Nonoverlapping Domain Decomposition Methods for Saddle Point Problems

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1 Introduction

Domain decomposition methods have been applied extensively for the saddle point problems arising from the mixed finite element discretizations. Overlapping methods are studied by many researchers such as [15, 6, 7, 4, 3, 1]. Some of these algorithms can be applied for both continuous and discontinuous pressure discretizations, however, the convergence analyses are available only for the methods with discontinuous pressure, to the best of our knowledge.

Most nonoverlapping domain decomposition methods are based on the benign subspace idea which is successfully used by [21] for the Stokes problem, followed by [10, 16, 18, 24, 26, 11, 22, 14, 12] for different nonoverlapping domain decomposition algorithms and different saddle point problems. In this approach, the original saddle point problems can be reduced to positive definite problems in the benign subspace with subdomain interface velocity and constant subdomain pressure variables. Therefore a conjugate gradient method (CG) can be used to accelerate the convergence. Most above-mentioned applications and analyses require discontinuous pressures to be used in the discretization. Several domain decomposition algorithms allow the use of continuous pressures such as [23, 2, 13], but the convergence rate analyses of those approaches are not available. [17, 27, 28] have proposed and analyzed a FETI-DP algorithm for solving incompressible Stokes equation, which allowed the use of both discontinuous and continuous pressures in the discretization. There, the Lagrange multipliers are introduced to enforce the continuity of the velocity variables across the subdomain interface. Recently, this FETI-DP algorithm has been applied to almost incompressible elasticity with isogeometric discretization by [32].

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In this paper, we show for both BDDC and FETI-DP algorithms how the original saddle point problems can be reduced to positive definite problems using either primal or dual variable approaches, outline their analyses, and make the connections between these two approaches.

The rest of this paper is organized as follows. The saddle problems are described in Section 2. In Section 3, the domain decomposition is introduced and the original system is reduced to Schur complements or a system of the Lagrange multiples and pressure. The positive definite formulations are discussed in Section 4 and the condition number estimates are outlined in Section 5. Finally, we summarize some differences and connections of these two methods in Section 6.

2 Problem setting

We consider the following saddle point problem: find $\mathbf{u}_h \in \mathbf{W}$ and $p_h \in Q$, such that,

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h), \ \forall\, \mathbf{v}_h \in \mathbf{W}, \\ b(\mathbf{u}_h, q_h) \qquad\qquad\quad = (g_h, q_h), \ \forall\, q_h \in Q, \end{cases} \tag{1}$$

where $\mathbf{W}$ and Q are finite element spaces. The continuous bilinear forms $a(\mathbf{u}_h, \mathbf{v}_h)$ and $b(\mathbf{u}_h, q_h)$ can come from the variational formulation of the Stokes equation or the Darcy problem. We call $\mathbf{u}_h$ velocity variables and p_h pressure variable, respectively.

The system (1) can be written as

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix}. \tag{2}$$

Here A is symmetric positive definite but B is rank deficient. $Ker(B^T)$, the kernel of B^T, includes all constant pressures in Q. $Im(B)$, the range of B, includes all vectors in Q with zero average. We note that $Im(B)$ is orthogonal to $Ker(B^T)$. Under the assumption that $g \in Im(B)$, i.e., g has zero average, the solution of (2) is uniquely determined if the pressure is restricted to the quotient space $Q/Ker(B^T)$.

We assume that $\mathbf{W}$ and Q are inf-sup stable: there exists a positive constant β, independent of h, such that

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{\langle q, B\mathbf{w}\rangle^2}{\langle \mathbf{w}, A\mathbf{w}\rangle} \geq \beta^2 \langle q, Zq\rangle, \qquad \forall q \in Q/Ker(B^T), \tag{3}$$

where Z is the so called mass matrix on Q, i.e., $\|q\|^2_{L^2} = \langle q, Zq\rangle$, $\forall q \in Q$.

3 Domain decomposition

We decompose the domain Ω into N nonoverlapping polygonal/polyhedral subdomains Ω_i, $i = 1, 2, ..., N$. We assume that each subdomain is a union of a bounded number of elements, with typical diameter of H. The subdomain interface nodes $\Gamma = (\cup\partial\Omega_i)\backslash\partial\Omega$. Γ includes the subdomain faces, which are open sets and shared by two subdomains, the subdomain edges, which are open sets and shared by more than two subdomains; and the subdomain vertices, which are end points of edges.

Denote the subdomain interior velocity spaces by $\mathbf{W}_I^{(i)}$ and subdomain interior pressure spaces by $Q_I^{(i)}$, respectively. The subdomain boundary velocity space is denoted by $\mathbf{W}_\Gamma$, which is shared by neighboring subdomains, while Q_Γ contains the subdomain boundary pressure degrees of freedom shared by neighboring subdomains. Let

$$\mathbf{W}_I = \bigoplus_{i=1}^{N} \mathbf{W}_I^{(i)}, \quad Q_I = \bigoplus_{i=1}^{N} Q_I^{(i)}.$$

We decompose the velocity and pressure finite element spaces $\mathbf{W}$ and Q into the subdomain interior and interface subspaces,

$$\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma, \quad Q = Q_I \bigoplus Q_\Gamma,$$

respectively, and write (2) as

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Gamma} & B_{\Gamma I}^T \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ A_{I\Gamma}^T & B_{I\Gamma}^T & A_{\Gamma\Gamma} & B_{\Gamma\Gamma}^T \\ B_{\Gamma I} & 0 & B_{\Gamma\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I \\ g_I \\ \mathbf{f}_\Gamma \\ g_\Gamma \end{bmatrix}, \tag{4}$$

which can be assembled from the subdomain problems, defined as below

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{I\Gamma}^{(i)} & B_{\Gamma I}^{(i)^T} \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\ A_{I\Gamma}^{(i)^T} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} & B_{\Gamma\Gamma}^{(i)^T} \\ B_{\Gamma I}^{(i)} & 0 & B_{\Gamma\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ p_I^{(i)} \\ \mathbf{u}_\Gamma^{(i)} \\ p_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I^{(i)} \\ g_I^{(i)} \\ \mathbf{f}_\Gamma^{(i)} \\ g_\Gamma^{(i)} \end{bmatrix}. \tag{5}$$

We note that the blocks corresponding to $\mathbf{u}_I$ and p_I in (4) can be arranged in subdomain wise. As long as p_Γ contains at least one pressure variables from each subdomain, we can eliminate $\mathbf{u}_I$ and p_I by solving independent subdomain problems and obtain the following global Schur complement system

$$\begin{bmatrix} S_\Gamma & T_{\Gamma\Gamma}^T \\ T_{\Gamma\Gamma} & -C_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{f}_s \\ g_s \end{bmatrix}, \tag{6}$$

where

$$S_\Gamma = A_{\Gamma\Gamma} - [A_{\Gamma I}\ B_{I\Gamma}^T] \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{I\Gamma} \\ B_{I\Gamma} \end{bmatrix}, \tag{7}$$

$$C_{\Gamma\Gamma} = [B_{\Gamma I}\ 0] \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \begin{bmatrix} B_{\Gamma I}^T \\ 0 \end{bmatrix}, \tag{8}$$

$$T_{\Gamma\Gamma} = B_{\Gamma\Gamma} - [B_{\Gamma I}\ 0] \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{I\Gamma} \\ B_{I\Gamma} \end{bmatrix}, \tag{9}$$

and

$$\begin{bmatrix} \mathbf{f}_s \\ g_s \end{bmatrix} = \begin{bmatrix} \mathbf{f}_\Gamma \\ g_\Gamma \end{bmatrix} - \begin{bmatrix} A_{I\Gamma}^T & B_{I\Gamma}^T \\ B_{\Gamma I} & 0 \end{bmatrix} \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}_I \\ g_I \end{bmatrix}.$$

We note that S_Γ can be assembled from the local subdomain Schur complements $S_\Gamma^{(i)}$ defined from (5) as:

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{I\Gamma}^{(i)} \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} \\ A_{I\Gamma}^{(i)^T} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ p_I^{(i)} \\ \mathbf{u}_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ S_\Gamma^{(i)} \mathbf{u}_\Gamma^{(i)} \end{bmatrix}. \tag{10}$$

We call (6) the primal approach. To formulate the preconditioners of (6) and introduce the domain decomposition algorithms using the dual approach, we introduce a partially sub-assembled interface velocity space

$$\widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_\Pi \bigoplus \mathbf{W}_\Delta = \mathbf{W}_\Pi \bigoplus \left(\bigoplus_{i=1}^{N} \mathbf{W}_\Delta^{(i)} \right).$$

Here, $\mathbf{W}_\Pi$ is the continuous coarse level velocity space, whose elements are shared by neighboring subdomains. The complimentary space $\mathbf{W}_\Delta$ is the direct sum of subdomain remaining interface velocity spaces $\mathbf{W}_\Delta^{(i)}$, whose elements vanish at the primal degrees of freedom. In general the functions $\mathbf{w}_\Delta$ in $\mathbf{W}_\Delta$ are not continuous across the subdomain interface Γ and we need to introduce Lagrange multipliers to enforce their continuity. We construct a boolean matrix J_Δ such that $J_\Delta \mathbf{w}_\Delta = 0$ implies the continuity of $\mathbf{w}_\Delta$ cross subdomain interface, see [8, 9] for details. We choose J_Δ to have full row rank and denote the range of J_Δ applied on $\mathbf{W}_\Delta$ by Λ.

The original fully assembled linear system (2) is equivalent to: find $(\mathbf{u}_I,\ p_I,\ \mathbf{u}_\Delta,\ \mathbf{u}_\Pi,\ p_\Gamma,\ \lambda) \in \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi \bigoplus Q_\Gamma \bigoplus \Lambda$, such that

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} & B_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} & 0 & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} & B_{\Gamma\Delta}^T & J_\Delta^T \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} & B_{\Gamma\Pi}^T & 0 \\ B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} & 0 & 0 \\ 0 & 0 & J_\Delta & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_\Pi \\ p_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I \\ g_I \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \\ g_\Gamma \\ 0 \end{bmatrix}, \tag{11}$$

which can be reduced to

$$G \begin{bmatrix} p_\Gamma \\ \lambda \end{bmatrix} = g_g, \tag{12}$$

where

$$G = B_C \widetilde{A}^{-1} B_C^T, \quad g_g = B_C \widetilde{A}^{-1} f - \begin{bmatrix} g_\Gamma \\ 0 \end{bmatrix}, \tag{13}$$

$$\widetilde{A} = \begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix}, \quad B_C = \begin{bmatrix} B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & J_\Delta & 0 \end{bmatrix}, \quad f = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{bmatrix}. \tag{14}$$

Since (12) is a system related to the Lagrange multipliers λ, we call it the dual approach.

4 Positive definite formulations

We have reduced the original saddle point problem into two systems: the primal system (6) and the dual system (12). Even though none of them is positive definite, they can be reduced to positive definite problems in certain special subspaces.

4.1 The primal system (6)

For a general pressure space Q, it is not easy to formalate the Schur complement system (6) as a positive definition system. However, when Q is a discontinuous finite element space, one can decompose Q properly and make (6) positive definite in a special subspace.

When p is discontinuous, subdomains do not share any pressure degrees of freedom on the subdomain boundary. We can take Q_Γ as the subspace of Q with constant values $p_0^{(i)}$, which is the average of the pressure in the subdomain Ω_i and satisfy $\sum_{i=1}^N p_0^{(i)} m(\Omega_i) = 0$, where $m(\Omega_i)$ is the measure of the subdomain Ω_i. The elements of $Q_I^{(i)}$ are the restrictions of the pressure variables to Ω_i which satisfy $\int_{\Omega_i} p_I^{(i)} = 0$. Since p_Γ is a constant pressure on each subdomain, $B_{\Gamma I} = 0$. Using this fact in (8) and (9), we have $C_{\Gamma\Gamma} = 0$ and $T_{\Gamma\Gamma} = B_{\Gamma\Gamma}$ and therefore the system (6) can be simplified as

$$\begin{bmatrix} S_\Gamma & B_{\Gamma\Gamma}^T \\ B_{\Gamma\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{f}_s \\ g_s \end{bmatrix}. \tag{15}$$

For the applications with $g_s \neq 0$, one can find a special $\mathbf{u}_\Gamma^*$ such that $B_{\Gamma\Gamma}\left(\mathbf{u}_\Gamma - \mathbf{u}_\Gamma^*\right) = 0$, see [25, Section 4.8] for details. From now on we assume $g_s = 0$.

The system matrix of (6) is positive definite in the space with $B_{\Gamma\Gamma}\mathbf{u}_\Gamma = 0$. Since p_Γ contains pressure variables which are constant in each subdomain, to make $B_{\Gamma\Gamma}\mathbf{u}_\Gamma = 0$, we only need require $\int_{\partial\Omega_i} \mathbf{u}_\Gamma^{(i)} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the normal direction to $\partial\Omega_i$.

We still need to construct a preconditioner to solve (6). Let $\overline{R}_\Gamma^{(i)}$ map $\widetilde{\mathbf{W}}_\Gamma$ to $\mathbf{W}_\Delta^{(i)} \bigoplus \mathbf{W}_\Pi^{(i)}$ and $\overline{R}_\Gamma$ is a direct sum of $\overline{R}_\Gamma^{(i)}$. We can define

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T \mathrm{diag}\left(S_\Gamma^{(1)}, \cdots, S_\Gamma^{(N)}\right) \overline{R}_\Gamma.$$

$\widetilde{B}_{\Gamma\Gamma}$ is defined on $\widetilde{\mathbf{W}}_\Gamma$ and is assembled from $B_{\Gamma\Gamma}^{(i)}$ given in (5). The BDDC preconditioned system of (6) can be written as

$$M_B^{-1} S \begin{bmatrix} \mathbf{u}_\Gamma \\ p_\Gamma \end{bmatrix} = M_B^{-1} \begin{bmatrix} \mathbf{f}_s \\ g_s \end{bmatrix}, \tag{16}$$

where $M_B^{-1} = \begin{bmatrix} \widetilde{R}_{D,\Gamma} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{\Gamma\Gamma}^T \\ \widetilde{B}_{\Gamma\Gamma} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{R}_{D,\Gamma} & 0 \\ 0 & I \end{bmatrix}$, $S = \begin{bmatrix} S_\Gamma & B_{\Gamma\Gamma}^T \\ B_{\Gamma\Gamma} & 0 \end{bmatrix}$, $\widetilde{R}_\Gamma$ maps $\mathbf{W}_\Gamma$ to $\widetilde{\mathbf{W}}_\Gamma$ and $\widetilde{R}_{D,\Gamma}$ is scaled operator obtained from $\widetilde{R}_\Gamma$ with the scaling D. The matrix D should provide a partition of unity:

$$\widetilde{R}_{D\Gamma}^T \widetilde{R}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{R}_{D\Gamma} = I.$$

See [5, 19, 18] for more details about the construction of the BDDC preconditioners. See [33, 34, 20, 31] for different scaling options.

We define two subspaces of $\mathbf{W}_\Gamma$ and $\widetilde{\mathbf{W}}_\Gamma$, respectively, as

$$\mathbf{W}_{\Gamma,B} = \{\mathbf{u}_\Gamma \in \mathbf{W}_\Gamma \mid B_{\Gamma\Gamma}\mathbf{u}_\Gamma = 0\}, \quad \widetilde{\mathbf{W}}_{\Gamma,B} = \{\mathbf{u}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma \mid \widetilde{B}_{\Gamma\Gamma}\mathbf{u}_\Gamma = 0\}.$$

They are called benign subspaces.

It is easy to see that the BDDC preconditioned system (16) is positive definite in the benign subspace $\mathbf{W}_{\Gamma,B}$. In order to use the conjugate gradient method (CG) to solve (16), we need to ensure all CG iterates in $\mathbf{W}_{\Gamma,B}$ with any initial guess in $\mathbf{W}_{\Gamma,B}$.

We can choose a proper $\mathbf{W}_\Pi$ such that

$$\int_{\partial\Omega_i} \mathbf{w}_\Delta^{(i)} \cdot \mathbf{n} = 0 \tag{17}$$

is satisfied for all $\mathbf{w}_\Delta^{(i)} \in \mathbf{W}_\Delta^{(i)}$. By [18, Lemma 6.2], all CG iterates will stay in $\mathbf{W}_{\Gamma,B}$ if the initial initial guess lies in $\mathbf{W}_{\Gamma,B}$.

The choice of $\mathbf{W}_\Pi$ to satisfy (17) depends on the original problem (1) and the finite element spaces $\mathbf{W}$, namely the discretization methods. See [18, Section 7] for incompressible Stokes problems; [29] for Stokes with the weak Galerkin discretization and [30] for the hybridizable discontinuous Galerkin discretizations; [24, 34] for Darcy problem.

4.2 The dual system (12)

Similar to (6), (12) can be positive definite in a special subspace.

If $\widetilde{A}$, defined in (14), is nonsingular, by the Sylvester law of inertia, we know that G is symmetric positive semi-definite. Let 1_v denote the constant vector 1 which has the same dimension as v and $J_{\Delta,D}$ is obtained by scaling J_Δ with the scaling matrix D. The null space of G is given by

$$\left(1_{p_\Gamma}, -J_{\Delta,D}[B_{I\Delta}^T \;\; B_{\Gamma\Delta}^T]\begin{bmatrix}1_{p_I}\\ 1_{p_\Gamma}\end{bmatrix}\right).$$

Let $X = Q_\Gamma \bigoplus \Lambda$ and $Im(G)$ be the range space of G, which is a subspace of X. $Im(G)$ is orthogonal to the null space of G and can be written as

$$Im(G) = \left\{\begin{bmatrix}g_{p_\Gamma}\\ g_\lambda\end{bmatrix} \in X : g_{p_\Gamma}^T 1_{p_\Gamma} - g_\lambda^T\left(J_{\Delta,D}[B_{I\Delta}^T \;\; B_{\Gamma\Delta}^T]\begin{bmatrix}1_{p_I}\\ 1_{p_\Gamma}\end{bmatrix}\right) = 0\right\}. \tag{18}$$

The restriction of G to its range space $Im(G)$ is positive definite. By [27], we know g_g, defined in (13), belongs to $Im(G)$. All CG iterates will be in $Im(G)$ if the CG method is used to solve (12) with zero initial guess.

Block preconditioners, proposed in [17, 27, 28], are used to solve (12). The preconditioned system can be written as

$$M_F^{-1}G\begin{bmatrix}p_\Gamma\\ \lambda\end{bmatrix} = M_F^{-1}g_g, \quad M_F^{-1} = \begin{bmatrix}M_p^{-1} & \\ & M_\lambda^{-1}\end{bmatrix}. \tag{19}$$

$M_p^{-1} = \frac{1}{h^n}I_{p_\Gamma}$ for the Stokes problem and M_λ^{-1} can be either lumped or Dirichlet preconditioners for λ. [32] defines M_p^{-1} to be a BDDC preconditioner for isogeometric discretization for almost incompressible elasticity and deluxe scaling is used. All these additional techniques ensure the algorithms robust in the presence of discontinuous material parameters, which is not considered for the algorithms in [17, 27, 28] for the Stokes problem. In [34], deluxe scaling and local generalized eigenvalue problems are also used to further enhance the performance of algorithms for (16). However, some special designs of these techniques are needed to make sure these additional primal variables lie in the benign subspace.

We note that for (12), we do not require that the pressure be discontinuous for the positive definite formulation. Moreover, we do not need to choose proper primal space $\mathbf{W}_\Pi$ to ensure the CG iterates in the subspace. The choices of $\mathbf{W}_\Pi$ for (12) only ensure the nice bound for the condition number of the preconditioned operator. This fact makes the algorithms much simpler, especially for three dimensional problems.

However, we do need to define a subspace $\widetilde{V}_0$ for the convergence analysis only, which plays a similar role as the benign subspaces. Let $\widetilde{V} = \mathbf{W}_I \bigoplus Q_I \bigoplus \widetilde{\mathbf{W}}_\Gamma$ and its subspace

$$\widetilde{V}_0 = \left\{ v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V} \;\middle|\; B_{II}\mathbf{w}_I + B_{I\Delta}\mathbf{w}_\Delta + B_{I\Pi}\mathbf{w}_\Pi = 0 \right\}. \tag{20}$$

For any $v \in \widetilde{V}_0$, $\langle \cdot, \cdot \rangle_{\widetilde{A}}$ defines a semi-inner product on $\widetilde{V}_0$, see [28] for details.

5 Condition number estimates

Since both (16) and (19) are symmetric positive definite in the special subspaces, we can use the CG methods to solve them. For the convergence analysis of the CG methods, we only need to bound the maximum and minimum eigenvalues of the preconditioned operators. Here we only outline the analyses, see, for example, [18] and [28] for details.

We first define two useful operators E_D and P_D. Different from the E_D and P_D defined for elliptic problems in [19], our E_D and P_D are defined on different subspaces. The matrix S in (16) are defined with S_Γ and $B_{\Gamma\Gamma}$, which is for the variables $\mathbf{u}_\Gamma$ and p_Γ. The matrix G in (19) are defined with $\widetilde{A}$, which is for the variables $\mathbf{u}_I$, p_I, and $\mathbf{u}_\Gamma$.

E_D is an averaging operator, defined by

$$E_D = \widetilde{R}\widetilde{R}_D^T = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix} \begin{bmatrix} \widetilde{R}_{D,\Gamma}^T & \\ & I \end{bmatrix}.$$

It maps $\widetilde{\mathbf{W}}_\Gamma \times Q_\Gamma$ to itself and computes a weighted average for the velocity across the subdomain interface Γ, and then distributes the average back to the original degree of freedoms on the interfaces while keeping the pressure variables unchanged.

Similarly, P_D is a jump operator, which maps $\widetilde{V}$ to itself. Here we only define the jump operator related to solving a Dirichlet problem on each subdomain. For any given $v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V}$, $P_D v = (\mathbf{u}_I,\ 0,\ \mathbf{u}_\Delta,\ 0) \in \widetilde{V}$, where each $\mathbf{u}_I^{(i)}$ is the harmonic extension, with given subdomain boundary velocity $\mathbf{u}_\Delta^{(i)} = J_{\Delta,D}^{(i)^T} J_\Delta \mathbf{w}_\Delta$ and $\mathbf{u}_\Pi^{(i)} = 0$. Here $J_{\Delta,D}^{(i)^T}$ represents restriction of $J_{\Delta,D}^T$ on subdomain Ω_i and is a map from Λ to $\mathbf{W}_\Delta^{(i)}$.

We assume that the interface averaging operator E_D and the jump operator P_D satisfy the following bounds:

$$|E_D\mathbf{w}|_{\widetilde{S}}^2 \le C_{ED}(H,h)|\mathbf{w}|_{\widetilde{S}}^2, \quad \forall\, \mathbf{w} = (\mathbf{u}_\Gamma,\ q_0) \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_\Gamma, \tag{21}$$

and

$$|P_D v|_{\widetilde{A}}^2 \le C_{PD}(H,h)|v|_{\widetilde{A}}^2, \quad \forall\, v \in \widetilde{V}_0, \tag{22}$$

where $C_{ED}(H,h)$ and $C_{PD}(H,h)$ are positive constants dependent on the subdomain size H and mesh size h.

Theorem 1 *For any* $\mathbf{w} = (\mathbf{u}_\Gamma,\ p_\Gamma) \in \mathbf{W}_{\Gamma,B} \times Q_\Gamma$,

$$\langle \mathbf{w},\, \mathbf{w}\rangle_S \le \left\langle \mathbf{w},\, M_B^{-1} S\mathbf{w}\right\rangle_S \le C_{ED}\, \langle \mathbf{w},\, \mathbf{w}\rangle_S\,,$$

where $C_{ED}(H,h)$ is the bound of the average operator, given in (21).

Theorem 2 *For any x in the range of $M_F^{-1}G$,*

$$c(\beta)\,\langle M_F x, x\rangle \le \langle Gx, x\rangle \le (CC_{PD}(H,h))\,\langle M_F x, x\rangle\,,$$

where $c(\beta)$ is a function of the inf-sup constan β, defined in (3), C is a positive constant, and C_{PD} is the bound of the jump operator, given in (22).

6 Connections and differences

One of the big advantages of using (19) is that the formulation can be applied to both continuous and discontinuous pressure discretizations. The algorithms can be applied to the problems discretized with widely used Taylor-Hood finite elements and isogeometric discretizations. Moreover, since the formulation does not put any constraints on velocity variable $\mathbf{u}$ for its positive definite formulation, we can relax the divergence free constraints defined in (17), which can be quite complicated to be enforced, see [18, Section 7]. The coarse problem resulting from (19) can be positive definite, which can be the same as those for simple elliptic problems.

Both (16) and (19) can be applied to discontinuous pressures. When Q is discontinuous, there are two choices of p_Γ in (19), as discussed in details in [27]. When p_Γ is taken as an empty set, (19) become a system for the Lagrange multiplier λ only. If the Stokes extension is used in the jump operator P_D instead of harmonic extension, the divergence free condition will be required and it has been proved in [18, Theorem 8.1] that both (16) and (19) have the same nonzero eigenvalues with the possible exception of 1. However, the Stokes extension and divergence free condition are not necessary for (19). Harmonic extension will make the algorithms more efficient.

From the analysis point of view, if (16) can be applied, the minimal eigenvalues of the preconditioned operator is always 1 as stated in Theorem 1. One only needs to estimate the bound C_{ED} of the average operator E_D, defined in (21). For the analysis of (19), one needs to estimate the bound C_{PD} of the jump operator P_D, defined in (22), which is similar to the estimate of E_D. Moreover, the lower bound in Theorem 2 has to be established, which is not as easy as for (16).

There are many discretizations with discontinuous pressure spaces such as the family of discontinuous Galerkin methods. (16) has been applied in [29, 30] for some of these discretizations, where the primal constraints, required by the bound of E_D, also ensure the divergence free conditions, which makes the algorithms simpler than those with standard finite element discretizations, especially in three dimensions. The difficulty for those applications is to estimate the bound for the average operator E_D, where properties of the discretizations have to be explored carefully.

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