Non-overlapping Spectral Additive Schwarz Methods for HDG and Multiscale Discretizations

Yi Yu, Maksymilian Dryja, and Marcus Sarkis

1 Introduction

In this paper, we design and state some theoretical results for the exact and inexact versions of Non-overlapping Spectral Additive Schwarz Methods (NOSAS) in the framework of Hybridizable Discontinuous Galerkin (HDG) discretizations and multiscale discretizations for the following elliptic problem:

\[ \begin{align*}
\rho(x)^{-1} \mathbf{q} + \nabla u &= 0 & \text{in } \Omega, \\
\nabla \cdot \mathbf{q} &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*} \]

where \( \rho(x) \in L^\infty(\Omega) \), \( \rho(x) \geq \rho_0 > 0 \), \( f \in L^2(\Omega) \) and \( \Omega \) is a polyhedral domain in \( \mathbb{R}^d \) \((d \geq 2)\). The problem (1) has a unique solution \((\mathbf{q}, u) \in H(\text{div}, \Omega) \times H^1_0(\Omega)\), where \( H(\text{div}, \Omega) := \{\mathbf{q} \in L^2(\Omega)^d, \text{div} \mathbf{q} \in L^2(\Omega)\} \).

We begin by describing the HDG discretization. Consider a partitioning of the domain \( \Omega \) into a conforming mesh \( T_h \) with elements \( K \). We assume that the partition \( T_h \) is shape regular and quasi-uniform of size \( O(h) \). A face of \( K \) is denoted by \( F \) and let \( E_h \) be the set of all faces of \( T_h \) excluding the ones on \( \partial \Omega \). The HDG yields a scalar approximation \( u_h \) to \( u \), a vector approximation \( \mathbf{q}_h \) to \( \mathbf{q} \), and a scalar approximation \( \lambda_h \) to the trace of \( u \) on element faces, in the spaces of \( \mathbf{Q}_h = \{p \in L^2(T_h)^d : p|_K \in \mathbf{P}_k(K), \forall K \in T_h\} \), \( W_h = \{w \in L^2(T_h) : w|_K \in \mathbf{P}_k(K), \forall K \in T_h\} \) and \( M_h = \{\mu \in L^2(E_h) : \mu|_F \in \mathbf{P}_k(F), \forall F \in E_h\} \), respectively. Here \( \mathbf{P}_k(K) = P_k(K)^d \) and \( P_k(K) \) is the space of polynomials of order at most \( k \) on \( K \).

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To cope with the heterogeneous coefficients for each element, we define the numerical flux \( \hat{q}_h \) which is a double-valued vector function on mesh interfaces as follow:

\[
\hat{q}_h \cdot n = q_h \cdot n + \tau \rho_K (u_h - \lambda_h) \quad \text{on } \mathcal{E}_h. \tag{2}
\]

Here \( \tau := \tau_K \rho_K \) is called stabilizer, and \( \rho_K \) is a constant which approximates \( \rho(x) \) in element \( K \), the nonnegative constant function \( \tau_K \) defined on \( \mathcal{E}_h \) can be either a single or a double valued function on the element interfaces and \( \tau_K \) above denotes the \( \tau \)-value on the \( \partial K \). The novelty of writing \( \tau = \tau_K \rho_K \) is that we will introduce an equivalent norm of \( a(\cdot, \cdot) \) independently of the coefficients. With the definitions of the numerical flux \( \hat{q}_h \), the HDG discretization of problem (1) can be written as: find \( (q_h, u_h, \lambda_h) \in Q_h \times W_h \times M_h, \) such that for all \( (p, w, \mu) \in Q_h \times W_h \times M_h \):

\[
\begin{align*}
(u_h, \nabla \cdot p)_{\Omega_K} - (\rho(x)^{-1} q_h, p)_{\Omega_K} - \langle \lambda_h, p \cdot n \rangle_{\partial \Omega_K} &= 0, \\
- \langle \hat{q}_h, \nabla w \rangle_{\Omega_K} + \langle q_h \cdot n, w \rangle_{\partial \Omega_K} &= \langle f, w \rangle_{\Omega_K}, \\
\langle \hat{q}_h \cdot n, \mu \rangle_{\partial \Omega_K} &= 0.
\end{align*} \tag{3}
\]

It is proved in [1] that the system (3) is uniquely solvable and can be reduced into the matrix form of the following problem: find \( \lambda_h \in M_h \) such that

\[
a(\lambda_h, \mu) = b(\mu), \quad \forall \mu \in M_h. \tag{4}
\]

Here

\[
a(\eta, \mu) = \sum_{K \in \mathcal{T}_h} a_K(\eta, \mu) = \sum_{K \in \mathcal{T}_h} (\rho_K^{-1} Q \eta, Q \mu)_K + \langle \tau_K \rho_K (U \eta - \eta), (U \mu - \mu) \rangle_{\partial K}
\]

and \( b(\mu) = \sum_{K \in \mathcal{T}_h} b_K(\mu) = \sum_{K \in \mathcal{T}_h} (f, U \mu)_K \), where \( Q \eta \in Q_h \) and \( U \eta \in W_h \) are the unique solution of the local element problem (3) with \( \lambda_h = \eta \) and right hand side \( f = 0 \). We note that once we get \( \lambda_h \), the solution of (3) can be completed by computing \( q_h \) and \( u_h \) in each element separately. Note that the bilinear form \( a(\cdot, \cdot) \) is positive definite. Let us define the norm \( ||| \cdot \|||_{\rho, h} \) as follows:

\[
||| \lambda |||_{\rho, h} = \left( \sum_{K \in \mathcal{T}_h} \frac{\rho_K}{h} ||| \lambda - m_K(\lambda) \|||_{L^2(\partial K)}^2 \right)^{1/2}, \tag{5}
\]

where \( m_K(\lambda) = \frac{1}{|\partial K|} \int_{\partial K} \lambda ds \). The next theorem shows the norm \( ||| \cdot \|||_{\rho, h} \) is equivalent to the energy norm \( a(\cdot, \cdot) \), for the proof see [1].

**Theorem 1** For all \( \lambda \in M_h \), there are positive constants \( C_1, C_2 \), independent of \( h \) and \( \rho_K \), such that

\[
C_1 ||| \lambda |||_{\rho, h}^2 \leq a(\lambda, \lambda) \leq C_2 \gamma ||| \lambda |||_{\rho, h}^2,
\]

where \( \gamma = 1 + \max_{K \in \mathcal{T}_h} \tau_K^* h \), and \( \tau_K^* \) denotes the second largest value of \( \tau_K \) on \( \partial K \).

NOSAS were first introduced for Continuous Galerkin (CG) discretizations in [7, 8] as domain decomposition preconditioners designed to elliptic problems with
NOSAS for HDG and Multiscale Discretizations

highly heterogeneous coefficients. NOSAS are non-overlapping Schwarz preconditioners where the subdomain interactions are via the coarse problem. The coarse problem involves local and global interactions. The global component is introduced to guarantee the robustness of the preconditioners for any coefficients $\rho(x)$ and number of subdomains. The proposed global problem is built from generalized eigenfunctions on the subdomains. The size of the global problem is equal to the total number of those eigenfunctions and is only related to the number of islands or channels with high-contrast coefficients that touch the boundary of the subdomains, see [9]. Additionally, the inexact version of NOSAS has good parallelization properties. The main goal of this paper is to design and show results of NOSAS for HDG and multiscale discretizations. We note that other kinds of domain decomposition preconditioners for HDG were introduced in [1, 6].

2 Domain decomposition setting

We decompose $\Omega$ into $N$ non-overlapping polygonal subdomains $\Omega_i$ of size $O(H)$. The local spaces $V_i$, $(1 \leq i \leq N)$ are the restriction of $M_h$ on $\Omega_i$ and vanishing on $\partial \Omega_i$ and coarse space $V_0$ is the restriction of $M_h$ on the interface of all subdomain. Then $M_h$ admits the following direct sum decomposition:

$$M_h = R^T_0 V_0 \oplus R^T_1 V_1 \oplus \cdots \oplus R^T_N V_N.$$  

The local extrapolation operators $R^T_i : V_i \rightarrow M_h(1 \leq i \leq N)$ is the extension by zero outside of $\Omega_i$. The coarse extrapolation operators $R^T_0 : V_0 \rightarrow M_h$ is the core of NOSAS which we will define and state theoretical results in Section 3.

For $1 \leq i \leq N$, denote matrix $A_i$ corresponding to the exact local bilinear form:

$$a_i(u, v) = v^T A_i u = a(R^T_i u, R^T_i v) \quad u, v \in V_i,$$

For $i = 0$, we first consider matrix $A_0$ corresponding to the exact bilinear form:

$$a_0(u, v) = v^T A_0 u = a(R^T_0 u, R^T_0 v) \quad u, v \in V_0.$$

We will also consider inexact bilinear form $\hat{a}_0(\cdot, \cdot)$ later in this paper. Then the non-overlapping Schwarz preconditioner have the following forms:

$$T_A = B^{-1} A,$$

$$B^{-1} = R^T_0 A_0^{-1} R_0 + \sum_{i=1}^N R^T_i A_i^{-1} R_i.$$

We note that if we had chosen $R^T_0$ as the $a$-discrete harmonic extension, then the above preconditioner would become a direct solver and it would be too expensive to solve the coarse problem. The core of NOSAS is to use a low-rank $a$-discrete harmonic extension $R^T_i$, which is inexpensive to solve the coarse problem, also guarantees good condition numbers.
3 NOSAS with exact and inexact solver

The linear system \( A \mathbf{u}_h = b \) corresponding to (4) can be assembled by the Neumann matrix \( A^{(i)} \) and \( b^{(i)} \) in each subdomain \( \Omega_i \). We decompose \( A^{(i)} \) into blocked matrix \( (A^{(i)}_{\Gamma \Gamma}, A^{(i)}_{\Gamma I}; A^{(i)}_{I \Gamma}, A^{(i)}_{II}) \) and \( b^{(i)} \) into \( (b^{(i)}_{\Gamma}; b^{(i)}_{I}) \), where subscript \( \Gamma, I \) denote the parts associated with the interface of subdomain and interior of subdomain, respectively. The Schur complement of \( A \) and \( A^{(i)} \) denote as \( S \) and \( S^{(i)} \), respectively.

For the NOSAS exact solver, solve the generalized eigenvalue problem in each subdomain \( (i = 1, \ldots, N) \) separately:

\[
S^{(i)} \xi_j^{(i)} := (A^{(i)}_{\Gamma \Gamma} - A^{(i)}_{\Gamma I}(A^{(i)}_{II})^{-1} A^{(i)}_{I \Gamma})\xi_j^{(i)} = \lambda_j^{(i)} A^{(i)}_{\Gamma \Gamma} \xi_j^{(i)} \quad (1 \leq j \leq n_i),
\]

where \( n_i \) is the degrees of freedom on \( \Gamma_i := \Gamma \cap \partial \Omega_i \). Note that the eigenvalue lies in \([0, 1]\) for the above generalized eigenvalue problem. We fix a threshold \( \delta < 1 \) and pick the smallest \( k_i \) eigenvalues \( \leq \delta \) and corresponding eigenvectors to construct eigenfunctions space \( Q^{(i)} \) and harmonic extension \( P^{(i)} \) as follow:

\[
Q^{(i)} = \{ \xi_1^{(i)}, \xi_2^{(i)}, \ldots, \xi_k^{(i)} \} \quad \text{and} \quad P^{(i)} = -(A^{(i)}_{II})^{-1} A^{(i)}_{I \Gamma} Q^{(i)}.
\]

We also define \( D^{(i)} = \text{diagonal}(1 - \Lambda_1^{(i)}, 1 - \Lambda_2^{(i)}, \ldots, 1 - \Lambda_k^{(i)}) = I - \Lambda^{(i)} \).

For \( u_0 \in V_0 \), we define the global extension \( R^T_0 : V_0 \to \mathcal{M}_h \) as:

\[
R^T_0 u_0 = \left[ \begin{array}{c} u_0 \\ \sum_{i=1}^{N} p^{(i)}(P^{(i)^T} A^{(i)}_{II} P^{(i)})^{-1} P^{(i)^T} A^{(i)}_{I \Gamma} u_0 \end{array} \right] = \left[ \begin{array}{c} u_0 \\ \sum_{i=1}^{N} p^{(i)}(Q^{(i)^T} A^{(i)}_{I \Gamma} Q^{(i)})^{-1} Q^{(i)^T} A^{(i)}_{II} u_0 \end{array} \right],
\]

where \( u_0^{(i)} \) is the restriction of \( u_0 \) on \( \Gamma_i \). Below \( v_0^{(i)} \) denotes the restriction of \( v_0 \) to \( \Gamma_i \). Next \( \forall u_0, v_0 \in V_0 \), we define the exact coarse bilinear form as:

\[
a_0(u_0, v_0) = a(R^T_0 u_0, R^T_0 v_0) = \sum_{i=1}^{N} v_0^{(i)^T} (A^{(i)}_{\Gamma \Gamma} - A^{(i)}_{\Gamma I}(A^{(i)}_{II})^{-1} A^{(i)}_{I \Gamma})^{-1} P^{(i)^T} A^{(i)}_{II} u_0^{(i)}
\]

\[
= \sum_{i=1}^{N} a_0^{(i)}(u_0^{(i)}, v_0^{(i)}) = \sum_{i=1}^{N} v_0^{(i)^T} (A^{(i)}_{\Gamma \Gamma} - A^{(i)}_{\Gamma I} Q^{(i)} D^{(i)} Q^{(i)^T} A^{(i)}_{II} Q^{(i)^T} A^{(i)}_{I \Gamma}) u_0^{(i)}.
\]

Above, \( a_0(\cdot, \cdot) \) is the global bilinear form and \( a_0^{(i)}(\cdot, \cdot) \) is the bilinear form on \( \Gamma_i \) locally. The next lemma shows that \( a_0^{(i)}(\cdot, \cdot) \) is equivalent to Schur complement \( S^{(i)} \) in the span of \( Q^{(i)} \), and an extension by zero for the orthogonal complement subspace.

**Lemma 1** ([9]) Let \( \Pi^{(i)}_{S} u^{(i)} \) be the projection of \( u^{(i)}_0 \) onto \( \text{Span}(Q^{(i)}) \). That is, \( \Pi^{(i)}_{S} u^{(i)} : = Q^{(i)^T} Q^{(i)} A^{(i)}_{\Gamma \Gamma} Q^{(i)^T} A^{(i)}_{I \Gamma} u_0^{(i)} \). Then:
\[ a^{(i)}_0(u^{(i)}_0, v^{(i)}_0) = (\Pi_S^{(i)} v^{(i)}_0)^T S^{(i)} (\Pi_S^{(i)} u^{(i)}_0) + (v^{(i)}_0 - \Pi_S^{(i)} v^{(i)}_0)^T A_{TT}^{(i)} (u^{(i)}_0 - \Pi_S^{(i)} u^{(i)}_0). \]

**Lemma 2 ([9])** Let \( u_0 \in V_0 \) then

\[ a_0(u_0, u_0) = \sum_{i=1}^N a^{(i)}_0(u^{(i)}_0, u^{(i)}_0) \leq \sum_{i=1}^N \frac{1}{2} a^{(i)}_0(u^{(i)}_0, u^{(i)}_0) = \frac{1}{2} u_0^T S u_0. \]

Using Lemma 1 and Lemma 2 and the classical Schwarz Theory [5] we have:

**Theorem 2 ([9])** For any \( u \in M_h \), the following holds:

\[ (2 + \frac{3}{\delta})^{-1} a(u, u) \leq a(T_A u, u) \leq 2a(u, u) \implies k(T_A) \leq 2(2 + \frac{3}{\delta}). \]

In the implementation of NOSAS, the complexity of the coarse problem involves computing \( A_{TT}^{-1} \) and this complexity can be reduced if we replace \( A_{TT} \) by its diagonal \( \hat{A}_{TT} \). This version is called the inexact NOSAS with \( \tilde{T}_A \) as the preconditioner. The generalized eigenvalue problem is now given by:

\[ S^{(i)} e^{(i)}_i := (A^{(i)}_{TT} - A_{TT}^{(i)} (A_H^{(i)})^{-1} A_{TT}^{(i)}) e^{(i)}_j = \hat{\lambda}_j^{(i)} \hat{A}_{TT}^{(i)} e^{(i)}_j. \]

And for \( u_0, v_0 \in V_0(\Omega) \), we define the inexact coarse solver as:

\[ a_0(u_0, v_0) = \sum_{i=1}^N v_{0}^{(i)^T} (A_{TT}^{(i)} - \hat{A}_{TT}^{(i)} \hat{D}^{(i)} \hat{Q}^{(i)} (\hat{Q}^{(i)^T} \hat{A}_{TT}^{(i)} \hat{Q}^{(i)})^{-1} \hat{Q}^{(i)^T} \hat{A}_{TT}^{(i)}) u^{(i)}_0. \]

where \( \hat{Q}^{(i)} \) are the generalized eigenvectors and \( \hat{D}^{(i)} \) is diagonal \( (1 - \hat{\lambda}_1^{(i)}, \ldots, 1 - \hat{\lambda}_k^{(i)}) \).

Then, we obtain the following condition number estimate:

**Theorem 3 ([9])** For any \( u \in M_h \), the following holds:

\[ (2 + \frac{5}{\delta})^{-1} a(u, u) \leq a(T_A u, u) \leq 3a(u, u) \implies k(T_A) \leq 3(2 + \frac{5}{\delta}). \]

### 4 Multiscale discretizations methods

The idea of multiscale methods [2, 3] is to use \( \lambda_{ms} \in V_{off} \) to approximate the exact solution \( \lambda_{h} \) from \( a(\lambda_{h}, \mu) = b(\mu), \forall \mu \in M_h \), where \( V_{off} \) is the space of multiscale basis functions. The following procedures show how we construct \( V_{off} \). The first step, a snapshot space \( V_{\text{snapshots}} \) is constructed by the solutions of local problems. In our NOSAS methods, we construct the snapshot space by \( V_{\text{snapshots}} = \mathcal{H}^T V_0 \), where \( \mathcal{H}^T \) is the a-discrete harmonic extension. Notice that the dimension of \( V_{\text{snapshots}} \) can be extremely large. The next step is to construct \( V_{off} \) from \( V_{\text{snapshots}} \) which can be used to generate an efficient and accurate approximation to the multiscale solution. We choose offline space \( V_{off} = R_0^T V_0 \) where \( R_0^T \) is the global extension for NOSAS.
We use the following outline of the Generalized Multiscale Finite Element Method (GMsFEM) to show coarse space of NOSAS is a multiscale discretization.

Offline stages:
1. Mesh partitioning to obtain the subdomains.
2. Construct $V_{\text{snapshots}}$ that will be used to compute an offline space.
3. Construct a small dimensional offline space $V_{\text{off}}$ by performing dimension reduction in the space of local snapshots. This is done by choosing a threshold $\delta$ and then compute multiscale basis functions.
4. Build the coarse and local matrices and factorize.

Online stages:
1. Given $f$, solve the local problems inside each subdomain and update the residual on the interfaces.
2. Solve a coarse problem. Add coarse and local solutions.

In the offline stages, we construct the $V_{\text{off}} = R_0^T V_0$ by NOSAS and also compute the factorization of matrices $A_0$ and $A_i (1 \leq i \leq N)$. In the online stage, we first solve $N$ local problems in parallel:

$$A_i \lambda_i = R_i b = b_i \quad 1 \leq i \leq N.$$

Then, we using the local solutions to form and solve the following coarse problem:

$$A_0 \lambda_0 = R_0(b - A \sum_{i=1}^{N} R_i^T \lambda_i).$$

Finally, $\lambda_{\text{ms}} = R_0^T \lambda_0 + \sum_{i=1}^{N} R_i^T \lambda_i$ is obtained.

We note that we have similar numerical results if using $A_0 \lambda_0 = R_0 b$ as the coarse problem. Additionally, we can also develop a multiscale technique to reduce the dimension of the local problems, see [4].

**Theorem 4** For NOSAS methods with $\lambda_{\text{ms}} = R_0^T \lambda_0 + \sum_{i=1}^{N} R_i^T \lambda_i$, holds

$$a(\lambda_h - \lambda_{\text{ms}}, \lambda_h - \lambda_{\text{ms}}) \leq (1 - \delta) a(\mathcal{H}^T \lambda_{\Gamma}, \mathcal{H}^T \lambda_{\Gamma}),$$

where $\mathcal{H}$ is the $a$-discrete harmonic extension and $\lambda_{\Gamma}$ the restriction of $\lambda_h$ on $\Gamma$.

The proof follows from Lemma 1. This bound is not sharp since we can see on numerical experiments for heterogeneous coefficient that a small $\delta$ can give small relative errors.

### 5 Numerical Experiments

We first show results of problem (1) for square domain with side length 1, $f \equiv 1$ and with highly heterogeneous coefficients in the following mesh (see Figure 1).
We choose $k = 0$ for HDG spaces $Q_h$, $W_h$, and $M_h$. We divide the square domain into $H \times H$ congruent square subdomains, and we fix the number of stripes in each subdomain. That means we always have two horizontal stripes and two vertical stripes in each subdomain. The coefficients $\rho(x) = 1$ in the green stripes and $\rho(x) = 10^6$ in the white regions. We do not consider $\rho(x) = 10^6$ in stripes and $\rho(x) = 1$ elsewhere. Because it is robust without the generalized eigenfunctions. In Table 2 we show numerical results for HDG with different values of $\tau = \tau_K \rho_K$ in this mesh using NOSAS with exact solvers. By choosing $\delta = \frac{1}{4} \frac{H}{H}$ the NOSAS will not deteriorate due to the small eigenvalues related to the jumps of coefficients, and the condition number is $O(H/h)$. We also note that we see little difference in numerical results when we use inexact diagonal solvers or exact solvers (We do not include the results here.) In Table 1, we show that the size of the global part for the coarse problem of NOSAS is proportional to the number of subdomains and does not depend on $H/h$.

![Fig. 1: Coefficients $\rho(x) = 1$ in green stripes and $\rho(x) = 10^6$ in white regions.](image)

<table>
<thead>
<tr>
<th>$H/h=8$</th>
<th>$H/h=4$</th>
<th>$H/h=1/2$</th>
<th>$H/h=1/8$</th>
<th>$H/h=1/16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>84</td>
<td>420</td>
<td>1860</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>84</td>
<td>420</td>
<td>1860</td>
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</table>

Table 1: The number of all eigenfunctions $N_E$ for NOSAS with the exact solver. The size of the global problem is $N_E \times N_E$.

Finally, Table 3 shows results for multiscale in HDG. We use a similar mesh in Figure 1. However, the number of stripes fixed in the whole domain. That means we always have eight horizontal stripes and eight vertical stripes with width $h$ in the whole domain. The coefficients $\rho(x) = 1$ in the stripes and $\rho(x) = 10^6$ elsewhere. We fix $\delta$ and $\delta = \rho_K$, and choose different $\delta$ to show the relative error of $\lambda_h - \lambda_{ms}$ and the number of coarse basis functions per subdomain. Since we use the exact local solver in each subdomain, we expect that the relative error of $\lambda_h - \lambda_{ms}$ will increase if we decrease $H$ because the error arises from approximating the exact local solution in each subdomain.

References

\[ \tau = \rho_k \]

<table>
<thead>
<tr>
<th>( H = \frac{1}{4} )</th>
<th>( H = \frac{1}{2} )</th>
<th>( H = \frac{3}{4} )</th>
<th>( H = \frac{5}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = 8 )</td>
<td>14 (10.2596)</td>
<td>17 (10.2686)</td>
<td>17 (10.2732)</td>
</tr>
<tr>
<td>( H = 16 )</td>
<td>20 (20.8039)</td>
<td>25 (20.8062)</td>
<td>26 (20.8074)</td>
</tr>
</tbody>
</table>

\( (a) \ \tau = \tau_k \rho_k \) with \( \tau_k = 1 \).

\[ \tau = \frac{\tau_k}{\sigma_k} \]

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<thead>
<tr>
<th>( H = \frac{1}{4} )</th>
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<th>( H = \frac{3}{4} )</th>
<th>( H = \frac{5}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = 8 )</td>
<td>14 (10.2516)</td>
<td>17 (10.2646)</td>
<td>18 (10.2713)</td>
</tr>
<tr>
<td>( H = 16 )</td>
<td>20 (20.7990)</td>
<td>24 (20.8038)</td>
<td>27 (20.8062)</td>
</tr>
</tbody>
</table>

\( (b) \ \tau \) is the harmonic mean of \( \rho_k \) in adjacent elements.

\[ \tau = \frac{\tau_k}{\sigma_k} \]

<table>
<thead>
<tr>
<th>( H = \frac{1}{4} )</th>
<th>( H = \frac{1}{2} )</th>
<th>( H = \frac{3}{4} )</th>
<th>( H = \frac{5}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = 8 )</td>
<td>14 (10.2562)</td>
<td>17 (10.2669)</td>
<td>17 (10.2724)</td>
</tr>
<tr>
<td>( H = 16 )</td>
<td>19 (20.8019)</td>
<td>25 (20.8052)</td>
<td>26 (20.8069)</td>
</tr>
</tbody>
</table>

\( (c) \ \tau \) is the arithmetic mean of \( \rho_k \) in adjacent elements.

**Table 2**: NOSAS with exact solver for HDG with different choices of \( \tau \). The number of iterations of the PCG required to reduced the residual by \( 10^{-6} \) and the condition number (in parenthesis).

<table>
<thead>
<tr>
<th>( \delta = 10^{-5} )</th>
<th>( \delta = 1/2 )</th>
<th>( \delta = 3/4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = \frac{1}{4} )</td>
<td>7 (3.21e-5)</td>
<td>24.25 (3.21e-5)</td>
</tr>
<tr>
<td>( H = \frac{1}{2} )</td>
<td>5.25 (7.17e-5)</td>
<td>18.56 (7.15e-5)</td>
</tr>
<tr>
<td>( H = \frac{3}{4} )</td>
<td>3.06 (1.55e-4)</td>
<td>11.26 (1.54e-4)</td>
</tr>
<tr>
<td>( H = \frac{5}{4} )</td>
<td>1.89 (7.54e-4)</td>
<td>5.05 (7.50e-4)</td>
</tr>
</tbody>
</table>

**Table 3**: NOSAS as a multiscale methods with fix \( h = 1/64 \). The number of global basis functions per subdomain and the relative energy error \( A_h - A_{ms} \) with respect to \( A_h \) (in parenthesis).