Happy 25^{th} Anniversary DDM! ... But How Fast Can the Schwarz Method Solve Your Logo?

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1 The ddm logo problem and the Schwarz method

"Vous n'avez vraiment rien à faire"![1] This was the smiling reaction of Laurence Halpern when the first author told her about our wish to accurately estimate the convergence rate of the Schwarz method for the solution of the ddm logo[2], see Figure 1 (left). Anyway, here we are: to honor the 25^{th} anniversary of the domain decomposition conference, we study the convergence rate of the alternating Schwarz method for the solution of Laplace's equation defined on the ddm logo. This method was invented by H.A. Schwarz in 1870 [12] for the solution of the Laplace problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{1}$$

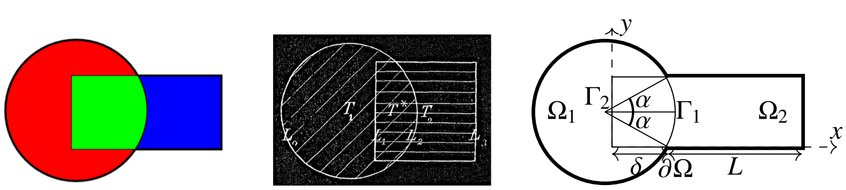

Fig. 1: Left: ddm logo. Center: Original drawing of Schwarz from 1870 [12]. Right: Geometric parametrization of the ddm logo.

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[1] "You have really nothing to do"!

[2] This logo was created by Benjamin Stocker, a friend for over 30 years of the second author and a computer scientist and web designer for SolNet.

Here g is a sufficiently regular function and Ω is the ddm logo, obtained from the union of a disc Ω_1 and a rectangle Ω_2, as historically considered by Schwarz [12]; see Figure 1 (center). In this paper, we assume that Ω_1 is a unit disc, and Ω_2 has length $\delta + L$ and height $2\cos\alpha$. Here, δ, L and α are used to parametrize Ω; see Figure 1 (right). In particular, δ and L measure the overlapping and non-overlapping parts of Ω_2, and α is the angle that parametrizes the interface $\Gamma_1 := \overline{\partial\Omega_1 \cap \Omega_2}$. The other interface $\Gamma_2 := \overline{\partial\Omega_2 \cap \Omega_1}$ is clearly parametrized by δ and α, and it is composed by three segments whose vertices are $(\delta, 0)$, $(0, 0)$, $(0, 2\sin\alpha)$, and $(\delta, 2\sin\alpha)$. To avoid meaningless geometries (e.g., $\Omega_2 \setminus \Omega_1$ becomes a disjoint set), we assume that δ and α are non-negative and satisfy $\delta < 2\cos\alpha$.

In error form, the classical alternating Schwarz method for the solution to (1) is

$$\begin{aligned}
\Delta e_1^n &= 0 \quad \text{in } \Omega_1, & \Delta e_2^n &= 0 \quad \text{in } \Omega_2,\\
e_1^n &= 0 \quad \text{on } \partial\Omega \cap \overline{\Omega}_1, & e_2^n &= 0 \quad \text{on } \partial\Omega \cap \overline{\Omega}_2,\\
e_1^n &= e_2^{n-1} \text{ on } \Gamma_1, & e_2^n &= e_1^n \text{ on } \Gamma_2,
\end{aligned} \tag{2}$$

where the left subproblem is a Laplace problem on the disc and the right one on the rectangle. Assuming that one begins with a sufficiently regular initial guess e^0, then solving iteratively (2) one obtains the sequence $(e_1^n)_{n\in\mathbb{N}^+}$ of errors on the disc Ω_1 and the sequence $(e_2^n)_{n\in\mathbb{N}^+}$ of errors on the rectangle Ω_2. The functions e_1^n and e_2^n are continuous in their (open) domain, but can have jumps at the two points where $\partial\Omega_1$ and $\partial\Omega_2$ intersect, except if the initial guess satisfies the boundary conditions. How fast do these two sequences converge to zero? The estimate of the convergence rate of the Schwarz method for this particular geometry is not easy. Over the course of time, different analysis techniques have been proposed to study the classical Schwarz method: maximum principle analysis, see, e.g., [12, 10, 3], Fourier analysis, see, e.g., [6, 2], variational analysis, see, e.g., [9, 4], and stochastic analysis [10]. In the spirit of this historical manuscript, we estimate the convergence rate by using tools that are considered "classical" in domain decomposition methods: maximum principle, the Riemann mapping theorem, the Poisson kernel, and the Schwarz-Christoffel mapping[3]. However, we wish to remark that, to the best of our knowledge the results presented in this work are new, and that the techniques used to prove them can be in principle used to study other domains with complicated geometries, whose subdomains can be mapped into circles and (semi-)infinite rectangles.

2 Convergence analysis

We begin our analysis noticing that maximum principle arguments, as done in [3, Theorem 7], allow us to obtain the following convergence result; see also [8].

[3] The Schwarz-Christoffel mapping was discovered independently by Christoffel in 1867 [1] and Schwarz in 1869 [11]; see [5] for a review.

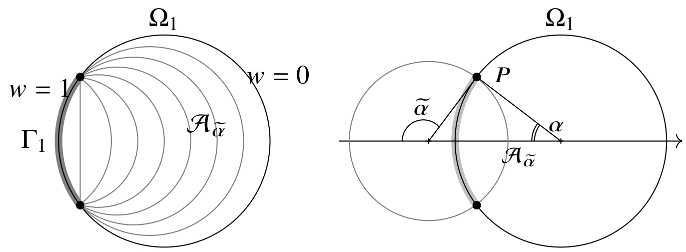

Fig. 2: Left: Level sets of w. Right: Geometric parametrization of a level set of w.

Theorem 1 (Convergence of the Schwarz method) *The Schwarz method* (2) *converges geometrically to the solution of* (1) *in the sense that there exists a convergence factor $\rho < 1$ such that*[4]

$$\max_{j=1,2} \|e_j^n\|_{\infty,\overline{\Omega}_j} \leq \rho^n \max_{j=1,2} \|e_j^0\|_{\infty,\overline{\Omega}_j}, \tag{3}$$

where $\rho = \big(\sup_{\Gamma_1} v_2\big)\big(\sup_{\Gamma_2} v_1\big)$, with v_j solving for $j = 1, 2$ the problem

$$\Delta v_j = 0 \text{ in } \Omega_j, \quad v_j = 1 \text{ on } \Gamma_j, \quad v_j = 0 \text{ on } \partial\Omega \cap \overline{\Omega}_j. \tag{4}$$

We thus have to study the two functions v_1 and v_2. Notice also the two sup in the definition of ρ could be replaced by max, as we see in what follows. We begin by studying v_1 and recalling the following result, which is proved in [3] by the Riemann mapping theorem and the Poisson kernel formula.

Lemma 1 *Problem* (4) *for $j = 1$ has a unique solution w which is harmonic in Ω_1 and constant on arcs of circles $\mathcal{A}_{\widetilde{\alpha}}$ passing through the two extrema of Γ_1 (see Fig. 2, left) and parametrized by angles $\widetilde{\alpha}$ between the horizontal line and the line that connects the center of the arc $\mathcal{A}_{\widetilde{\alpha}}$ to the point P (see Fig. 2, right), i.e.*

$$w(x, y) = \frac{\widetilde{\alpha} - \alpha}{\pi} \quad \forall (x, y) \in \mathcal{A}_{\widetilde{\alpha}}, \tag{5}$$

with $0 < w(x, y) < 1$ for any $(x, y) \in \Omega_1$ and $\alpha \leq \widetilde{\alpha} < \pi$. Moreover, it holds that $w(x, y) = \vartheta/\pi$ for all $(x, y) \in \mathcal{A}_{\widetilde{\alpha}}$, where ϑ is the angle between the tangent to $\mathcal{A}_{\widetilde{\alpha}}$ in P and the tangent of $\partial\Omega_2$ in P; see Fig. 3 (left).

Lemma 1 allows us to identify the sup of v_1 on Γ_2 with the max, that we estimate:

Lemma 2 (Estimated convergence factor on the disc) *Consider the function v_1 solving* (4) *for $j = 1$. It holds that*

$$\max_{\Gamma_2} v_1 = \begin{cases} \frac{1}{2} - \frac{\alpha}{\pi} & \text{if } \delta \geq \sin\alpha, \\ 1 - \frac{1}{\pi}\Big[\alpha + \arcsin\big(\frac{2\delta\sin\alpha}{\delta^2 + \sin^2\alpha}\big)\Big] & \text{if } \delta < \sin\alpha. \end{cases} \tag{6}$$

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[4] The convergence rate is $-\log\rho$, see [7, Section 11.2.5].

Proof Lemma 1 implies that v_1 decays monotonically in Ω_1, in the sense that, according to formula (5) as $\widetilde{\alpha}$ decreases, the arc $\mathcal{A}_{\widetilde{\alpha}}$ is closer to $\partial\Omega_1 \setminus \Gamma_1$, and $v_1|_{\mathcal{A}_{\widetilde{\alpha}}}$ decreases monotonically. Therefore, to estimate the maximum of v_1 on Γ_2 we must find the arc that intersects Γ_2 on which v_1 has the highest value. To do so, we distinguish two cases: $\delta \geq \sin\alpha$ and $\delta < \sin\alpha$.

If $\delta \geq \sin\alpha$, then there exists an arc $\mathcal{A}$ (a semi-circle) that lies in the closure of the overlapping domain $\Omega_1 \cap \Omega_2$ and that is tangent to Γ_2 in the two points $\partial\Omega_1 \cap \partial\Omega_2$. Notice that if $\delta = \sin\alpha$, then $\mathcal{A}$ intersects Γ_2 also in the midpoint of its vertical segment. By the monotonicity of v_1, $\mathcal{A}$ is the arc intersecting Γ_2 on which v_1 attains the highest value. Since $\mathcal{A}$ is tangent to Γ_2 in both the points in $\partial\Omega_1 \cap \partial\Omega_2$, a simple geometric argument and the formula $v_1(x,y) = \vartheta/\pi$ allow us to obtain that $\max_{\Gamma_2} v_1 = \max_{\partial\Omega_1\cap\partial\Omega_2} v_1 = \frac{1}{2} - \frac{\alpha}{\pi}$.

Consider now that $\delta < \sin\alpha$. In this case, the monotonicity of v_1 implies that the arc that intersects Γ_2 on which v_1 attains the highest value is the one that passes through the two points in $\partial\Omega_1 \cap \partial\Omega_2$ and the midpoint of the vertical segment of Γ_2. Once this arc is found, direct calculations using simple geometric arguments and the formula $v_1(x,y) = \vartheta/\pi$ allow us to obtain the claim. $\square$

Next, we focus on the function v_2 defined on the rectangle Ω_2. We begin recalling the following result proved in [3].

Lemma 3 *Let m denote the Möbius transformation that maps the half-plane $\mathcal{P} := \mathbb{R} \times \mathbb{R}^+$ onto the unit disc Ω_1. Recall the function w defined in Lemma 1. Then the function $\widehat{w}(\xi,\eta) := w(m(\xi,\eta))$ for all $(\xi,\eta) \in \mathcal{P}$ is harmonic in $\mathcal{P}$, it satisfies the boundary conditions $\widehat{w}(\xi,\eta) = 1$, for all (ξ,η) on the segment $m^{-1}(\Gamma_1)$ that lies on the horizontal line, and $\widehat{w}(\xi,\eta) = 0$, for all $(\xi,\eta) \in (\mathbb{R}\times\{0\}) \setminus m^{-1}(\Gamma_1)$. Moreover $\widehat{w}$ is constant on arcs of circles passing through the extrema of $m^{-1}(\Gamma_1)$. Let Q be one of the two extrema of $m^{-1}(\Gamma_1)$ and let ϑ be the external angle between the tangent to one of these arcs, denoted by $\mathcal{A}_\vartheta$, in Q and the horizontal axis, then $\widehat{w}|_{\mathcal{A}_\vartheta} = \vartheta/\pi$.*

Notice that Lemma 3 allows us to identify the sup of v_2 on Γ_1 with the max. We can then prove the following lemmas.

Lemma 4 *Consider a semi-infinite strip Ω_2^∞ obtained by extending Ω_2 from the right to infinity and recall the half-plane $\mathcal{P}$ from Lemma 3.*

(a) *The Schwarz-Christoffel function that maps the semi-infinite strip onto the half-plane, denoted as $g : \Omega_2^\infty \to \mathcal{P}$, is given by*

$$g(x,y) = \begin{bmatrix} \cosh(x\frac{\pi}{2\sin\alpha})\cos(y\frac{\pi}{2\sin\alpha}) \\ \sinh(x\frac{\pi}{2\sin\alpha})\sin(y\frac{\pi}{2\sin\alpha}) \end{bmatrix}.$$

Moreover, g maps the interface Γ_2 onto the set $[g(\delta,0), g(\delta,2\sin\alpha)] \times \{0\}$.

(b) *Let v_2^∞ be a harmonic function in Ω_2^∞ such that $v_2^\infty = 1$ on Γ_2, $v_2^\infty = 0$ on $\partial\Omega_2^\infty \setminus \Gamma_2$ and $v_2^\infty(x,y) \to 0$ as $x \to \infty$. Let v_2 be the solution of* (4) *for $j = 2$. Then $v_2(x,y) < v_2^\infty(x,y)$ for all $(x,y) \in \Omega_2$.*

Proof Part (a): Recall the Schwarz-Christoffel function $f(\zeta) = C + K \operatorname{arcosh}(\zeta)$ for $\zeta \in \mathbb{C}$, where C and K are two constants in $\mathbb{C}$. It is well known that f maps the half-plane into any semi-infinite strip. Therefore, it is sufficient to determine the constants C and K by requiring that $f(1) = 0$ and $f(-1) = i2\sin\alpha$, where i is the imaginary unit. These conditions imply that the two corners of Γ_2 are mapped onto the points $\{-1, 1\}$ that lie on the real line in $\mathbb{C}$. We get $C = 0$ and $K = 2(\sin\alpha)/\pi$. Hence, $f(\zeta) = (2(\sin\alpha)/\pi)\operatorname{arcosh}(\zeta)$. Now, for any $z = x + iy = f(\zeta)$, we have that $\zeta = \cosh((x+iy)\pi/(2\sin\alpha))$. The function g is then obtained by using the formula $\cosh(a(x+iy)) = \cosh(ax)\cos(ay) + i\sinh(ax)\sin(ay)$, with $a = \pi/(2\sin\alpha)$. The last claim follows by the fact that $(g(x,0))_2 = (g(x, 2\sin\alpha))_2 = 0$ for any x and $(g(0,y))_2 = 0$ for any y and the properties of cosh and cos.

Part (b): Consider the function $p := v_2^\infty|_{\overline{\Omega_2}}$. Clearly p is harmonic in Ω_2 and it satisfies $p = v_2$ on Γ_2, $p = 0$ on $\partial\Omega_2 \cap \Omega_2^\infty$. However, by the maximum principle $p(x,y) > 0$ for all $(x,y) \in \partial\Omega_2 \setminus \Omega_2^\infty$. We can then decompose p as $p = v_2 + \widetilde{p}$, where $\widetilde{p}$ is harmonic in Ω_2, $\widetilde{p} = 0$ on $\partial\Omega_2 \setminus \Omega_2^\infty$ and $\widetilde{p} = p$ on $\partial\Omega_2 \cap \Omega_2^\infty$. By the maximum principle $\widetilde{p}(x,y) > 0$ for all $(x,y) \in \Omega_2$. Hence, $v_2^\infty|_{\overline{\Omega_2}}(x,y) = p(x,y) = v_2(x,y) + \widetilde{p}(x,y) > v_2(x,y)$ for all $(x,y) \in \Omega_2$ and the claim follows. □

Next, we parametrize the arc Γ_1 by an angle $\varphi \in [0,\pi]$ such that every point P on Γ_1 can be obtained as

$$P(\varphi) = \begin{bmatrix} x_P(\varphi) \\ y_P(\varphi) \end{bmatrix} := \begin{bmatrix} \delta + r(\varphi)\sin\varphi \\ \sin\alpha - r(\varphi)\cos\varphi \end{bmatrix},$$

where $r(\varphi) = -\cos\alpha\sin\varphi + \sqrt{\sin^2\alpha + \cos^2\alpha\sin^2\varphi}$. Using the function g in Lemma 4, we can map the arc Γ_1 into the half-plane and define $\widehat{\Gamma}_1 := g(\Gamma_1) = \{(\xi,\eta) \in \mathcal{P} : (\xi,\eta) = g(x_P(\varphi), y_P(\varphi)) \text{ for } \varphi \in [0,\pi]\}$. Notice that $\widehat{\Gamma}_1$ is a curve in the half-plane $\mathcal{P}$ and intersects the horizontal axis in the two points $g(\delta,0)$ and $g(\delta, 2\sin\alpha)$. We consider the following conjecture.

Conjecture Consider the arc Γ of the circle passing through the points $g(\delta,0)$ and $g(\delta, 2\sin\alpha)$ and that intersects $\widehat{\Gamma}_1$ in $g(x_P(\pi/2), y_P(\pi/2))$. Then for any $\delta \geq 0$ and $\alpha \geq 0$ such that $\delta < 2\cos\alpha$, Γ is contained in the closure of the domain whose boundary is $\widehat{\Gamma}_1 \cup ([g(\delta,0), g(\delta,2\sin\alpha)] \times \{0\})$. □

A pictorial representation of Conjecture 1 is given in Fig. 3 (right). Notice that we have observed by direct numerical evaluation that Conjecture 1 always holds. We can then prove the following result.

Lemma 5 (Estimated convergence factor on the rectangle) *Let Conjecture 1 hold and recall the function v_2^∞ in Lemma 4. Then*

$$\max_{\Gamma_1} v_2 \leq v_2^\infty(x_P(\pi/2), y_P(\pi/2)) = \frac{1}{2} - \frac{1}{\pi}\arcsin(\kappa(\delta,\alpha)), \tag{7}$$

where

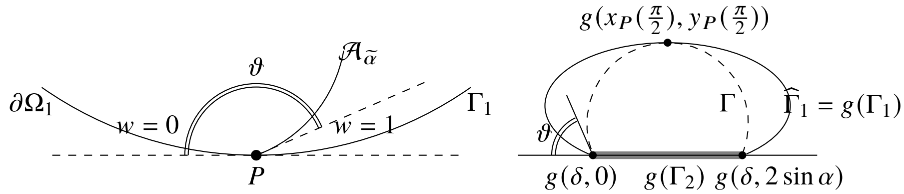

Fig. 3: Left: Geometry used in Lemma 1. Right: Geometric representation of Conjecture 1: the black solid curve represents $\widehat{\Gamma}_1 = g(\Gamma_1)$ and the black dashed arc of circle is Γ that passes through the points $g(\delta, 0)$, $g(x_P(\frac{\pi}{2}), y_P(\frac{\pi}{2}))$ and $g(\delta, 2\sin\alpha)$. The set $g(\Gamma_2)$ is $[g(\delta, 0), g(\delta, 2\sin\alpha)]\times\{0\}$ and is marked in grey.

$$\kappa(\delta,\alpha) = \frac{\sinh^2\left(\frac{\pi(1+\delta-\cos\alpha)}{2\sin\alpha}\right) - \cosh^2\left(\frac{\pi\delta}{2\sin\alpha}\right)}{\sinh^2\left(\frac{\pi(1+\delta-\cos\alpha)}{2\sin\alpha}\right) + \cosh^2\left(\frac{\pi\delta}{2\sin\alpha}\right)}. \tag{8}$$

Proof Lemma 4 (b) implies that $v_2(x, y) \leq v_2^\infty(x, y)$ for all $(x, y) \in \Gamma_1$. Using the function g in Lemma 4 (a), we define $w^\infty(\xi, \eta) := v_2^\infty(x, y)$ for all $(x, y) \in \Omega_2^\infty$ and $(\xi, \eta) = g(x, y)$. Notice that $\max_{\Gamma_1} v_2^\infty = \max_{\widehat{\Gamma}_1} w^\infty$. The function w^∞ is harmonic in $\mathcal{P}$ and satisfies the conditions $w^\infty(\xi, 0) = 1$ for $\xi \in [g(\delta, 0), g(\delta, 2\sin\alpha)]$ and $w^\infty(\xi, 0) = 0$ for $\xi \in \mathcal{R} \setminus [g(\delta, 0), g(\delta, 2\sin\alpha)]$. Hence, by using Lemma 3 we obtain that the function w^∞ is constant on arcs of circles $\mathcal{A}_\vartheta$ passing through the two points $g(\delta, 0)$ and $g(\delta, 2\sin\alpha)$. Moreover, the value of w^∞ on these arcs is given by $\widehat{w}|_{\mathcal{A}_\vartheta} = \vartheta/\pi$, where ϑ is defined in Lemma 3. This means that as ϑ decreases, the arc $\mathcal{A}_\vartheta$ becomes larger and the value $\widehat{w}|_{\mathcal{A}_\vartheta}$ decreases monotonically. Therefore, the value $\max_{\widehat{\Gamma}_1} w^\infty$ is given by the value of w^∞ on the arc Γ of the circle that passes through the two points $g(\delta, 0)$ and $g(\delta, 2\sin\alpha)$, that intersects $\widehat{\Gamma}_1$, and that corresponds to the largest ϑ; see Fig. 3 (right). By Conjecture 1, Γ is the arc of the circle that passes through the point $g(x_P(\frac{\pi}{2}), y_P(\frac{\pi}{2}))$. Notice that Γ is represented by a dashed line in Fig. 3 (right). Hence, $\max_{\widehat{\Gamma}_1} w^\infty = w^\infty(g(x_P(\pi/2), y_P(\pi/2)))$. The result follows by the formula $\widehat{w}|_{\mathcal{A}_\vartheta} = \vartheta/\pi$ and a direct calculation based on geometric arguments to obtain the angle ϑ characterizing Γ (see Fig. 3, right). □

We are now ready to prove our estimate of the convergence rate of the Schwarz method for the ddm logo.

Theorem 2 (Estimated convergence factor on the ddm logo) *The Schwarz method* (2) *converges in the sense of* (3)*, where*

$$\rho \leq \begin{cases} \left(\frac{1}{2} - \frac{1}{\pi}\arcsin(\kappa(\delta,\alpha))\right)\left(\frac{1}{2} - \frac{\alpha}{\pi}\right) & \text{if } \delta \geq \sin\alpha, \\ \left(\frac{1}{2} - \frac{1}{\pi}\arcsin(\kappa(\delta,\alpha))\right)\left[1 - \frac{1}{\pi}\left(\alpha + \arcsin\left(\frac{2\delta\sin\alpha}{\delta^2+\sin^2\alpha}\right)\right)\right] & \text{if } \delta < \sin\alpha, \end{cases} \tag{9}$$

with $\kappa(\delta,\alpha)$ *given in* (8).

Proof Recalling Theorem 1 and the formula $\rho = \left(\max_{\Gamma_1} v_2\right)\left(\max_{\Gamma_2} v_1\right)$, the estimate (9) follows using Lemmas 2 and 5. □

The estimated convergence factors obtained in Lemmas 2 and 5 and Theorem 2 are shown in Fig. 4. In particular, in Fig. 4 (left) the function (6) is shown. Fig. 4 (center)

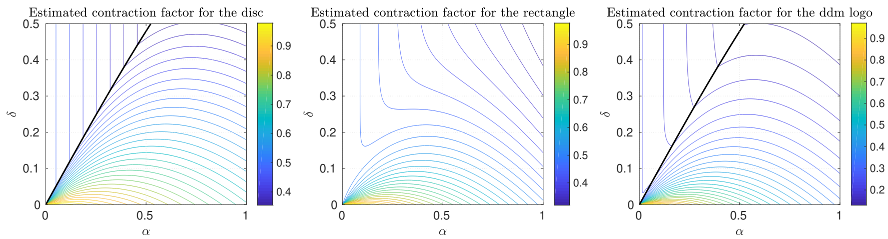

Fig. 4: Left: Values of $\max_{\Gamma_2} v_1$ as function of α and δ given in (6). Center: Estimate of $\max_{\Gamma_1} v_2$ given in (7). Right: Estimated convergence factor for the ddm logo given in (9).

represents the upper bound (7). Fig. 4 (right) shows the estimated convergence factor (9) for the ddm logo. The black curves in Fig. 4 (left and right) represent the function $\sin\alpha$ separating two regions according to (6) and (9).

3 Numerical experiments

We now compare our theoretical estimates with the numerical convergence behavior. We discretize the ddm logo by linear finite elements using Freefem[5]. Two finite element discretizations of the ddm logo are shown in Fig. 5. In order to accurately describe the behavior of the (continuous) Schwarz method, we used however in our experiments much finer meshes than the ones shown in Fig. 5. We solve problem (1) for a fixed $L = 2$ and different values of the parameters α and δ. Our results are shown in Fig. 6, where the decay of the error with respect to the number of iterations is represented. In particular, our theoretical estimates (solid lines) are compared with

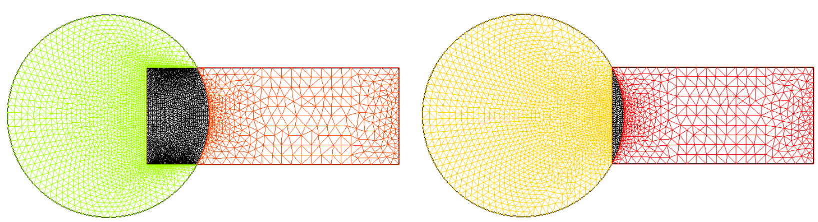

Fig. 5: Examples of finite element discretizations of the ddm logo obtained by Freefem. Left: $\alpha = 0.5$, $\delta = 0.5$ and $L = 2$. Right: $\alpha = 0.5$, $\delta = 0$ and $L = 2$.

[5] This finite-element code was designed by the first author and Felix Kwok for the DD Summer school organized by the second author at the University of Nice, June 19-21, 2018, and it was also used by the second author in his plenary lecture at the 25^{th} domain decomposition conference.

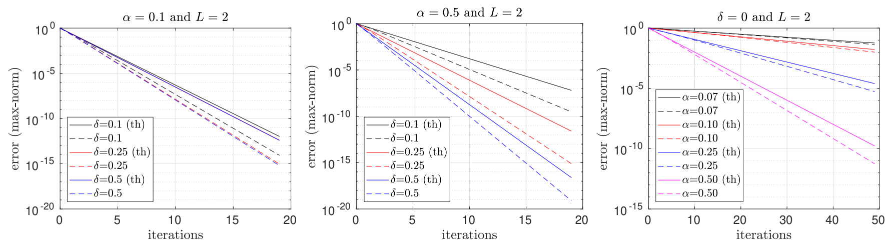

Fig. 6: Theoretical (solid line) and numerical (dashed line) convergences.

the numerical errors (dashed lines). The first two pictures in Fig. 6 (left and center) correspond to $\alpha = 0.1$ and $\alpha = 0.5$ and different values of $\delta > 0$. Notice that, even though our theoretical estimate is an upper bound for the true convergence rate, it describes very well the behavior of the method for different parameters. To study the sharpness of our results, we consider also the case with $\delta = 0$ and different values of α. The results of these experiments are shown in Fig. 6 (right), where one can clearly see that our results are very sharp for $\delta = 0$ and small values of α. The reason for this behavior is that our results are based on Theorem 1, where few estimates are present in the proof; see [3]. These are sharper when the dominating error is localized near the two points in $\partial\Omega_1 \cap \partial\Omega_2$ and the overlap is small.

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