A Local Coarse Space Correction Leading to a Well-Posed Continuous Neumann-Neumann Method in the Presence of Cross Points

Faycal Chaouqui, Martin J. Gander, and Kévin Santugini-Repiquet

1 Introduction

Neumann-Neumann methods (NNMs) are among the best parallel solvers for discretized partial differential equations, see [12] and references therein. Their common polylogarithmic condition number estimate shows their effectiveness for many discretized elliptic problems, see [9, 10, 5]. However, NNM was originally described in [1] as an iteration at the continuous level like the classical Schwarz method, but only for two subdomains, see also [11]. This is because in contrast to the Schwarz method, it does not converge for general decompositions into many subdomains when used as a stationary iteration [4, 3]. Furthermore, for decompositions presenting cross points, NNM is not well-posed in H^1 and has as a stationary iteration a convergence factor that deteriorates polylogarithmically in the mesh size h, see [4]. The iterates being discontinuous at the cross points also prevents NNM from being well-posed in H^2. We propose here a very specific local coarse space that leads to a well posed NNM at the continuous level for the model problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $f \in L^2(\Omega)$, and Ω can be decomposed as in Fig. 1, i.e. the decomposition can contain cross points. In Section 2 we present NNM at the continuous level for a 2×1 decomposition and show why it is always well-posed in H^1. In Section 3 we show why NNM for a 2×2 decomposition containing a cross point is not in general well-posed in H^1. To make it well-posed in H^2, we introduce a very specific

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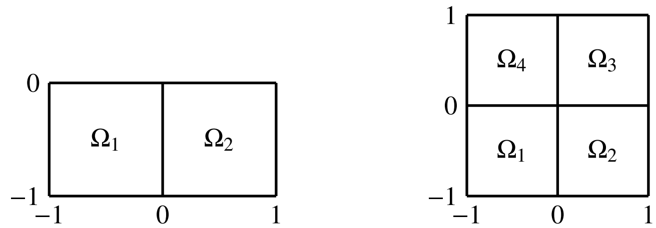

Fig. 1: Decomposition without a cross point (left) and with a cross point (right)

Algorithm 1: NNM for a 2×1 decomposition

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1. Set g_{12}^0 to zero or any inexpensive initial guess.
2. For $n = 0, 1, \ldots$ until convergence

   a. Solve the Dirichlet problems

$$\begin{aligned} -\Delta u_1^n &= f \text{ in } \Omega_1, & -\Delta u_2^n &= f \text{ in } \Omega_2,\\ u_1^n &= g_{12}^n \text{ on } \Gamma, & u_2^n &= g_{12}^n \text{ on } \Gamma,\\ u_1^n &= 0 \text{ on } \partial\Omega_1 \cap \partial\Omega, & u_2^n &= 0 \text{ on } \partial\Omega_2 \cap \partial\Omega. \end{aligned}$$

   b. Solve the Neumann problems

$$\begin{aligned} -\Delta \psi_1^n &= 0 \text{ in } \Omega_1, & -\Delta \psi_2^n &= 0 \text{ in } \Omega_2,\\ \frac{\partial \psi_1^n}{\partial n_1} &= \frac{1}{2}\left(\frac{\partial u_1^n}{\partial n_1} + \frac{\partial u_2^n}{\partial n_2}\right) \text{ on } \Gamma, & \frac{\partial \psi_2^n}{\partial n_2} &= \frac{1}{2}\left(\frac{\partial u_1^n}{\partial n_1} + \frac{\partial u_2^n}{\partial n_2}\right) \text{ on } \Gamma,\\ \psi_1^n &= 0 \text{ on } \partial\Omega_1 \cap \partial\Omega, & \psi_2^n &= 0 \text{ on } \partial\Omega_2 \cap \partial\Omega, \end{aligned}$$

   where n_i is the outward pointing normal on $\partial\Omega_i$, $i = 1, 2$.

   c. Update the trace $g_{12}^{n+1} = g_{12}^n - \frac{1}{2}(\psi_1^n + \psi_2^n)$ on Γ.

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local coarse space correction. Our new NNM then converges as a stationary iterative solver, also in the presence of cross points, and we show numerically that it is a better preconditioner than the classical NNM in the case of many cross points.

2 Existence of iterates for a 2×1 decomposition

For a decomposition as shown in Fig. 1 (left), let $\Gamma := \{0\} \times (0, 1)$ be the interface between Ω_1 and Ω_2. The NNM in Algorithm 1 is well-posed with iterates in H^1:

Theorem 1 *If* $g_{12}^0 \in H_{00}^{\frac{1}{2}}(\Gamma)$ *(the* Lions-Magenes *space defined in [7, Chapter 1]), then Algorithm 1 is well-posed and for all* $n \geq 0$ *we have* $u_i^n \in V_i$*, where* $V_i := \left\{v \in H^1(\Omega_i) : v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega\right\}$ *for* $i = 1, 2$.

To prove Theorem 1, we first need to prove

Lemma 1 *Denote by* $\gamma\colon V_1 \mapsto H_{00}^{\frac{1}{2}}(\Gamma)$ *the restriction map on* Ω_1*. There exists* $C_1 > 0$ *such that for all* $v_1 \in V_1$

$$\|\gamma v_1\|_{H_{00}^{\frac{1}{2}}(\Gamma)} \leq C_1 \|v_1\|_{V_1}. \tag{2}$$

Moreover, there exists $C_2 > 0$ *such that for all* $g \in H_{00}^{\frac{1}{2}}(\Gamma)$*, there exists* $\widetilde{v}_2$ *such that* $\widetilde{\gamma}\widetilde{v}_2 = g$*, and*

$$\|\widetilde{v}_2\|_{V_2} \leq C_2 \|g\|_{H_{00}^{\frac{1}{2}}(\Gamma)}, \tag{3}$$

where $\widetilde{\gamma}\colon V_2 \mapsto H_{00}^{\frac{1}{2}}(\Gamma)$ *denotes the restriction map on* Ω_2.

Proof The continuity and surjectivity of $\gamma\colon V_1 \mapsto H_{00}^{\frac{1}{2}}(\Gamma)$ comes from [8, Chapter 4,Th 2.3] and the definition of $H_{00}^{\frac{1}{2}}(\Gamma)$. Let $g \in H_{00}^{\frac{1}{2}}(\Gamma)$. The surjectivity of $\widetilde{\gamma}\colon V_2 \mapsto H_{00}^{\frac{1}{2}}(\Gamma)$ ensures the existence of $\widetilde{v}_2 \in V_2$ such that the equality $\widetilde{\gamma}\widetilde{v}_2 = g$ holds. Using then the open mapping theorem for $\widetilde{\gamma}$; see e.g. [2, Chapter 2,Th 2.6], we know that there exists $C_2 > 0$ such that Eq. (3) holds, which concludes the proof. □

Proof (of Theorem 1) Since g_{12}^0 satisfies the H^1-compatibility relations, we know by the Lax-Millgram Lemma that $u_1^0 \in V_1$ and $u_2^0 \in V_2$. Now it suffices to show that ψ_1^0 and ψ_2^0 are also in V_1 and V_2. We know that ψ_1^0 and ψ_2^0 satisfy

$$\begin{aligned}
\int_{\Omega_1} \nabla\psi_1^0 \nabla v_1 &= \int_\Gamma \frac{1}{2}\left(\frac{\partial u_1^0}{\partial n_1} + \frac{\partial u_2^0}{\partial n_2}\right) v_1, \quad \text{for all } v_1 \in V_1,\\
\int_{\Omega_2} \nabla\psi_1^0 \nabla v_2 &= \int_\Gamma \frac{1}{2}\left(\frac{\partial u_1^0}{\partial n_1} + \frac{\partial u_2^0}{\partial n_2}\right) v_2, \quad \text{for all } v_2 \in V_2.
\end{aligned}$$

In order to apply the Lax-Milgram Lemma, it suffices to show that $b_1(v_1) := \int_\Gamma \frac{1}{2}(\frac{\partial u_1^0}{\partial n_1} + \frac{\partial u_2^0}{\partial n_2})v_1$ and $b_2(v_2) := \int_\Gamma \frac{1}{2}(\frac{\partial u_1^0}{\partial n_1} + \frac{\partial u_2^0}{\partial n_2})v_2$, define a continuous map on V_1 and V_2. It suffices to prove this for b_1, and the same then holds for b_2. Indeed, we have for all $v_1 \in V_1$

$$\begin{aligned}
b_1(v_1) &= \left\langle \frac{\partial u_1^0}{\partial n_1}, v_1 \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H_{00}^{\frac{1}{2}}(\Gamma)} + \left\langle \frac{\partial u_2^0}{\partial n_2}, v_1 \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H_{00}^{\frac{1}{2}}(\Gamma)} \\
&= \left\langle \frac{\partial u_1^0}{\partial n_1}, \gamma v_1 \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H_{00}^{\frac{1}{2}}(\Gamma)} + \left\langle \frac{\partial u_2^0}{\partial n_2}, \widetilde{\gamma v}_2 \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H_{00}^{\frac{1}{2}}(\Gamma)} \\
&= -\int_{\Omega_1} f v_1 + \int_{\Omega_1} \nabla u_1^0 \nabla v_1 - \int_{\Omega_2} f \widetilde{v}_2 + \int_{\Omega_2} \nabla u_2^0 \nabla \widetilde{v}_2.
\end{aligned}$$

Hence, $|b_1(v_1)| \leq C(\|v_1\|_{V_1} + \|\widetilde{v}_2\|_{V_2}) \leq C(1 + C_1 C_2)\|v_1\|_{V_1}$. We deduce then that b_1 is a continuous map on V_1. In the same manner, we prove that b_2 is continuous on V_2, and by applying the Lax-Milgram Lemma, we obtain that ψ_1^0 and ψ_2^0 are in V_1 and V_2. Finally, we conclude that $g_{12}^1 \in H_{00}^{\frac{1}{2}}(\Gamma)$. Repeating then the same arguments, we conclude that $g_{12}^n \in H_{00}^{\frac{1}{2}}(\Gamma)$ for all $n \geq 0$. □

3 Existence of iterates for a 2 × 2 decomposition

We now study the well-posedness of NNM for a 2×2 decomposition, see Fig. 1 (right). The well-posedness in this case cannot be treated as in Section 2. In fact, let $\Gamma_{12} := \{0\} \times (-1, 0)$, $\Gamma_{23} := (0, 1) \times \{0\}$, $\Gamma_{34} := \{0\} \times (0, 1)$, $\Gamma_{41} := (-1, 0) \times \{0\}$ be the shared interfaces. Then $g_{12}^0 \in H^{\frac{1}{2}}(\Gamma_{12})$, $g_{23}^0 \in H^{\frac{1}{2}}(\Gamma_{23})$, $g_{34}^0 \in H^{\frac{1}{2}}(\Gamma_{34})$, $g_{41}^0 \in H^{\frac{1}{2}}(\Gamma_{41})$ is not sufficient for the first iterates to exist: the traces need to satisfy additional assumptions which are known as the H^1-compatibility relations ($\mathcal{CR}_1$) which are

$$\int_0^{\varepsilon} \left|g_{12}^0(-\sigma) - g_{41}^0(-\sigma)\right|^2 \frac{\mathrm{d}\sigma}{\sigma} < \infty, \quad \int_0^{\varepsilon} \left|g_{12}^0(-\sigma) - g_{23}^0(\sigma)\right|^2 \frac{\mathrm{d}\sigma}{\sigma} < \infty,$$

$$\int_0^{\varepsilon} \left|g_{23}^0(-\sigma) - g_{34}^0(-\sigma)\right|^2 \frac{\mathrm{d}\sigma}{\sigma} < \infty, \quad \int_0^{\varepsilon} \left|g_{34}^0(-\sigma) - g_{41}^0(\sigma)\right|^2 \frac{\mathrm{d}\sigma}{\sigma} < \infty,$$

for $\varepsilon > 0$ small enough; see [8, chapter 4,Th 2.3]. However, even if the initial iterates satisfy $\mathcal{CR}_1$, this does in general not hold for the following iterates. This explains why NNM is in general not well-defined for a 2×2 decomposition with a cross point. This is also the reason why NNM does not converge iteratively and has a convergence factor that grows logarithmically with respect to the mesh size after discretization as we mentioned in Section 1. We propose here to add a very specific local coarse space correction such that NNM becomes well-posed. Since the $\mathcal{CR}_1$ are global, it is not clear how to define a coarse space such that NNM with the additional coarse correction satisfies these conditions systematically. We thus look for a coarse space correction such the iterates are not in H^1 but rather in H^2. However, even the condition $g_{12}^0 \in H^{\frac{3}{2}}(\Gamma_{12})$, $g_{23}^0 \in H^{\frac{3}{2}}(\Gamma_{23})$, $g_{34}^0 \in H^{\frac{3}{2}}(\Gamma_{34})$, $g_{41}^0 \in H^{\frac{3}{2}}(\Gamma_{41})$ does not ensure the existence of H^2 iterates, and one needs to satisfy the so-called

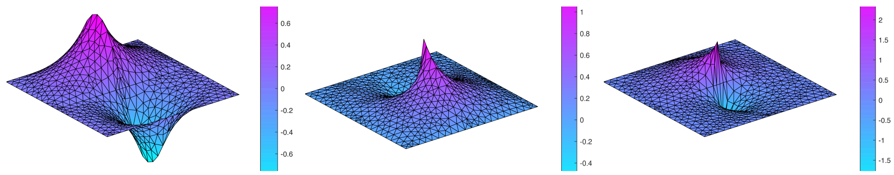

Fig. 2: Iterates 1,2,3 of NNM for the solution of Eq. (1) (note the different scale).

Algorithm 2: NNM for a 2×2 decomposition

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1. Initialize $g_{12}^0, g_{23}^0, g_{34}^0, g_{41}^0$.
2. For $n = 0, 1, \ldots$ until convergence
   - Compute $g_{12}^{n+\frac{1}{2}}, g_{23}^{n+\frac{1}{2}}, g_{34}^{n+\frac{1}{2}}, g_{41}^{n+\frac{1}{2}}$ using NNM (which used superscript $n+1$).
   - Find $\varphi_{12}, \varphi_{23}, \varphi_{34}, \varphi_{41} \in H^{\frac{3}{2}}$ in a given local coarse space, e.g. in (4), s.t.

$$g_{12}^{n+1} := g_{12}^{n+\frac{1}{2}} + \varphi_{12},\ \ g_{23}^{n+1} := g_{23}^{n+\frac{1}{2}} + \varphi_{23},$$
$$g_{34}^{n+1} := g_{34}^{n+\frac{1}{2}} + \varphi_{34},\ \ g_{41}^{n+1} := g_{41}^{n+\frac{1}{2}} + \varphi_{41},$$

   satisfiy by solving (5) the compatibility conditions

$$g_{12}^{n+1}(0) = g_{23}^{n+1}(0) = g_{34}^{n+1}(0) = g_{41}^{n+1}(0).$$

H^2-compatibility relations ($\mathcal{CR}_2$). This can also be illustrated numerically: we show in Fig. 2 the first iterates of NNM for Eq. (1) with $f = 1$ discretized by P_1 finite elements with a mesh size $h = 0.1$, and starting with smooth traces along the shared edges. The iterates in Fig. 2 show that NNM does not converge iteratively and has a discontinuity that forms at the origin. This discontinuity cannot happen if the iterates are in H^2 since their traces are in $H^{\frac{3}{2}}$, hence continuous at the cross point. One can show that this is the only problem that needs to be fixed in order to have a well-posed method. We thus propose to add a coarse space correction consisting of functions that are in $H^{\frac{3}{2}}$ on the common edges such that we enforce the continuity of the iterates at the origin. The NNM with this local coarse space correction is given in Algorithm 2. The next theorem ensures the well-posedness of Algorithm 2.

Theorem 2 *If* $(g_{12}^0, g_{23}^0, g_{34}^0, g_{41}^0) \in H^{\frac{3}{2}}(\Gamma_{12}) \times H^{\frac{3}{2}}(\Gamma_{23}) \times H^{\frac{3}{2}}(\Gamma_{34}) \times H^{\frac{3}{2}}(\Gamma_{41})$ *satisfy* $g_{12}^0(0) = g_{23}^0(0) = g_{34}^0(0) = g_{41}^0(0)$, *then Algorithm 2 is well-posed and for all* $n \geq 0$ *we have* $u_i^n \in H^2(\Omega_i) \cap V_i$, *where* $V_i := \left\{ v \in H^1(\Omega_i) : v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \right\}$ *for* $i = 1, \ldots, 4$.

We first state a result for the H^2 compatibility relations ($\mathcal{CR}_2$) which can be found in [8, chapter 4,Th 2.3].

Theorem 3 *Define the trace mapping*

$$\begin{aligned}\gamma : H^2(\Omega_1) \cap V_1 &\mapsto H^{\frac{3}{2}}(\Gamma_{12}) \times H^{\frac{1}{2}}(\Gamma_{12}) \times H^{\frac{3}{2}}(\Gamma_{41}) \times H^{\frac{1}{2}}(\Gamma_{41})\\ u &\mapsto \big(u(0,\cdot), \partial_x u(0,\cdot), u(\cdot,0), \partial_y u(\cdot,0)\big).\end{aligned}$$

Then $(g_{12}, h_{12}, g_{41}, h_{41}) \in \mathrm{Im}\,(\gamma)$ *iff* $g_x(0) = g_y(0)$ *and*

$$\int_0^{\varepsilon} \left|g'_{12}(-\sigma) - h_{41}(-\sigma)\right|^2 \frac{\mathrm{d}\sigma}{\sigma} < \infty, \quad \int_0^{\varepsilon} \left|g'_{41}(-\sigma) - h_{12}(-\sigma)\right|^2 \frac{\mathrm{d}\sigma}{\sigma} < \infty,$$

for $\varepsilon > 0$ *sufficiently small.*

From Theorem 3, we obtain the corollaries

Corollary 1 *Define the mapping*

$$\begin{aligned}\gamma_D : H^2(\Omega_1) \cap V_1 &\mapsto H^{\frac{3}{2}}(\Gamma_{12}) \times H^{\frac{3}{2}}(\Gamma_{41})\\ u &\mapsto (u(0,\cdot), u(\cdot,0)).\end{aligned}$$

Then $(g_{12}, g_{41}) \in \mathrm{Im}\,(\gamma_D)$ *iff* $g_{12}(0) = g_{41}(0)$.

Proof In fact, it suffices to define $h_{41} := g'_{12}(\sigma) \in H^{\frac{1}{2}}(\Gamma_{41})$ and $h_{12} := g'_{41}(\sigma) \in H^{\frac{1}{2}}(\Gamma_{12})$ and apply Theorem 3 to $(g_{12}, h_{12}, g_{41}, h_{41})$. □

Corollary 2 *Define the mapping*

$$\begin{aligned}\gamma_N : H^2(\Omega_1) \cap V_1 &\mapsto H^{\frac{1}{2}}(\Gamma_{12}) \times H^{\frac{1}{2}}(\Gamma_{41})\\ u &\mapsto \big(\partial_x u(0,\cdot), \partial_y u(\cdot,0)\big).\end{aligned}$$

Then γ_N *is onto.*

Proof Here again, it suffices to define $g_{12} := -\psi(\sigma)\int_{\sigma}^{0} h_{41}(\sigma')\ d\sigma'$ and $g_{41} := -\psi(\sigma)\int_{\sigma}^{0} h_{12}(\sigma')\ d\sigma'$, where $\psi(\sigma) \in C^{\infty}[-1,0]$ such that $\psi(\sigma) = 1$ on $(-\varepsilon, 0]$ and $\psi(\sigma) = 0$ on $[-1, -2\epsilon)$, and apply Theorem 3 to $(g_{12}, h_{12}, g_{41}, h_{41})$. □

Proof (of Theorem 2) We start by showing that $u_i^0 \in H^2(\Omega_i) \cap V_i$ for $i = 1, \ldots, 4$. We prove it for u_1^0 and the proof for the remaining u_i^0 is exactly the same. We have that $g_{12}^0 \in H^{\frac{3}{2}}(\Gamma_{12})$ and $g_{14}^0 \in H^{\frac{3}{2}}(\Gamma_{14})$ and they satisfy $g_{12}^0(0) = g_{14}^0(0)$, hence using Corollary 1 we know that there exists $w_1 \in H^2(\Omega_1) \cap V_1$ such that $\widetilde{u}_1 := u_1^0 - w_1 \in H_0^1(\Omega_1)$ is the solution of the variational problem

$$\int_{\Omega_1} \nabla\widetilde{u}_1 \nabla v = \int_{\Omega_1} (f + \Delta w_1) v, \text{ for all } v \in H_0^1(\Omega_1),$$

which using the result in [6, Chapter 3, p 147] has a unique solution in $H^2(\Omega_1) \cap H_0^1(\Omega_1)$, and it follows that $u_1^0 = \widetilde{u}_1 + w_1 \in H^2(\Omega_1) \cap V_1$. In the same manner we can show that u_2^0, u_3^0, u_4^0 are in $H^2(\Omega_2) \cap V_2$, $H^2(\Omega_3) \cap V_3$ and $H^2(\Omega_4) \cap V_4$. Now, since

$$\frac{\partial \psi_1^0}{\partial n_1}_{|\Gamma_{12}} = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial n_1} + \frac{\partial u_2^0}{\partial n_2}\right) \in H^{\frac{1}{2}}(\Gamma_{12}), \quad \frac{\partial \psi_1^0}{\partial n_1}_{|\Gamma_{14}} = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial n_1} + \frac{\partial u_4^0}{\partial n_4}\right) \in H^{\frac{1}{2}}(\Gamma_{14}),$$

we know by Corollary 2 that there exists again a function $\widetilde{w}_1 \in H^2(\Omega_1) \cap V_1$ such that $\widetilde{\psi}_1 := \psi_1^0 - \widetilde{w}_1 \in H^2(\Omega_1) \cap V_1$ is the solution of the variational problem

$$\int_{\Omega_1} \nabla\widetilde{\psi}_1 \nabla v = \int_{\Omega_1} \Delta\widetilde{w}_1 v \text{ for all } v \in V_1,$$

which has a solution $\widetilde{\psi}_1 \in H^2(\Omega_1) \cap V_1$, hence $\psi_1^0 = \widetilde{\psi}_1 + \widetilde{w}_1 \in H^2(\Omega_1) \cap V_1$. The same conclusion can be drawn for ψ_2^0, ψ_3^0, ψ_4^0 using the same reasoning. It follows then that $g_{12}^1, g_{23}^1, g_{34}^1, g_{41}^1$ are in $H^{\frac{3}{2}}(\Gamma_{12})$, $H^{\frac{3}{2}}(\Gamma_{23})$, $H^{\frac{3}{2}}(\Gamma_{34})$ and $H^{\frac{3}{2}}(\Gamma_{41})$ respectively. Since the coarse functions $\phi_{12}, \phi_{23}, \phi_{34}, \phi_{41}$ in Algorithm 2 are chosen such that $g_{12}^1(0) = g_{23}^1(0) = g_{34}^1(0) = g_{41}^1(0)$, we can apply again Corollary 1. We proceed again as before to prove that the next iterates are well defined, and so on. Finally, we conclude that Algorithm 2 is well defined with iterates $u_i^n \in H^2(\Omega_i) \cap V_i$ for $i = 1, \ldots, 4$. This finishes the proof. □

It remains to choose the coarse basis, and a first idea is to use linear functions,

$$\begin{aligned} \varphi_{12} &:= \alpha_{12}(1+y),\ \varphi_{23} := \alpha_{23}(1-x), \\ \varphi_{34} &:= \alpha_{34}(1-y),\ \varphi_{41} := \alpha_{41}(1+x), \end{aligned} \tag{4}$$

where the coefficients α are determined using the pseudo inverse,

$$\begin{bmatrix} \alpha_{12} \\ \alpha_{23} \\ \alpha_{34} \\ \alpha_{41} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}^{\dagger} \begin{bmatrix} g_{23}^{n+\frac{1}{2}}(0) - g_{12}^{n+\frac{1}{2}}(0) \\ g_{34}^{n+\frac{1}{2}}(0) - g_{23}^{n+\frac{1}{2}}(0) \\ g_{41}^{n+\frac{1}{2}}(0) - g_{34}^{n+\frac{1}{2}}(0) \end{bmatrix}, \tag{5}$$

i.e. we compute the smallest correction to obtain continuous traces at the cross point. The plots in Fig. 3 (top) show that this local linear coarse correction is sufficient to obtain a convergent iterative method which does not form a singularity at the cross point any more. To investigate how the convergence depends on the basis chosen, we now use exponentially decaying functions of the form $e^{-\mu x}$ and $e^{-\mu y}$. Choosing $\mu := 3$, we obtain the results shown in Fig. 3 (bottom): convergence is much faster than with the linear coarse basis; see also Fig. 4 (left) for a comparison. The number of iterations required for NNM with our local coarse correction to reach a tolerance of 10^{-6} for mesh size $h = 0.4, 0.2, 0.1, 0.05, 0.03$ is 9, 15, 19, 23, 26 with the linear coarse functions, and 7, 7, 7, 7, 10 with the exponential ones. We finally test Algorithm 2 with Krylov acceleration (GMRES), for the case of nine cross points and the exponential coarse basis functions: the result is shown in Figure 3 (right), and we see that the fact to be well posed in function space leads to a more effective preconditioner.

We thus answered an interesting question in this short manuscript, namely why NNM only appears in the literature for two subdomains at the continuous level, and

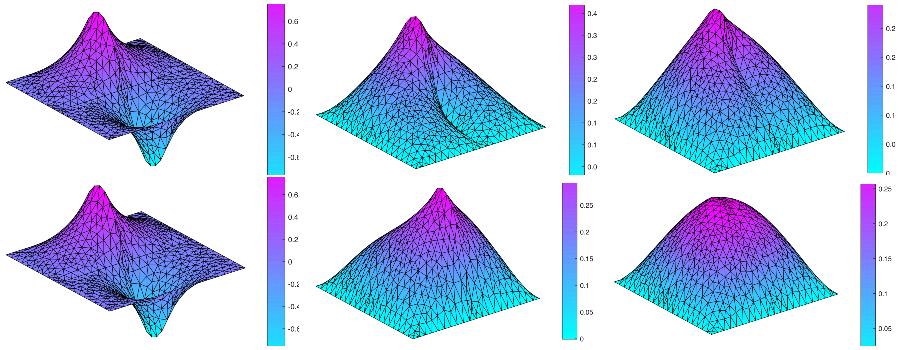

Fig. 3: Iterates 1,2,3 of Algorithm 2 for Eq. (1) using a linear coarse basis (top) and an exponentially decaying coarse basis (bottom)

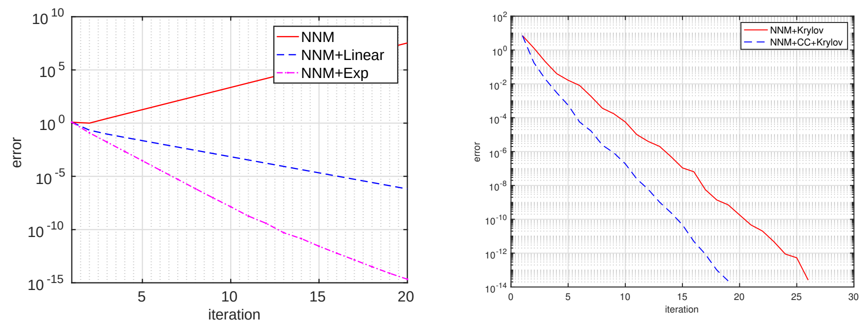

Fig. 4: Error curves of NNM with and without coarse correction for one cross point (left), and with Krylov acceleration for nine cross points (right)

otherwise only at the discrete level as a preconditioner: it is because it is not well posed at the continuous level in the many subdomain case with cross points. We then showed that a specific local coarse space can make NNM well posed at the continuous level, which both leads to a convergent iterative NNM algorithm, and a better preconditioner in the presence of cross points. We are currently investigating if coarse basis functions exist for which we can prove that the convergence factor of NNM becomes independent of the mesh size h like for 2 subdomains.

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