Non-overlapping Spectral Additive Schwarz Methods for HDG and Multiscale Discretizations

Yi Yu, Maksymilian Dryja, and Marcus Sarkis

1 Introduction

In this paper, we design and state some theoretical results for the exact and inexact versions of Non-overlapping Spectral Additive Schwarz Methods (NOSAS) in the framework of Hybridizable Discontinuous Galerkin (HDG) discretizations and multiscale discretizations for the following elliptic problem:

$$\begin{aligned} \rho(x)^{-1}\mathbf{q} + \nabla u &= 0 && \text{in } \Omega, \\ \nabla \cdot \mathbf{q} &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\rho(x) \in L^\infty(\Omega)$, $\rho(x) \geq \rho_0 > 0$, $f \in L^2(\Omega)$ and Ω is a polyhedral domain in $\mathbb{R}^d (d \geq 2)$. The problem (1) has a unique solution $(\mathbf{q}, u) \in \mathbf{H}(\mathrm{div}, \Omega) \times H_0^1(\Omega)$, where $\mathbf{H}(\mathrm{div}, \Omega) := \{\mathbf{q} \in L^2(\Omega)^d,\ \mathrm{div}\ \mathbf{q} \in L^2(\Omega)\}$.

We begin by describing the HDG discretization. Consider a partitioning of the domain Ω into a conforming mesh $\mathcal{T}_h$ with elements K. We assume that the partition $\mathcal{T}_h$ is shape regular and quasi-uniform of size $O(h)$. A face of K is denoted by F and let $\mathcal{E}_h$ be the set of all faces of $\mathcal{T}_h$ excluding the ones on $\partial\Omega$. The HDG yields a scalar approximation u_h to u, a vector approximation $\mathbf{q}_h$ to $\mathbf{q}$, and a scalar approximation λ_h to the trace of u on element faces, in the spaces of $\mathbf{Q}_h = \{\mathbf{p} \in L^2(\mathcal{T}_h)^d : \mathbf{p}|_K \in \mathbf{P}_k(K), \forall K \in \mathcal{T}_h\}$, $W_h = \{w \in L^2(\mathcal{T}_h) : w|_K \in P_k(K), \forall K \in \mathcal{T}_h\}$ and $M_h = \{\mu \in L^2(\mathcal{E}_h) : \mu|_F \in P_k(F), \forall F \in \mathcal{E}_h\}$, respectively. Here $\mathbf{P}_k(K) = P_k(K)^d$ and $P_k(K)$ is the space of polynomials of order at most k on K.

Yi Yu
Worcester Polytechnic Institute, 100 Institute Rd, Worcester, USA, e-mail: yyu5@wpi.edu

Maksymilian Dryja
Warsaw University,Banacha 2, 00-097 Warsaw, Poland, e-mail: dryja@mimuw.edu.pl

Marcus Sarkis
Worcester Polytechnic Institute, 100 Institute Rd, Worcester, USA, e-mail: msarkis@wpi.edu.

To cope with the heterogeneous coefficients for each element, we define the numerical flux $\hat{\mathbf{q}}_h$ which is a double-valued vector function on mesh interfaces as follow:

$$\hat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau_K \rho_K (u_h - \lambda_h) \qquad \text{on } \mathcal{E}_h. \tag{2}$$

Here $\tau := \tau_K \rho_K$ is called stabilizer, and ρ_K is a constant which approximates $\rho(x)$ in element K, the nonnegative constant function τ_K defined on $\mathcal{E}_h$ can be either a single or a double valued function on the element interfaces and τ_K above denotes the $\tau-$value on the ∂K. The novelty of writing $\tau = \tau_K \rho_K$ is that we will introduce an equivalent norm of $a(\cdot,\cdot)$ independently of the coefficients. With the definitions of the numerical flux $\hat{\mathbf{q}}_h$, the HDG discretization of problem (1) can be written as: find $(\mathbf{q}_h, u_h, \lambda_h) \in \mathbf{Q}_h \times W_h \times M_h$, such that for all $(\mathbf{p}, w, \mu) \in \mathbf{Q}_h \times W_h \times M_h$:

$$(u_h, \nabla \cdot \mathbf{p})_{\mathcal{T}_h} - (\rho(x)^{-1}\mathbf{q}_h, \mathbf{p})_{\mathcal{T}_h} - < \lambda_h, \mathbf{p} \cdot \mathbf{n} >_{\partial \mathcal{T}_h} = 0, \tag{3a}$$

$$-(\mathbf{q}_h, \nabla w)_{\mathcal{T}_h} + < \hat{\mathbf{q}}_h \cdot \mathbf{n}, w >_{\partial \mathcal{T}_h} = (f, w)_{\mathcal{T}_h}, \tag{3b}$$

$$< \hat{\mathbf{q}}_h \cdot \mathbf{n}, \mu >_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0. \tag{3c}$$

It is proved in [1] that the system (3) is uniquely solvable and can be reduced into the matrix form of the following problem: find $\lambda_h \in M_h$ such that

$$a(\lambda_h, \mu) = b(\mu), \quad \forall \mu \in M_h. \tag{4}$$

Here

$$a(\eta, \mu) = \sum_{K \in \mathcal{T}_h} a_K(\eta, \mu) = \sum_{K \in \mathcal{T}_h} (\rho_K^{-1} Q\eta, Q\mu)_K + <\tau_K \rho_K (U\eta - \eta), (U\mu - \mu)>_{\partial K}$$

and $b(\mu) = \sum_{K \in \mathcal{T}_h} b_K(\mu) = \sum_{K \in \mathcal{T}_h} (f, U\mu)_K$, where $Q\nu \in \mathbf{Q}_h$ and $U\nu \in W_h$ are the unique solution of the local element problem (3) with $\lambda_h = \nu$ and right hand side $f = 0$. We note that once we get λ_h, the solution of (3) can be completed by computing $\mathbf{q}_h$ and u_h in each element separately. Note that the bilinear form $a(\cdot,\cdot)$ is positive definite. Let us define the norm $|||\cdot|||_{\rho,h}$ as follows:

$$|||\lambda|||_{\rho,h} = \Big(\sum_{K \in \mathcal{T}_h} \frac{\rho_K}{h} ||\lambda - m_k(\lambda)||^2_{L^2(\partial K)} \Big)^{1/2}, \tag{5}$$

where $m_K(\lambda) = \frac{1}{|\partial K|} \int_{\partial K} \lambda ds$. The next theorem shows the norm $|||\cdot|||_{\rho,h}$ is equivalent to the energy norm $a(\cdot,\cdot)$, for the proof see [1].

Theorem 1 *For all $\lambda \in M_h$, there are positive constants C_1, C_2, independent of h and ρ_K, such that*

$$C_1 |||\lambda|||^2_{\rho,h} \le a(\lambda, \lambda) \le C_2 \gamma \, |||\lambda|||^2_{\rho,h},$$

where $\gamma = 1 + \max_{K \in \mathcal{T}_h} \tau_K^* h$, and τ_K^* denotes the second largest value of τ_K on ∂K.

NOSAS were first introduced for Continuous Galerkin (CG) discretizations in [7, 8] as domain decomposition preconditioners designed to elliptic problems with

highly heterogeneous coefficients. NOSAS are non-overlapping Schwarz preconditioners where the subdomain interactions are via the coarse problem. The coarse problem involves local and global interactions. The global component is introduced to guarantee the robustness of the preconditioners for any coefficients $\rho(x)$ and number of subdomains. The proposed global problem is built from generalized eigenfunctions on the subdomains. The size of the global problem is equal to the total number of those eigenfunctions and is only related to the number of islands or channels with high-contrast coefficients that touch the boundary of the subdomains, see [9]. Additionally, the inexact version of NOSAS has good parallelization properties. The main goal of this paper is to design and show results of NOSAS for HDG and multiscale discretizations. We note that other kinds of domain decomposition preconditioners for HDG were introduced in [1, 6].

2 Domain decomposition setting

We decompose Ω into N non-overlapping polygonal subdomains Ω_i of size $O(H)$. The local spaces $V_i, (1 \leq i \leq N)$ are the restriction of M_h on Ω_i and vanishing on $\partial\Omega_i$ and coarse space V_0 is the restriction of M_h on the interface of all subdomain. Then M_h admits the following direct sum decomposition:

$$M_h = R_0^T V_0 \oplus R_1^T V_1 \oplus \cdots \oplus R_N^T V_N .$$

The local extrapolation operators $R_i^T : V_i \rightarrow M_h (1 \leq i \leq N)$ is the extension by zero outside of Ω_i. The coarse extrapolation operators $R_0^T : V_0 \rightarrow M_h$ is the core of NOSAS which we will define and state theoretical results in Section 3.

For $1 \leq i \leq N$, denote matrix A_i corresponding to the exact local bilinear form:

$$a_i(u, v) = v^T A_i \, u = a(R_i^T u, R_i^T v) \qquad u, v \in V_i,$$

For $i = 0$, we first consider matrix A_0 corresponding to the exact bilinear form:

$$a_0(u, v) = v^T A_0 \, u = a(R_0^T u, R_0^T v) \qquad u, v \in V_0.$$

We will also consider inexact bilinear form $\hat{a}_0(\cdot, \cdot)$ later in this paper. Then the non-overlapping Schwarz preconditioner have the following forms:

$$T_A = B^{-1} A, \qquad B^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i .$$

We note that if we had chosen R_0^T as the a-discrete harmonic extension, then the above preconditioner would become a direct solver and it would be too expensive to solve the coarse problem. The core of NOSAS is to use a low-rank a-discrete harmonic extension R_0^T, which is inexpensive to solve the coarse problem, also guarantees good condition numbers.

3 NOSAS with exact and inexact solver

The linear system $A\lambda_h = b$ corresponding to (4) can be assembled by the Neumann matrix $A^{(i)}$ and $b^{(i)}$ in each subdomain Ω_i. We decompose $A^{(i)}$ into blocked matrix $(A^{(i)}_{\Gamma\Gamma}\ A^{(i)}_{\Gamma I}\ ;\ A^{(i)}_{I\Gamma}\ A^{(i)}_{II})$ and $b^{(i)}$ into $(b^{(i)}_{\Gamma}\ ;\ b^{(i)}_{I})$, where subscript Γ, I denote the parts associated with the interface of subdomain and interior of subdomain, respectively. The Schur complement of A and $A^{(i)}$ denote as S and $S^{(i)}$, respectively.

For the NOSAS exact solver, solve the generalized eigenvalue problem in each subdomain $(i = 1, \cdots, N)$ separately:

$$S^{(i)}\xi^{(i)}_j := (A^{(i)}_{\Gamma\Gamma} - A^{(i)}_{\Gamma I}(A^{(i)}_{II})^{-1}A^{(i)}_{I\Gamma})\xi^{(i)}_j = \Lambda^{(i)}_j A^{(i)}_{\Gamma\Gamma}\xi^{(i)}_j \quad (1 \le j \le n_i), \tag{6}$$

where n_i is the degrees of freedom on $\Gamma_i := \Gamma \cap \partial\Omega_i$. Note that the eigenvalue lies in $[0, 1]$ for the above generalized eigenvalue problem. We fix a threshold $\delta < 1$ and pick the smallest k_i eigenvalues $\le \delta$ and corresponding eigenvectors to construct eigenfunctions space $Q^{(i)}$ and harmonic extension $P^{(i)}$ as follow:

$$Q^{(i)} = [\xi^{(i)}_1, \xi^{(i)}_2, \cdots, \xi^{(i)}_{k_i}] \qquad \text{and} \qquad P^{(i)} = -(A^{(i)}_{II})^{-1}A^{(i)}_{I\Gamma}Q^{(i)}.$$

We also define $D^{(i)} = \text{diagonal}(1 - \Lambda^{(i)}_1, 1 - \Lambda^{(i)}_2, \cdots, 1 - \Lambda^{(i)}_{k_i}) = I - \Lambda^{(i)}$.

For $u_0 \in V_0$, we define the global extension $R_0^T : V_0 \to M_h$ as:

$$R_0^T u_0 = \begin{bmatrix} u_0 \\ -\sum_{i=1}^{N} P^{(i)}(P^{(i)^T}A^{(i)}_{II}P^{(i)})^{-1}P^{(i)^T}A^{(i)}_{I\Gamma}u^{(i)}_0 \end{bmatrix} = \begin{bmatrix} u_0 \\ \sum_{i=1}^{N} P^{(i)}(Q^{(i)^T}A^{(i)}_{\Gamma\Gamma}Q^{(i)})^{-1}Q^{(i)^T}A^{(i)}_{\Gamma\Gamma}u^{(i)}_0 \end{bmatrix},$$

where $u^{(i)}_0$ is the restriction of u_0 on Γ_i. Below $v^{(i)}_0$ denotes the restriction of v_0 to Γ_i. Next $\forall u_0, v_0 \in V_0$, we define the exact coarse bilinear form as:

$$\begin{aligned} a_0(u_0, v_0) = a(R_0^T u_0, R_0^T v_0) &= \sum_{i=1}^{N} v^{(i)^T}_0 \big(A^{(i)}_{\Gamma\Gamma} - A^{(i)}_{\Gamma I}P^{(i)}(P^{(i)^T}A^{(i)}_{II}P^{(i)})^{-1}P^{(i)^T}A^{(i)}_{I\Gamma}\big)u^{(i)}_0 \\ &= \sum_{i=1}^{N} a^{(i)}_0(u^{(i)}_0, v^{(i)}_0) = \sum_{i=1}^{N} v^{(i)^T}_0 \big(A^{(i)}_{\Gamma\Gamma} - A^{(i)}_{\Gamma\Gamma}Q^{(i)}D^{(i)}(Q^{(i)^T}A^{(i)}_{\Gamma\Gamma}Q^{(i)})^{-1}Q^{(i)^T}A^{(i)}_{\Gamma\Gamma}\big)u^{(i)}_0. \end{aligned}$$

Above, $a_0(\cdot,\cdot)$ is the global bilinear form and $a^{(i)}_0(\cdot,\cdot)$ is the bilinear form on Γ_i locally. The next lemma shows that $a^{(i)}_0(\cdot,\cdot)$ is equivalent to Schur complement $S^{(i)}$ in the span of $Q^{(i)}$, and an extension by zero for the orthogonal complement subspace.

Lemma 1 *([9]) Let $\Pi^{(i)}_S u^{(i)}_0$ be the projection of $u^{(i)}_0$ onto Span$\{Q^{(i)}\}$. That is, $\Pi^{(i)}_S u^{(i)}_0 := Q^{(i)}(Q^{(i)}A^{(i)}_{\Gamma\Gamma}Q^{(i)})^{-1}Q^{(i)}A^{(i)}_{\Gamma\Gamma}u^{(i)}_0$. Then:*

$$a_0^{(i)}(u_0^{(i)}, v_0^{(i)}) = (\Pi_S^{(i)} v_0^{(i)})^T S^{(i)} (\Pi_S^{(i)} u_0^{(i)}) + (v_0^{(i)} - \Pi_S^{(i)} v_0^{(i)})^T A_{\Gamma\Gamma}^{(i)} (u_0^{(i)} - \Pi_S^{(i)} u_0^{(i)}).$$

Lemma 2 *([9]) Let $u_0 \in V_0$ then*

$$a_0(u_0, u_0) = \sum_{i=1}^{N} a_0^{(i)}(u_0^{(i)}, u_0^{(i)}) \leq \sum_{i=1}^{N} \frac{1}{\delta} u_0^{(i)^T} S^{(i)} u_0^{(i)} = \frac{1}{\delta} u_0^T S u_0.$$

Using Lemma 1 and Lemma 2 and the classical Schwarz Theory [5] we have:

Theorem 2 *([9]) For any $u \in M_h$, the following holds:*

$$(2 + \frac{3}{\delta})^{-1} a(u, u) \leq a(T_A u, u) \leq 2a(u, u) \Longrightarrow k(T_A) \leq 2(2 + \frac{3}{\delta}).$$

In the implementation of NOSAS, the complexity of the coarse problem involves computing $A_{\Gamma\Gamma}^{-1}$ and this complexity can be reduced if we replace $A_{\Gamma\Gamma}$ by its diagonal $\hat{A}_{\Gamma\Gamma}$. This version is called the inexact NOSAS with $\tilde{T}_A$ as the preconditioner. The generalized eigenvalue problem is now given by:

$$S^{(i)} \hat{\xi}_i^{(i)} := (A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} (A_{II}^{(i)})^{-1} A_{I\Gamma}^{(i)}) \hat{\xi}_j^{(i)} = \hat{\Lambda}_j^{(i)} \hat{A}_{\Gamma\Gamma}^{(i)} \hat{\xi}_j^{(i)}.$$

And for $u_0, v_0 \in V_0(\Omega)$, we define the inexact coarse solver as:

$$\hat{a}_0(u_0, v_0) = \sum_{i=1}^{N} v_0^{(i)^T} (\hat{A}_{\Gamma\Gamma}^{(i)} - \hat{A}_{\Gamma\Gamma}^{(i)} \hat{Q}^{(i)} \hat{D}^{(i)} (\hat{Q}^{(i)^T} \hat{A}_{\Gamma\Gamma}^{(i)} \hat{Q}^{(i)})^{-1} \hat{Q}^{(i)^T} \hat{A}_{\Gamma\Gamma}^{(i)}) u_0^{(i)},$$

where $\hat{Q}^{(i)}$are the generalized eigenvectors and $\hat{D}^{(i)}$=diagonal$(1 - \hat{\Lambda}_1^{(i)}, \cdots, 1 - \hat{\Lambda}_{k_i}^{(i)})$. Then, we obtain the following condition number estimate:

Theorem 3 *([9]) For any $u \in M_h$, the following holds:*

$$(2 + \frac{5}{\delta})^{-1} a(u, u) \leq a(\tilde{T}_A u, u) \leq 3a(u, u) \Longrightarrow k(\tilde{T}_A) \leq 3(2 + \frac{5}{\delta}).$$

4 Multiscale discretizations methods

The idea of multiscale methods [2, 3] is to use $\lambda_{ms} \in V_{\text{off}}$ to approximate the exact solution λ_h from $a(\lambda_h, \mu) = b(\mu), \forall \mu \in M_h$, where V_{off} is the space of multiscale basis functions. The following procedures show how we construct V_{off}. The first step, a snapshot space $V_{\text{snapshots}}$ is constructed by the solutions of local problems. In our NOSAS methods, we construct the snapshot space by $V_{\text{snapshots}} = \mathcal{H}^T V_0$, where $\mathcal{H}^T$ is the a-discrete harmonic extension. Notice that the dimension of $V_{\text{snapshots}}$ can be extremely large. The next step is to construct V_{off} from $V_{\text{snapshots}}$ which can be used to generate an efficient and accurate approximation to the multiscale solution. We choose offline space $V_{\text{off}} = R_0^T V_0$ where R_0^T is the global extension for NOSAS.

We use the following outline of the Generalized Multiscale Finite Element Method (GMsFEM) to show coarse space of NOSAS is a multiscale discretization.

Offline stages:

1. Mesh partitioning to obtain the subdomains.
2. Construct $V_{\text{snapshots}}$ that will be used to compute an offline space.
3. Construct a small dimensional offline space V_{off} by performing dimension reduction in the space of local snapshots. This is done by choosing a threshold δ and then compute multiscale basis functions.
4. Build the coarse and local matrices and factorize.

Online stages:

1. Given f, solve the local problems inside each subdomain and update the residual on the interfaces.
2. Solve a coarse problem. Add coarse and local solutions.

In the offline stages, we construct the $V_{\text{off}} = R_0^T V_0$ by NOSAS and also compute the factorization of matrices A_0 and $A_i (1 \leq i \leq N)$. In the online stage, we first solve N local problems in parallel:

$$A_i \lambda_i = R_i b = b_i \qquad 1 \leq i \leq N.$$

Then, we using the local solutions to form and solve the following coarse problem:

$$A_0 \lambda_0 = R_0 (b - A \sum_{i=1}^{N} R_i^T \lambda_i).$$

Finally, $\lambda_{ms} = R_0^T \lambda_0 + \sum_{i=1}^{N} R_i^T \lambda_i$ is obtained.

We note that we have similar numerical results if using $A_0 \lambda_0 = R_0 b$ as the coarse problem. Additionally, we can also develop a multiscale technique to reduce the dimension of the local problems, see [4].

Theorem 4 *For NOSAS methods with $\lambda_{ms} = R_0^T \lambda_0 + \sum_{i=1}^{N} R_i^T \lambda_i$, holds*

$$a(\lambda_h - \lambda_{ms}, \lambda_h - \lambda_{ms}) \leq (1 - \delta)\, a(\mathcal{H}^T \lambda_\Gamma, \mathcal{H}^T \lambda_\Gamma),$$

where $\mathcal{H}^T$ is the a-discrete harmonic extension and λ_Γ the restriction of λ_h on Γ.

The proof follows from Lemma 1. This bound is not sharp since we can see on numerical experiments for heterogeneous coefficient that a small δ can give small relative errors.

5 Numerical Experiments

We first show results of problem (1) for square domain with side length 1, $f \equiv 1$ and with highly heterogeneous coefficients in the following mesh (see Figure 1).

We choose $k = 0$ for HDG spaces $\mathbf{Q}_h$, W_h, and M_h. We divide the square domain into $H \times H$ congruent square subdomains, and we fix the number of stripes in each subdomain. That means we always have two horizontal stripes and two vertical stripes in each subdomain. The coefficients $\rho(x) = 1$ in the green stripes and $\rho(x) = 10^6$ in the white regions. We do not consider $\rho(x) = 10^6$ in stripes and $\rho(x) = 1$ elsewhere. Because it is robust without the generalized eigenfunctions. In Table 2 we show numerical results for HDG with different values of $\tau = \tau_K \rho_K$ in this mesh using NOSAS with exact solvers. By choosing $\delta = \frac{1}{4}\frac{h}{H}$ the NOSAS will not deteriorate due to the small eigenvalues related to the jumps of coefficients, and the condition number is $O(H/h)$. We also note that we see little difference in numerical results when we use inexact diagonal solvers or exact solvers (We do not include the results here.) In Table 1, we show that the size of the global part for the coarse problem of NOSAS is proportional to the number of subdomains and does not depend on H/h.

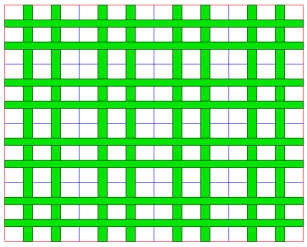

Fig. 1: Coefficients $\rho(x) = 1$ in green stripes and $\rho(x) = 10^6$ in white regions.

	H=1/2	H=1/4	H=1/8	H=1/16
H/h=8	12	84	420	1860
H/h=16	12	84	420	1860

Table 1: The number of all eigenfunctions N_E for NOSAS with the exact solver. The size of the global problem is $N_E \times N_E$.

Finally, Table 3 shows results for multiscale in HDG. We use a similar mesh in Figure 1. However, the number of stripes fixed in the whole domain. That means we always have eight horizontal stripes and eight vertical stripes with width h in the whole domain. The coefficients $\rho(x) = 1$ in the stripes and $\rho(x) = 10^6$ elsewhere. We fix h and $\tau = \rho_K$, and choose different δ to show the relative error of $\lambda_h - \lambda_{ms}$ and the number of coarse basis functions per subdomain. Since we use the exact local solver in each subdomain, we expect that the relative error of $\lambda_h - \lambda_{ms}$ will increase if we decrease H because the error arises from approximating the exact local solution in each subdomain.

References

1. Bernardo Cockburn, Olivier Dubois, Jay Gopalakrishnan, and Shuguang Tan. Multigrid for an HDG method. *IMA Journal of Numerical Analysis*, 34(4):1386–1425, 2014.
2. Yalchin Efendiev, Juan Galvis, and Thomas Y Hou. Generalized multiscale finite element methods (GMsFEM). *Journal of Computational Physics*, 251:116–135, 2013.

$\tau = \rho_K$	$H = \frac{1}{2}$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$
$\frac{H}{h} = 8$	14 (10.2596)	17 (10.2686)	17 (10.2732)	19 (10.2756)
$\frac{H}{h} = 16$	20 (20.8039)	25 (20.8062)	26 (20.8074)	29 (20.8080)

(a) τ=$\tau_K\rho_K$ with τ_K= 1.

$\tau = \frac{2}{\rho_{K_1}^{-1}+\rho_{K_2}^{-1}}$	$H = \frac{1}{2}$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$
$\frac{H}{h} = 8$	14 (10.2516)	17 (10.2646)	18 (10.2713)	18 (10.2746)
$\frac{H}{h} = 16$	20 (20.7990)	24 (20.8038)	27 (20.8062)	27 (20.8074)

(b) τ is the harmonic mean of ρ_K in adjacent elements.

$\tau = \frac{\rho_{K_1}+\rho_{K_2}}{2}$	$H = \frac{1}{2}$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$
$\frac{H}{h} = 8$	14 (10.2562)	17 (10.2669)	17 (10.2724)	17 (10.2752)
$\frac{H}{h} = 16$	19 (20.8019)	25 (20.8052)	26 (20.8069)	26 (20.8078)

(c) τ is the arithmetic mean of ρ_K in adjacent elements.

Table 2: NOSAS with exact solver for HDG with different choices of τ. The number of iterations of the PCG required to reduced the residual by 10^{-6} and the condition number (in parenthesis).

	$H = \frac{1}{2}$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$
δ =10^{-5}	7 ($3.21e^{-5}$)	5.25 ($7.17e^{-5}$)	3.06 ($1.55e^{-4}$)	1.89 ($7.54e^{-4}$)
$\delta = 1/2$	24.25 ($3.21e^{-5}$)	18.56 ($7.15e^{-5}$)	11.26 ($1.54e^{-4}$)	5.05 ($7.50e^{-4}$)
$\delta = 3/4$	63 ($4.10e^{-18}$)	45.75 ($1.40e^{-14}$)	24.93 ($1.14e^{-13}$)	11.48 ($8.03e^{-13}$)

Table 3: NOSAS as a multiscale methods with fix $h = 1/64$. The number of global basis functions per subdomain and the relative energy error $\lambda_h - \lambda_{ms}$ with respect to λ_h (in parenthesis).

3. Shubin Fu, Eric Chung, and Guanglian Li. Edge multiscale methods for elliptic problems with heterogeneous coefficients. *Journal of Computational Physics*, 396:228–242, 2019.
4. Alexandre L. Madureira and Marcus Sarkis. Hybrid localized spectral decomposition for multiscale problems. *SIAM J. Numer. Anal.*, 59(2):829–863, 2021.
5. Andrea Toselli and Olof Widlund. *Domain decomposition methods-algorithms and theory*, volume 34. Springer Science & Business Media, 2006.
6. Xuemin Tu and Bin Wang. A BDDC algorithm for second-order elliptic problems with hybridizable discontinuous Galerkin discretizations. *Electronic Transactions on Numerical Analysis*, 45:354–370, 2016.
7. Yi Yu, Maksymilian Dryja, and Marcus Sarkis. Non-overlapping spectral additive Schwarz methods. In *International Conference on Domain Decomposition Methods Proceedings of DD25*, pages 375–382. Springer, 2018.
8. Yi Yu, Maksymilian Dryja, and Marcus Sarkis. From additive average Schwarz methods to non-overlapping spectral additive Schwarz methods. *SIAM Journal on Numerical Analysis*, 2021.
9. Yi Yu, Maksymilian Dryja, and Marcus Sarkis. Non-overlapping spectral additive schwarz methods for HDG discretizations. In preparation, 2021.