0^T plays an essential role into approximating functions in V_0. Two different approaches have been studied so far: we can construct V_0 based on linear interpolants [4], or we can use projections onto polynomial spaces of degree of at least two [3], which we will discuss in this manuscript.

Instead of having a triangular mesh and a FEM discretization for problem (1), we could also consider a discretization based on VEM. There is a lack of literature on DD methods for such type of problems. At the DD25 Conference, held in Saint John's, Canada on July 2018, interesting talks by Yunrong Zhu (Auxiliary Space Preconditioners for Virtual Element Discretization) and Daniele Prada (FETI-DP for Three Dimensional VEM) addressed this problem with different approaches as ours. We note that the theory developed in [3, 4] is also useful for designing Schwarz operators for discretizations obtained by VEM, and it is possible to obtain similar bounds for the condition number of the preconditioned system.

2 Description of the preconditioner

In this section we describe the discretizacion of the model problem and the construction of the additive preconditioner. We refer [13] for general details on VEM, [3, 4] for a detailed explanation on the coarse space definition, and [11, Chapter 3] for a complete study of overlapping preconditioners.

The usual weak form for problem (1) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} = (f, v) \quad \forall \, v \in H_0^1(\Omega), \tag{2}$$

where $(\cdot, \cdot)$ is the usual inner product in $L^2(\Omega)$. When using nodal Lagrange triangular elements, we consider the lowest-order finite-dimensional Lagrange space V_h, which consists of continuous piecewise-linear functions on each element, and Problem (2) becomes: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h. \tag{3}$$

When using VEM, we can consider a general triangulation $\mathcal{T}_h$ composed by general polygons as in Figure 1 (not necessarily similar or with the same number of edges), and V_h now contains piecewise-linear continuous functions on the boundary of each element that are harmonic in its interior. We omit further details on how to modify the bilinear form $a(\cdot,\cdot)$ and the right-hand side in Equation (3) when VEM are used; see, e.g., [13]. We then obtain a linear system $Au = f$, for which we will describe the construction of an additive preconditioner.

2.1 Virtual coarse space

We present the coarse spaces considered in [3, 4]. We first define the lowest-order virtual element space on the polygonal decomposition $\{\Omega_i\}_{i=1}^N$ of Ω. For each Ω_i, consider the set

$$\mathcal{B}(\partial\Omega_i) := \left\{ v \in C^0(\partial\Omega_i) : v|_e \in \mathcal{P}_1(e)\ \forall\, e \subset \partial\Omega_i \right\},$$

where e represents any straight segment of the boundary of the polygon Ω_i. The local virtual space is then defined as

$$V^{\Omega_i} := \left\{ v \in H^1(\Omega_i) : v|_{\partial\Omega_i} \in \mathcal{B}(\partial\Omega_i),\ \Delta v = 0 \right\}.$$

A natural choice for the coarse space of the two-level algorithm is the global virtual space

$$V_0 := \left\{ v \in H^1(\Omega) : v|_{\Omega_i} \in V^{\Omega_i} \right\}.$$

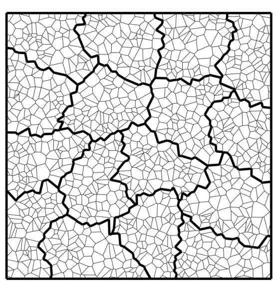

Fig. 1: General polygonal mesh for VEM with irregular subdomains.

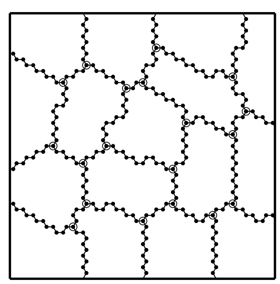

Fig. 2: Decomposition $\{\Omega_i\}$. The coarse space V_0 has one degree of freedom per polygonal vertex (black dots). The reduced coarse space V_0^R has only one degree of freedom per subdomain vertex (black circles)

Hence, a function in V_0 is continuous, piecewise-linear on the boundary of each Ω_i, and harmonic in the interior of each subdomain. Thus, it is completely determined by its values at the vertices of the polygonal domain Ω_i and the dimension of V_0 can be quite large; see Figure 2 for an example with an hexagonal mesh. Therefore, we define a reduced coarse space as follows.

For each subdomain vertex $\boldsymbol{x}_0$ we define a coarse function $\psi^H_{\boldsymbol{x}_0} \in V_0$ by choosing appropriately its degrees of freedom, a construction modified from [5]. First, we set $\psi^H_{\boldsymbol{x}_0}(\boldsymbol{x}) = 0$ for all the subdomain vertices $\boldsymbol{x}$, except at $\boldsymbol{x}_0$ where $\psi^H_{\boldsymbol{x}_0}(\boldsymbol{x}_0) = 1$. Second, we set the degrees of freedom related to the nodal values on each subdomain edge. If $\boldsymbol{x}_0$ is not an endpoint of $\mathcal{E}$, then $\psi^H_{\boldsymbol{x}_0}$ vanishes on that edge. If $\mathcal{E}$ has endpoints $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$, let $\boldsymbol{d}_{\mathcal{E}}$ be the unit vector with direction from $\boldsymbol{x}_1$ to $\boldsymbol{x}_0$. For any node $\widetilde{\boldsymbol{x}} \in \mathcal{E}$ set

$$\psi^H_{\boldsymbol{x}_0}(\widetilde{\boldsymbol{x}}) = \begin{cases} 0, \text{ if } & (\widetilde{\boldsymbol{x}} - \boldsymbol{x}_1) \cdot \boldsymbol{d}_{\mathcal{E}} < 0 \\ \dfrac{(\widetilde{\boldsymbol{x}} - \boldsymbol{x}_1) \cdot \boldsymbol{d}_{\mathcal{E}}}{|\boldsymbol{x}_0 - \boldsymbol{x}_1|}, \text{ if } & 0 \le (\widetilde{\boldsymbol{x}} - \boldsymbol{x}_1) \cdot \boldsymbol{d}_{\mathcal{E}} \le |\boldsymbol{x}_0 - \boldsymbol{x}_1| \\ 1, \text{ if } & (\widetilde{\boldsymbol{x}} - \boldsymbol{x}_1) \cdot \boldsymbol{d}_{\mathcal{E}} > |\boldsymbol{x}_0 - \boldsymbol{x}_1| \end{cases}$$

It is clear that $\psi^H_{\boldsymbol{x}_0}(\boldsymbol{x}_0) = 1$, $\psi^H_{\boldsymbol{x}_0}(\boldsymbol{x}_1) = 0$, and that the function varies linearly in the direction of $\boldsymbol{d}_{\mathcal{E}}$ for such nodes. In this way, we define all the degrees of freedom of $\psi^H_{\boldsymbol{x}_0} \in V_0$. By construction, $0 \le \psi^H_{\boldsymbol{x}_0} \le 1$ and $\sum_{\boldsymbol{x}_0} \psi^H_{\boldsymbol{x}_0} \equiv 1$.

We then define the *reduced coarse space* as the span of $\{\psi^H_{\boldsymbol{x}_0}\}$, i.e.,

$$V_0^R := \left\{ v \in H_0^1(\Omega) : v = \sum_{\boldsymbol{x}_0} \alpha_{\boldsymbol{x}_0} \psi^H_{\boldsymbol{x}_0} \right\} \subset V_0$$

for some real coefficients $\alpha_{\boldsymbol{x}_0}$; see [4, Section 6]. We point out that in the case where the partition $\{\Omega_i\}$ is composed by triangles or squares, $V_0 = V_0^R$ and they reduce to the usual linear or bilinear finite element space, respectively. We can naturally define a linear interpolant $I^H : V_h \to V_0^R$ by

$$I^H u := \sum_{\boldsymbol{x}_0} u(\boldsymbol{x}_0) \psi^H_{\boldsymbol{x}_0},$$

and it is easy to deduce that I^H reproduces linear polynomials. We can prove the following lemma, where we present an upper bound for the energy of coarse functions:

Lemma 1 *Given $u \in V_h$, let $u_0 := I^H u \in V_0^R$. Then, there exists a constant C such that*

$$|u_0|^2_{H^1(\Omega_i)} \le C \left(1 + \log \frac{H_i}{h_i} \right) |u|^2_{H^1(\Omega_i)},$$

where H_i is the diameter of Ω_i and h_i is the smallest element diameter of the triangulation of Ω_i. Here, C depends only on the aspect ratio of Ω_i and the number of subdomain vertices on $\partial\Omega_i$.

Proof See [3, Lemma 4.4 and Theorem 6.1], [4, Lemma 5.6]. □

Since virtual coarse functions cannot be evaluated at internal nodes of the subdomains, we still need to define an appropriate operator $R_0^T : V_0^R \to V_h$, such that each function in V_0^R is well-approximated in V_h. We could:

(1) Solve a Dirichlet problem on each subdomain in order to compute the discrete harmonic extension of the values on the boundary of each Ω_i, as it is done in [5, 14, 6].
(2) Triangulate each subdomain Ω_i and define R_0^T as a piecewise-linear interpolant onto such triangulations; see [4, Section 3.1] for further details and assumptions that are required.
(3) Construct a projection $\Pi^{\nabla}_{\Omega_i,k} u_0$ for a given function u_0 in V^{Ω_i}, onto the polynomial space defined on Ω_i of degree $k \geq 2$, and this operator can be constructed by knowing only the degrees of freedom of the virtual functions; see [3, Section 6.1] for implementation details. The main advantage in this approach is that in order to compute all the internal degrees of freedom, we only need to solve a linear system with $k(k-1)/2$ unknowns. Thus, in the interior of each subdomain we approximate u_0 by $\Pi^{\nabla}_{\Omega_i,k} u_0$, avoiding discrete harmonic extensions.

It can be shown that the following estimates hold:

Lemma 2 *Given $u \in V_h$, let $u_0 := I^H u \in V_0^R$. Then there exists a constant C such that*

$$\|u - R_0^T u_0\|^2_{L^2(\Omega_i)} \leq C H_i^2 \left(1 + \log \frac{H_i}{h_i}\right) |u|^2_{H^1(\Omega_i)},$$
$$|R_0^T u_0|^2_{H^1(\Omega_i)} \leq C \left(1 + \log \frac{H_i}{h_i}\right) |u|^2_{H^1(\Omega_i)},$$

where C is independent of H_i and h_i.

Proof See [4, Lemma 3, Lemma 4] and [3, Lemma 5.7] for cases (2) and (3), respectively. For case (1), similar estimates holds; see the proof in [5, Theorem 3.1], [6, Theorem 3.1] □

2.2 Local spaces and preconditioner

For each subdomain Ω_i, we construct the overlapping subdomain Ω_i' by adding layers of elements to Ω_i and denote by δ_i the size of the overlap. The local virtual space is then defined by

$$V_i := \left\{ v \in H_0^1(\Omega_i') : v|_K \in \mathcal{B}(\partial K),\ \Delta v|_K = 0 \text{ in } K,\ \forall\, K \subset \Omega_i' \right\}.$$

Thus, the degrees of freedom are the values at all the nodes in the interior of Ω_i', and it is straightforward to define zero extension operators $R_i^T : V_i \to V_h$. Consider the matrix representation of the operators R_i^T denoted again by R_i^T. We use exact local solvers and define $\tilde{A}_i = R_i A R_i^T$, $0 \leq i \leq N$. Schwarz projections are given by

$$P_i = R_i^T \tilde{A}_i^{-1} R_i A, \ 0 \le i \le N.$$

The additive preconditioned operator is defined by

$$P_{ad} := \sum_{i=0}^{N} P_i = A_{ad}^{-1} A, \text{ with } A_{ad}^{-1} = \sum_{i=0}^{N} R_i^T \tilde{A}_i^{-1} R_i. \tag{4}$$

Multiplicative and hybrid preconditioners can be considered as well; see [11, Section 2.2]. We can then prove the following result:

Theorem 1 *There exists a constant C, independent of H, h and ρ, such that the condition number of the preconditioned system $\kappa(A_{ad}^{-1}A)$ satisfies*

$$\kappa(A_{ad}^{-1}A) \le C \left(1 + \log \frac{H}{h}\right)\left(1 + \frac{H}{\delta}\right),$$

where the ratios H/h and H/δ denote their maximum value over all the subdomains.

Proof See [4, Theorem 6.1], [3, Theorem 4.1]. □

3 Some numerical results

We first provide a comparison of the running time when assembling R_0^T by discrete harmonic extensions and by quadratic and cubic polynomial approximations; see Figure 3 where we have used a serial implementation in MATLAB with $N = 4$ METIS subdomains and triangular elements.

We also include an experiment with a different application of the virtual coarse spaces. We approximate accurately harmonic functions with given Dirichlet boundary conditions in a domain Ω, by using the projector $\Pi^{\nabla}_{\Omega,k}$ for sufficiently large k. Instead of solving the resulting ill-conditioned linear system $Au = f$ that arises from FEM or VEM, we can approximate the nodal values in the interior nodes of $\mathcal{T}_h$ by evaluating $u_h := \Pi^{\nabla}_{\Omega,k} u$. In order to do so, we just need to solve a linear system with $k(k-1)/2$ unknowns. We remark that in the construction of the preconditioner (4),

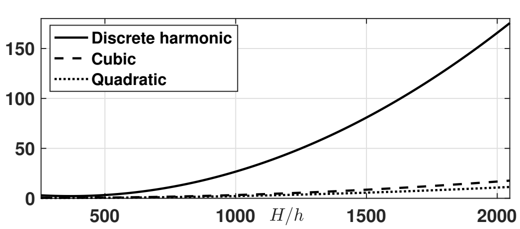

Fig. 3: Time (in seconds) required for computing R_0^T with discrete harmonic extensions, quadratic and cubic projections, as a function of H/h, with $N = 4$ irregular subdomains.

a competitive number of iterations can be obtained with just $k = 2$ or $k = 3$, since they provide good-enough approximations for functions in the virtual coarse space. Here instead, we construct the projection onto the domain Ω, obtaining directly u_h.

For simplicity, we consider the unit square $\Omega = [0, 1]^2$ with boundary conditions such that the exact solution is $u(x, y) = (e^{2x}+e^{-2x})\sin(2y)$. We consider a triangular partition for Ω; the inf-norm of the error in the approximation is shown in Figure 4, for different values of k and mesh size h. As we observe, for a fixed k, the error decreases quadratically as a function of h, and it reaches a minimum value that depends on k, for which $\Pi^{\nabla}_{\Omega,k}$ cannot improve the approximation. We remark that further exploration is required, and this approach is being studied for problems in two and three dimensions.

For further experiments on the performance of the preconditioner (4), we refer to the numerical experiments shown in [3, 4].

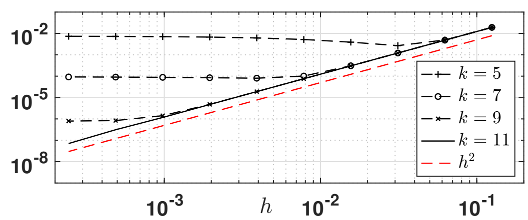

Fig. 4: Inf-norm of the error, $\|u - u_h\|_\infty$, as a function of h, in the approximation of the solution of Laplace's equation in the unit square by computing $\Pi^{\nabla}_{\Omega,k} u$. Convergence is quadratic as a function of h.

4 Conclusions

We note that the main advantage of our approach with respect to previous studies is that no discrete harmonic extensions are required in the algorithm, saving computational time. We also aim to contribute and enrich the literature related to iterative solvers for VEM discretizations, since there is a lack of theoretical analysis for such problems. Even though theory does not include the case of a discontinuous coefficient in the interior of each subdomain, a reasonable number of iterations is obtained even for extreme cases of discontinuities and high-contrast jumps across the elements; see [3, Section 6.2.4]. For higher values of k, we can directly obtain more accurate approximations of harmonic functions, as shown in Figure 4. For preconditioning, experimentally we have found that using quadratic or cubic polynomials is sufficient, but we can use higher degree spaces in order to improve accuracy in the approximation of harmonic functions.

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