# OSDS: A Sweeping Preconditioner for the Helmholtz Equation

Nacime Bouziani and Frédéric Nataf

Domain decomposition algorithms have become popular in solving the Helmholtz equation since the seminal Després paper [3]. Although it is known that the presence of overlaps helps to speed up the convergence for domain decomposition methods, nonoverlapping based methods are often used to avoid to deal with the construction of the normal derivative of the solution. For decompositions into vertical strips, a sweeping algorithm was first proposed and analyzed in [6] for convection-diffusion operators. Recently, sweeping methods have gained interested due to their capability to achieve nearly-linear asymptotic complexity, see e.g. the double sweep preconditionner of Vion and Geuzaine for non overlapping decomposition with high order interface conditions [8, 9], the PML-based sweep method of Stolk [2], and the polarized traces method of Zepeda-Núñez and Demanet [10].

We consider a decomposition of the domain into layers where the local subproblems are equipped with interface conditions, also called absorbing boundary conditions (ABC). In practice, the exact ABC (which are also the optimal interface conditions, see [7]) procedure is tedious to implement and computationally expensive. As a consequence, the boundary conditions at the interfaces produce spurious reflected waves that significantly increase the number of iterations to converge, in particular for heterogeneous media and high frequency regimes.

We propose to precondition the discrete Helmholtz system by an overlapping splitting double-sweep algorithm that allows for overlapping subdomains and prevents spurious interface reflections from hindering the convergence. Using overlapping subdomains allows us to leverage its beneficial effect on the damping of high frequency modes of the error, while splitting prevents its adversary effect on the convergence of propagative modes. This is highly beneficial since in the non-overlapping

Nacime Bouziani
Imperial College London, Department of Mathematics, London, SW7 2AZ, UK, e-mail: n.bouziani18@imperial.ac.uk,nacime.bouziani@gmail.com

Frédéric Nataf
Laboratory J.L. Lions, Sorbonne Université, Paris e-mail: frederic.nataf@sorbonne-universite.fr

approach [8, 9], the quality of the ABC is nearly the only way of impacting the convergence of the algorithm, and when dealing with more complex problems such as Maxwell equations high order ABCs are harder to handle.

## 1 Statement of the problem and some algorithms

We consider the Helmholtz equation in a bounded domain $\Omega \subset \mathbb{R}^2$ with frequency $\omega$, velocity $c$ and wavenumber $k$ defined by $k^2 = \omega^2/c^2$:

$$\begin{aligned} &\left(-k^2 - \Delta\right) u = f \text{ in } \Omega \\ &+ \text{appropriate boundary conditions on } \partial\Omega\,. \end{aligned} \tag{1}$$

We consider a layered decomposition of $\Omega$ into $N$ slices $(\Omega_i)_{1\leq i\leq N}$, with or without overlap, see Figure 1. More precisely, for each $1 \leq i \leq N$, $\Omega \setminus \Omega_i$ is written as the disjoint union of two open subsets $\Omega_{i,l}$ and $\Omega_{i,r}$ where $\Omega_{i,l}$ is on the left of $\Omega_i$ and $\Omega_{i,r}$ on its right. The boundary $\partial\Omega_i \setminus \partial\Omega$ is written as the disjoint union of $\Gamma_{i,l}$ and $\Gamma_{i,r}$ where $\Gamma_{i,l}$ is on the left of $\Omega_i$ and $\Gamma_{i,r}$ is on its right ($\Omega_{1,l} = \emptyset$ and $\Omega_{N,r} = \emptyset$) (see Figure 2). The outward normal from $\Omega_i$ on $\Gamma_{i,l}$ (resp. $\Gamma_{i,r}$) is denoted by $\boldsymbol{n}_{i,l}$ (resp. $\boldsymbol{n}_{i,r}$). The problem (1) can be solved iteratively using a domain decomposi-

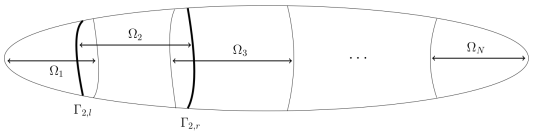


**Fig. 1:** Decomposition into vertical strips

tion method where we solve locally on each subdomain $\Omega_i$ the equation (1) with appropriate boundary conditions on the physical boundaries and interfaces [3]. The method reads:

Solve in parallel:

$$\begin{cases} \left(-k^2 - \Delta\right) u_i^{n+1} = f \ \text{ in } \Omega_i,\ 1 \leq i \leq N \\ \quad \mathcal{B}_{i,l}\left(u_i^{n+1}\right) = \mathcal{B}_{i,l}\left(u_{i-1}^{n}\right) \ \text{ on } \Gamma_{i,l},\ 2 \leq i \leq N \\ \quad \mathcal{B}_{i,r}\left(u_i^{n+1}\right) = \mathcal{B}_{i,r}\left(u_{i+1}^{n}\right) \ \text{ on } \Gamma_{i,r},\ 1 \leq i \leq N-1 \\ \qquad\qquad + \ \text{appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i\,, \end{cases} \tag{2}$$

where $\mathcal{B}_{i,l}$ and $\mathcal{B}_{i,r}$ are the interface conditions. For sake of simplicity, we consider first-order ABC as interface conditions:

$$\begin{cases} \mathcal{B}_{i,l} = \partial_{\boldsymbol{n}_{i,l}} + Ik \\ \mathcal{B}_{i,r} = \partial_{\boldsymbol{n}_{i,r}} + Ik \end{cases} \tag{3}$$

where $I^2 = -1$ and $\boldsymbol{n}_{i,r}$ (resp. $\boldsymbol{n}_{i,l}$) is the outward normal to domain $\Omega_i$ on $\Gamma_{i,r}$ (resp. $\Gamma_{i,l}$). It is known that higher-order ABC lead to significant improvement of the convergence speed, see e.g. [4, 1].

A more efficient variant of algorithm 2 was introduced in [6]. It consists in double sweeps over the subdomains:

**Left to right sweep:**

$$\begin{cases} \left(-k^2 - \Delta\right) u_i^{n+1/2} = f \ \text{ in } \Omega_i, \ 1 \le i \le N \\ \quad \mathcal{B}_{i,l}\left(u_i^{n+1/2}\right) = \mathcal{B}_{i,l}\left(u_{i-1}^{n+1/2}\right) \text{ on } \Gamma_{i,l}, \ 2 \le i \le N \\ \quad \mathcal{B}_{i,r}\left(u_i^{n+1/2}\right) = \mathcal{B}_{i,r}\left(u_{i+1}^{n}\right) \text{ on } \Gamma_{i,r}, \ 1 \le i \le N-1 \\ \qquad\qquad + \text{ appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i \, . \end{cases} \tag{4}$$

**Right to left sweep:**

$$\begin{cases} \left(-k^2 - \Delta\right) u_i^{n+1} = f \ \text{ in } \Omega_i, \ 1 \le i \le N \\ \quad \mathcal{B}_{i,l}\left(u_i^{n+1}\right) = \mathcal{B}_{i,l}\left(u_{i-1}^{n+1/2}\right) \text{ on } \Gamma_{i,l}, \ 2 \le i \le N \\ \quad \mathcal{B}_{i,r}\left(u_i^{n+1}\right) = \mathcal{B}_{i,r}\left(u_{i+1}^{n+1}\right) \text{ on } \Gamma_{i,r}, \ 1 \le i \le N-1 \\ \qquad\qquad + \text{ appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i \, . \end{cases} \tag{5}$$

## 2 Overlapping Splitting double sweep

In this section, we define a variant of algorithm (4)-(5) which has a superior convergence. Numerical results will show that it benefits better from the overlap and has better parallelism. This algorithm is written in terms of the substructured problem that we define first.

### 2.1 Substructuring

Substructuring algorithm (2), the iterative method can be reformulated considering only surfacic unknowns on the interfaces:

$$\begin{cases} h_{i,l}^n := \mathcal{B}_{i,l}\left(u_i^n\right), \ \text{ on } \ \Gamma_{i,l} \quad \text{for } \ 2 \le i \le N \, , \\ h_{i,r}^n := \mathcal{B}_{i,r}\left(u_i^n\right), \ \text{ on } \ \Gamma_{i,r} \quad \text{for } \ 1 \le i \le N-1 \, . \end{cases} \tag{6}$$

Considering the global vector $h^n := (h^n_{2,l}, \dots, h^n_{N,l}, h^n_{N-1,r}, \dots, h^n_{1,r})^T$, containing first the local unknowns $(h^n_{i,l})_{2\leq i\leq N}$ and then in reverse order $(h^n_{i,r})_{1\leq i\leq N-1}$, we can reformulate the parallel Schwarz method (2) as a Jacobi algorithm on $h^n$: $h^{n+1} := \mathscr{T}(h^n) + G$, where the iteration operator $\mathscr{T}$ can be written in the form of an operator valued matrix and $G$ refers to the contribution of the right-hand side $f$, see [7]. Therefore, we look for a vector $h$ such that

$$(Id - \mathscr{T})(h) = G\,. \tag{7}$$

In order to define more precisely the operator $\mathscr{T}$, we introduce for each subdomain an operator $S_i$ which takes three arguments, two surfacic functions $h_l$ and $h_r$ and a volume function $f$:

$$S_i(h_{i,l}, h_{i,r}, f) := v, \tag{8}$$

where $v : \Omega_i \mapsto \mathbb{C}$ satisfies:

$$\begin{cases} \left(-k^2 - \Delta\right) v = f \ \text{ in } \Omega_i \\ \qquad \mathcal{B}_{i,l}(v) = h_{i,l} \text{ on } \Gamma_{i,l} \quad (2 \leq i \leq N) \\ \qquad \mathcal{B}_{i,r}(v) = h_{i,r} \text{ on } \Gamma_{i,r} \quad (1 \leq i \leq N-1) \\ \qquad\qquad + \text{ appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i\,, \end{cases} \tag{9}$$

for $1 < i < N$ (see Figure 2). For $i = 1$, the definition of $S_1$ is similar except that it takes only the two arguments $(h_{1,r}, f)$ since domain $\Omega_1$ has no left interface and similarly operator $S_N$ takes only the two arguments $(h_{N,l}, f)$ since domain $\Omega_N$ has no right interface. *As of now, for sake of simplicity and by abuse of notation,* $S_1(h_{1,l},\ h_{1,r},\ f)$ *(resp.* $S_N(h_{N,l},\ h_{N,r},\ f)$*) will refer to* $S_1(h_{1,r}, f)$ *(resp.* $S_N(h_{N,l},\ f)$*).*

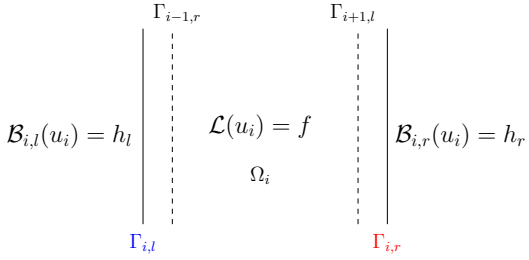


**Fig. 2:** Local problem on the subdomain $\Omega_i$

Next, we introduce the surfacic right hand-side $G$ by

$$\begin{aligned} G_{i+1,l} &:= \mathcal{B}_{i+1,l}(S_i(0, 0,\ f)),\ \ 1 \leq i \leq N-1\,, \\ G_{i-1,r} &:= \mathcal{B}_{i-1,r}(S_i(0, 0,\ f)),\ \ 2 \leq i \leq N\,, \end{aligned} \tag{10}$$

and the substructured operator $\mathscr{T}$ by:

$$\begin{aligned} \mathcal{T}(h)_{i+1,l} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l},\ h_{i,r},0)),\ \ 1 \le i \le N-1\,, \\ \mathcal{T}(h)_{i-1,r} &:= \mathcal{B}_{i-1,r}(S_i(h_{i,l},\ h_{i,r},0)),\ \ 2 \le i \le N\,. \end{aligned} \tag{11}$$

We can now write the substructured form of the *double sweep algorithm* as:

**Forward sweep**

$$\begin{aligned} h_{i+1,l}^{n+1/2} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l}^{n+1/2},\ h_{i,r}^{n},f))\,, \\ h_{i-1,r}^{n+1/2} &:= \mathcal{B}_{i-1,r}(S_i(h_{i,l}^{n+1/2},\ h_{i,r}^{n},f))\,, \end{aligned} \tag{12}$$

**followed by a Backward sweep**

$$\begin{aligned} h_{i+1,l}^{n+1} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l}^{n+1/2},\ h_{i,r}^{n+1},f))\,, \\ h_{i-1,r}^{n+1} &:= \mathcal{B}_{i-1,r}(S_i(h_{i,l}^{n+1/2},\ h_{i,r}^{n+1},f))\,. \end{aligned} \tag{13}$$

As for the Jacobi method, by introducing an operator $\mathcal{T}_{DS}$, this algorithm can be written in a compact form $h^{n+1} = h^n + (I - \mathcal{T}_{DS})^{-1}(G - (I - \mathcal{T})(h^n))$, see [6].

## 2.2 Overlapping Splitting Double Sweep preconditioner (OSDS)

We explain now the rationale behind the overlapping splitting double sweep preconditioner that we define in this section. Note first that by linearity of the operators $(S_i)_{1\le i\le N}$, the contribution of each subdomain can be split into two contributions, one for each of its two interfaces:

$$\begin{aligned} \mathcal{T}(h)_{i+1,l} &= \mathcal{B}_{i+1,l}(S_i(h_{i,l},\ 0,\ 0)) \ + \ \mathcal{B}_{i+1,l}(S_i(0,\ h_{i,r},\ 0)),\ \ 1 \le i \le N-1\,, \\ \mathcal{T}(h)_{i-1,r} &= \mathcal{B}_{i-1,r}(S_i(0,\ h_{i,r},\ 0)) \ + \ \mathcal{B}_{i-1,r}(S_i(h_{i,l},\ 0,\ 0)),\ \ 2 \le i \le N\,. \end{aligned} \tag{14}$$

Had we used exact absorbing (a.k.a transparent or non-reflecting) boundary conditions (EABC) $\mathcal{B}^{EABC}$ instead of the zero-th order ones (3) in equations (8)-(9), two terms in (14) would vanish, namely:

$$\begin{aligned} \mathcal{B}_{i+1,l}^{EABC}(S_i^{EABC}(0,\ h_{i,r},0)) &= 0,\ \ 1 \le i \le N-1\,, \\ \mathcal{B}_{i-1,r}^{EABC}(S_i^{EABC}(h_{i,l},0,0)) &= 0,\ \ 2 \le i \le N\,. \end{aligned} \tag{15}$$

The corresponding operator in (14) would thus only contain one term then,

$$\begin{aligned} \mathcal{T}^{EABC}(h)_{i+1,l} &= \mathcal{B}_{i+1,l}^{EABC}(S_i^{EABC}(h_{i,l},0,0)),\ \ 1 \le i \le N-1\,, \\ \mathcal{T}^{EABC}(h)_{i-1,r} &= \mathcal{B}_{i-1,r}^{EABC}(S_i^{EABC}(0,\ h_{i,r},0)),\ \ 2 \le i \le N\,. \end{aligned} \tag{16}$$

Then thanks to our numbering of $h$, the operator valued matrix $\mathcal{T}^{EABC}$ is $2\times 2$ block diagonal matrix where each block is subdiagonal. As a consequence, for some vector $G$, computing $(I - \mathcal{T}^{EABC})^{-1}\,G$ can be performed by two parallel forward substitutions, which amounts to a single double sweep over the subdomains.

In practice, the absorbing boundary conditions are non exact, therefore we have

$$\begin{aligned} \mathcal{B}_{i+1,l}(S_i(0,\ h_{i,r},0)) \neq 0,\ \ 1 \leq i \leq N-1\,, \\ \mathcal{B}_{i-1,r}(S_i(h_{i,l},0,0)) \neq 0,\ \ 2 \leq i \leq N\,, \end{aligned} \tag{17}$$

and we loose the block diagonal structure of $\mathscr{T}$. This led us to define a new operator $\mathscr{T}_{OSDS}$

$$\begin{aligned} \mathscr{T}_{OSDS}(h)_{i+1,l} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l},0,0)),\ \ 1 \leq i \leq N-1\,, \\ \mathscr{T}_{OSDS}(h)_{i-1,r} &:= \mathcal{B}_{i-1,r}(S_i(0,\ h_{i,r},0)),\ \ 2 \leq i \leq N\,, \end{aligned} \tag{18}$$

which by definition has the same structure than $\mathscr{T}^{EABC}$. We propose to use this newly defined operator to build a preconditioner for (7). The right-preconditioned solves reads: Find $\tilde{h}$ solution to

$$(Id - \mathscr{T})\,(Id - \mathscr{T}_{OSDS})^{-1}(\tilde{h}) = G\,, \tag{19}$$

followed by $h := (Id - \mathscr{T}_{OSDS})^{-1}(\tilde{h})$.

More intuitively, the key idea is to cancel out the reverse contribution at the interfaces that should not happen for the exact ABC case in order to prevent spurious interface reflections from hindering the convergence. In fact, these boundary conditions at the interfaces produce spurious reflected waves that significantly increase the number of iterations to converge, in particular for heterogeneous media and high frequency regimes. Note that for a non overlapping domain decomposition, the OSDS algorithm is similar to the double sweep method of [8, 9]. Our approach addresses the case of overlapping subdomains that benefits the convergence rate.

## 3 Numerical results

In this section, we present numerical results when solving the substructured equation (7) with the GMRES algorithm right preconditioned by $Id$ (Jacobi method), $(Id-\mathscr{T}_{DS})^{-1}$ (Double sweep algorithm) and $(Id-\mathscr{T}_{OSDS})^{-1}$ (Overlapping Splitting Double sweep algorithm). Note that the Jacobi method requires $N$ subdomain solves per iteration instead of $2\,N$ for the sweeping methods. The Helmholtz equation is discretized with a P1 finite element using FreeFem++ [5]. Note that we use a careful variational discretisation of the normal derivative ensuring that the solution obtained converges to the solution of the problem without domain decompositions.

### 3.1 Wedge test

We consider the classical test case of the wedge, see e.g. [8], a rectangular domain $[0, 600]\times[0, 1000]$ with three different velocities in regions separated by non-parallel boundaries (Fig. 3 left). Starting from the top, the velocities are $c = 2000$, $c = 1500$ and $c = 3000$. Sommerfeld conditions are imposed on the bottom, right and left

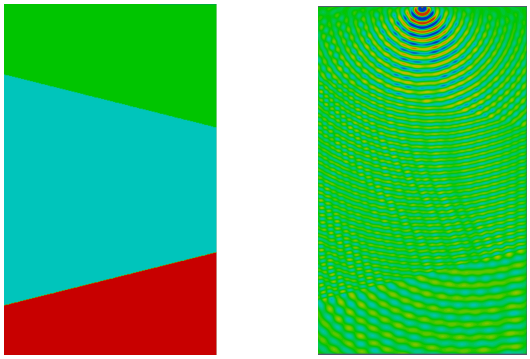

**Fig. 3:** Heterogeneous media: Wedge (**Left**: Velocity model, **Right**: Solution (real) for $\omega = 160\pi$)

boundaries. The abrupt variations of the wavenumber produce internal reflections in different directions. A typical solution is shown in Figure 3 right.

Iteration counts are given in Table 1. The OSDS method is clearly superior to the Jacobi and DS methods. When increasing the number of subdomains, the ratio in favor of the OSDS method compared to the DS method increases up to reaching a value of nearly 4 for a domain decomposition into 40 vertical strips. Interestingly, we see that for a low tolerance on the residual (TOL=$10^{-3}$), the OSDS iteration counts are almost independent of the number of subdomains.

| $N$ | $\omega = 40\pi$ | | | $\omega = 60\pi$ | | |
|---|---|---|---|---|---|---|
| | Jacobi | DS | OSDS | Jacobi | DS | OSDS |
| 5 | 28 (17) | 19 (11) | 13 (6) | 28 (15) | 18 (10) | 12 (5) |
| 10 | 55 (31) | 31 (16) | 14 (7) | 56 (30) | 31 (15) | 14 (6) |
| 20 | 110 (55) | 58 (29) | 18 (8) | 111 (53) | 57 (28) | 18 (7) |
| 40 | 203 (88) | 103 (47) | 27 (9) | 206 (85) | 111 (55) | 30 (9) |

**Table 1:** Wedge, $\omega = 40\pi$ and $60\pi$, $\delta = 16h$, TOL=$10^{-6}$($10^{-3}$), nppwl = 24 , P1

## 3.2 Influence of the overlap

We have also tested the effect of the width of the overlap on the convergence. We considered two test cases: the homogeneous waveguide and the wedge (see Table 2) that is defined in more detail in § 3.1. We observe that for the waveguide solved by the Overlapping Splitting Double Sweep method, the iteration count decreases significantly with increasing overlap. This monotonical decrease in the iteration count contrasts with the behaviour of the other two methods. We see that for the Jacobi and double sweep (DS) methods, the overlap has very little effect. For the Jacobi method it improves slightly the iteration counts whereas for the DS method, it might deteriorate the iteration count. For the wedge test case, all methods benefit

| $\delta$ | Homogeneous waveguide ($\omega = 20$) | | | Wedge ($\omega = 40\pi$) | | |
|---|---|---|---|---|---|---|
| | Jacobi | DS | OSDS | Jacobi | DS | OSDS |
| 2 | 159 | 69 | 27 | 259 | 127 | 97 |
| 4 | 165 | 74 | 23 | 245 | 117 | 83 |
| 8 | 160 | 76 | 20 | 221 | 105 | 69 |
| 16 | 143 | 73 | 18 | 202 | 91 | 53 |

**Table 2:** Influence of the overlap, $\delta$ varies, TOL=$10^{-6}$, nppwl = 24 , P1

monotonically from the size of the overlap but once again the reduction in the iteration count is more pronounced for the Overlapping Splitting Double Sweep method where the iteration count is reduced by a factor 1.83 when the overlap is increased from $2h$ to $16h$.

## 4 Conclusion

We have introduced an overlapping splitting double sweep algorithm which yields improved convergence for various problems. Many aspects deserve further investigations: higher-order ABC instead of the zero-th order one considered here and the introduction of a pipelining technique that can be applied to multiple right-hand sides problems to improve parallelism and achieve significant speed-ups, see [9].

## References

1. Xavier Antoine, Yassine Boubendir, and Christophe Geuzaine. A quasi-optimal non-overlapping domain decomposition algorithm for the Helmholtz equation. *Journal of Computational Physic*, 231(2):262–280, 2012.
2. Christiaan C. Stolk. A rapidly converging domain decomposition method for the Helmholtz equation. *Journal of Computational Physics*, 241:240–252, May 2013.
3. Bruno Després. Domain decomposition method and the helmholtz problem. In *Mathematical and numerical aspects of wave propagation phenomena (Strasbourg, 1991)*, pages 44–52, Philadelphia, PA, 1991. SIAM.
4. Martin J. Gander, Frédéric Magoulès, and Frédéric Nataf. Optimized Schwarz methods without overlap for the Helmholtz equation. *SIAM J. Sci. Comput.*, 24(1):38–60, 2002.
5. F. Hecht. New development in Freefem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.
6. Frédéric Nataf and Francis Nier. Convergence rate of some domain decomposition methods for overlapping and nonoverlapping subdomains. *Numerische Mathematik*, 75(3):357–77, 1997.
7. Frédéric Nataf, Francois Rogier, and Eric de Sturler. Optimal interface conditions for domain decomposition methods. Technical Report 301, CMAP (Ecole Polytechnique), 1994.
8. A. Vion and C. Geuzaine. Double sweep preconditioner for optimized Schwarz methods applied to the Helmholtz problem. *Journal of Computational Physics*, 266:171–190, 2014.
9. A. Vion and C. Geuzaine. Parallel Double Sweep Preconditioner for the Optimized Schwarz Algorithm Applied to High Frequency Helmholtz and Maxwell Equations. *Domain Decomposition Methods in Science and Engineering XXII*.
10. Leonardo Zepeda-Núñez and Laurent Demanet. The method of polarized traces for the 2D Helmholtz equation. *Journal of Computational Physics*, 308:347–388, March 2016.