# A Simple Finite Difference Discretization for Ventcell Transmission Conditions at Cross Points

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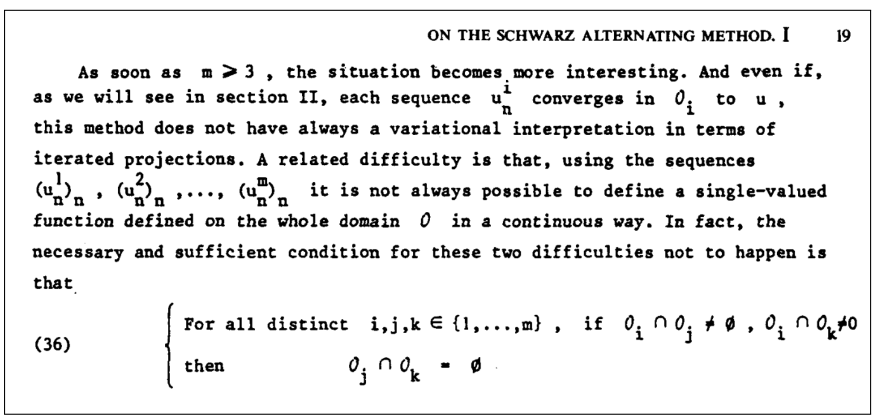
## 1 Introduction

Our main focus here is on cross points in non-overlapping domain decomposition methods, but our techniques can also be applied to cross points in overlapping domain decomposition methods, which can be an issue as indicated already by P.L Lions in his seminal paper [17], see Figure 1. The Additive Schwarz method [7] for



ON THE SCHWARZ ALTERNATING METHOD. I 19

As soon as $m \geq 3$, the situation becomes more interesting. And even if, as we will see in section II, each sequence $u_n^i$ converges in $\mathcal{O}_i$ to $u$, this method does not have always a variational interpretation in terms of iterated projections. A related difficulty is that, using the sequences $(u_n^1)_n$, $(u_n^2)_n$, ..., $(u_n^m)_n$ it is not always possible to define a single-valued function defined on the whole domain $\mathcal{O}$ in a continuous way. In fact, the necessary and sufficient condition for these two difficulties not to happen is that

$$(36)\qquad \begin{cases} \text{For all distinct } i,j,k \in \{1,\dots,m\}, \text{ if } \mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset,\ \mathcal{O}_i \cap \mathcal{O}_k \neq 0 \\ \text{then} \quad \mathcal{O}_j \cap \mathcal{O}_k = \emptyset \end{cases}$$

**Fig. 1:** Lions' comment from the first international conference on domain decomposition methods in Paris in 1987 on the difficulty of cross point situations for the parallel overlapping Schwarz method ($m$ is the number of subdomains, $O_i$ a subdomain, and $n$ the iteration index).

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example leaves the treatment of the divergent modes around cross points[1] to Krylov acceleration, which leads to the coloring constant in the condition number estimate. A partition of unity can however be used to make the method convergent, as in Restricted Additive Schwarz, see [10, 11] for more information.

For non-overlapping domain decomposition methods, Dean and Glowinski proposed in 1993 [4] already a cross point treatment with specific Lagrange multipliers for wave equations, and FETI-DP treats cross points by imposing continuity there [8, 20], see also [2] for the Helmholtz case. At the continuous level, the seminal energy estimates of Lions in [18] and Després in [5] showed that Robin transmission conditions do not pose any problem at cross points, but when discretized, standard energy estimates do not work any more [14], and one needs to use methods like auxiliary variables or complete communication to treat cross points [15], see also [19]. In an algebraic setting the optimized Robin parameter can also require a different weight at cross points [13]. Less was known historically for higher order transmission conditions containing also tangential derivatives in the presence of cross points, for an early approach at the continuous level, see [22][2]. More recently, cross points have become a focus of attention in the domain decomposition community: in [21] a new approach at the cross points based on a corner treatment developed for absorbing boundary conditions is proposed for higher order transmission conditions for lattice type partitions; in [6] a new technique with quasi-continuity relations is proposed for polygonal domains; and in [3] cross points are treated with a non-local problem in the context of a multi-trace formulation and non-local transmission conditions, an approach related to the algebraic non-local approach in [12] which leads to a direct solver without approximation, independent of the number of subdomains and type of PDE solved.

Often however in the above references, several difficulties are mixed: the domain decomposition method is for high frequency wave propagation instead of simple Laplace problems, or non-local transmission conditions instead of local ones are used, which can make the cross point difficulties which exist already for Laplace problems appear less clearly.

## 2 Optimized Schwarz with Ventcell Transmission Conditions

We consider an optimized Schwarz method (OSM) with Ventcell transmission conditions [9] for the Laplace problem and the decomposition of a square domain $\Omega$ into four square subdomains $\Omega_\ell$, as shown in Figure 2,

$$\begin{aligned} \Delta u_\ell^n &= f && \text{in } \Omega_\ell, \\ (\partial_{n_\ell} + p - q\partial_\tau^2)u_\ell^n &= (\partial_{n_\ell} + p - q\partial_\tau^2)u_l^{n-1} && \text{on } \Gamma_{\ell,l}, \end{aligned} \tag{1}$$

[1] For an illustration of these modes, see [10, Figure 3.2]

[2] Note that the term 'additive' in this reference does not refer to the additive Schwarz method!

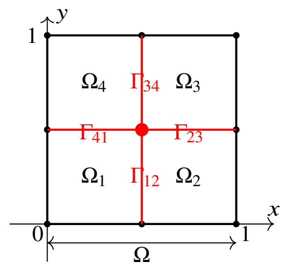
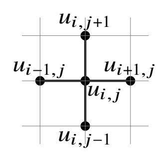


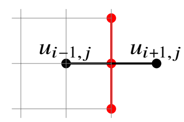

**Fig. 2:** Model problem domain and decomposition.

where $n$ is the iteration index, $\ell, l \in \{1,2,3,4\}$ are the subdomain indices, $\partial_{n_\ell}$ is the normal and $\partial_\tau$ the tangential derivative, and $p, q$ are the Ventcell transmission parameters (or Robin if $q = 0$) that can be optimized for best performance of the OSM [9]. A standard second order five point finite difference discretization, omitting the subdomain and iteration indices to avoid cluttering the notation, leads for generic grid point indices $i$ in $x$ and $j$ in $y$ to

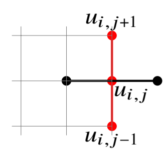
$$\Delta u \approx \frac{u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{h^2}$$




$$\partial_n u \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

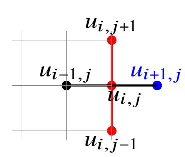



$$\partial_\tau^2 u \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$

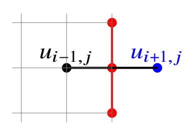



where we indicated the vertical interface in red. A problem is that for the normal derivative approximation, one point lies outside of the domain, here $u_{i+1,j}$ on the right. The value of this so called ghost point is however also involved in the interior five point finite difference stencil when evaluated at the interface,

$$\frac{{\color{blue}u_{i+1,j}} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{h^2} = f_{i,j}$$



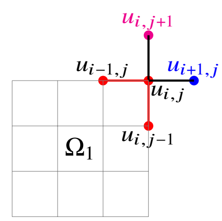

$$\frac{{\color{blue}u_{i+1,j}} - u_{i-1,j}}{2h} = g_j$$



Hence the ghost point value $u_{i+1,j}$ is determined by the approximation of the boundary condition $\partial_n u = g$ imposed in a centered fashion at the vertical red interface, and the scheme is complete. The same approach can naturally be used for a centered approximation of the more general Ventcell condition $(\partial_n + p + q\partial_\tau^2)u = g$.

Now at the $90^o$ cross point in Figure 2, something special happens with this discretization: we have for example for subdomain $\Omega_1$ the interior five point Laplacian at the cross point $(i, j)$ with the two Ventcell conditions

$$\frac{u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{h^2} = f_{i,j}$$
$$\frac{u_{i+1,j}-u_{i-1,j}}{2h} + pu_{i,j} - q\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2} = g_j$$
$$\frac{u_{i,j+1}-u_{i,j-1}}{2h} + pu_{i,j} - q\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} = \tilde{g}_i$$

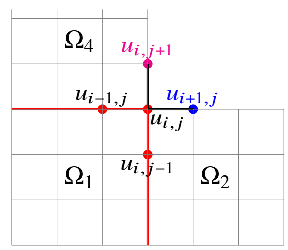



and thus the scheme is complete for the subdomain solve: we have the two equations from the two discretized boundary conditions for the two ghost points in color.

Once the subdomain solution is known, the ghost point values are known as well, and one can easily extract the values to be transmitted to the neighboring subdomains, again in the form of the centered discretized Ventcell conditions,

For $\Omega_2$: $$\frac{u_{i-1,j}-u_{i+1,j}}{2h} + pu_{i,j} - q\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}$$

For $\Omega_4$: $$\frac{u_{i,j-1}-u_{i,j+1}}{2h} + pu_{i,j} - q\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$$




The complete discrete OSM algorithm is thus for example for subdomain $\Omega_1$ given by solving at iteration $n$ for $i = 1 \ldots I$, $j = 1 \ldots J$ for $u^n_{1,i,j}$ the discrete equations

$$\frac{u^n_{1,i+1,j} + u^n_{1,i-1,j} - 4u^n_{1,i,j} + u^n_{1,i,j+1} + u^n_{1,i,j-1}}{h^2} = f_{i,j},$$

with the transmission condition for $j = 1 \ldots J$ on the right,

$$\frac{u^n_{1,i+1,j} - u^n_{1,i-1,j}}{2h} + pu^n_{1,i,j} - q\frac{u^n_{1,i,j+1} - 2u^n_{1,i,j} + u^n_{1,i,j-1}}{h^2} =$$
$$\frac{u^{n-1}_{2,i+1,j} - u^{n-1}_{2,i-1,j}}{2h} + pu^{n-1}_{2,i,j} - q\frac{u^{n-1}_{2,i,j+1} - 2u^{n-1}_{2,i,j} + u^{n-1}_{2,i,j-1}}{h^2},$$

and the transmission condition for $i = 1 \ldots I$ at the top,

$$\frac{u^n_{1,i,j+1} - u^n_{1,i,j-1}}{2h} + pu^n_{1,i,j} - q\frac{u^n_{1,i+1,j} - 2u^n_{1,i,j} + u^n_{1,i-1,j}}{h^2} =$$

$$\frac{u_{4,i,j+1}^{n-1} - u_{4,i,j-1}^{n-1}}{2h} + pu_{4,i,j}^{n-1} - q\frac{u_{4,i+1,j}^{n-1} - 2u_{4,i,j}^{n-1} + u_{4,i-1,j}^{n-1}}{h^2},$$

and analogously for the other three subdomains. We thus have a very simple finite difference scheme for the OSM with Ventcell transmission conditions, which takes advantage of the rectangular structure of the Laplace operator at such rectangular cross points.

## 3 Numerical Experiments

We show in Figure 3 the error in the first few iterates of the OSM with Ventcell transmission conditions in the left column, and for comparison in the right column the case of optimized Robin transmission conditions, i.e. $q = 0$. We used as mesh parameter $h = 1/16$, and solve directly the error equations, starting with a random initial guess; for the importance of this, see [10, Section 5.1]. We observe that the OSM is converging nicely also at the cross point, both for Robin and Ventcell transmission conditions, and convergence is much faster for the Ventcell transmission conditions. This appears even more clearly in the convergence plot shown in Figure 4. We see that the OSM with optimized Ventcell transmission conditions converges almost four times faster than with optimized Robin transmission conditions, at the same cost per iteration, and Krylov acceleration with GMRES only leads to little further improvement for the decay of the error in the iterations, especially for the Ventcell transmission condition (as already seen in [10, Fig 5.1] for the two subdomain case).

## 4 Conclusions

We presented a simple finite difference discretization of optimized Schwarz methods with Ventcell transmission conditions for the Laplace problem in the presence of cross points in the decomposition. The discretization takes advantage of the rectangular structure of the Laplace operator and only works for rectangular cross points as in Figure 2. For more general cross point situations, auxiliary variables or complete communication can be used [15], but only in the simpler case of Robin conditions.

For a rectangular cross point, our technique can also be used in a variational formulation: multiplying by a test function $v$ and integrating by parts on $\Omega_1$, we get using the Ventcell condition

$$\int_{\Omega_1} \nabla u_1 \nabla v + p\int_{\Gamma_{12}\cup\Gamma_{14}} u_1 v - q\int_{\Gamma_{12}\cup\Gamma_{14}} \partial_\tau^2 u_1 v,$$

and the last term gives when integrating by parts, and using the fact that the test function $v$ vanishes on the outer boundary due to the Dirichlet condition there

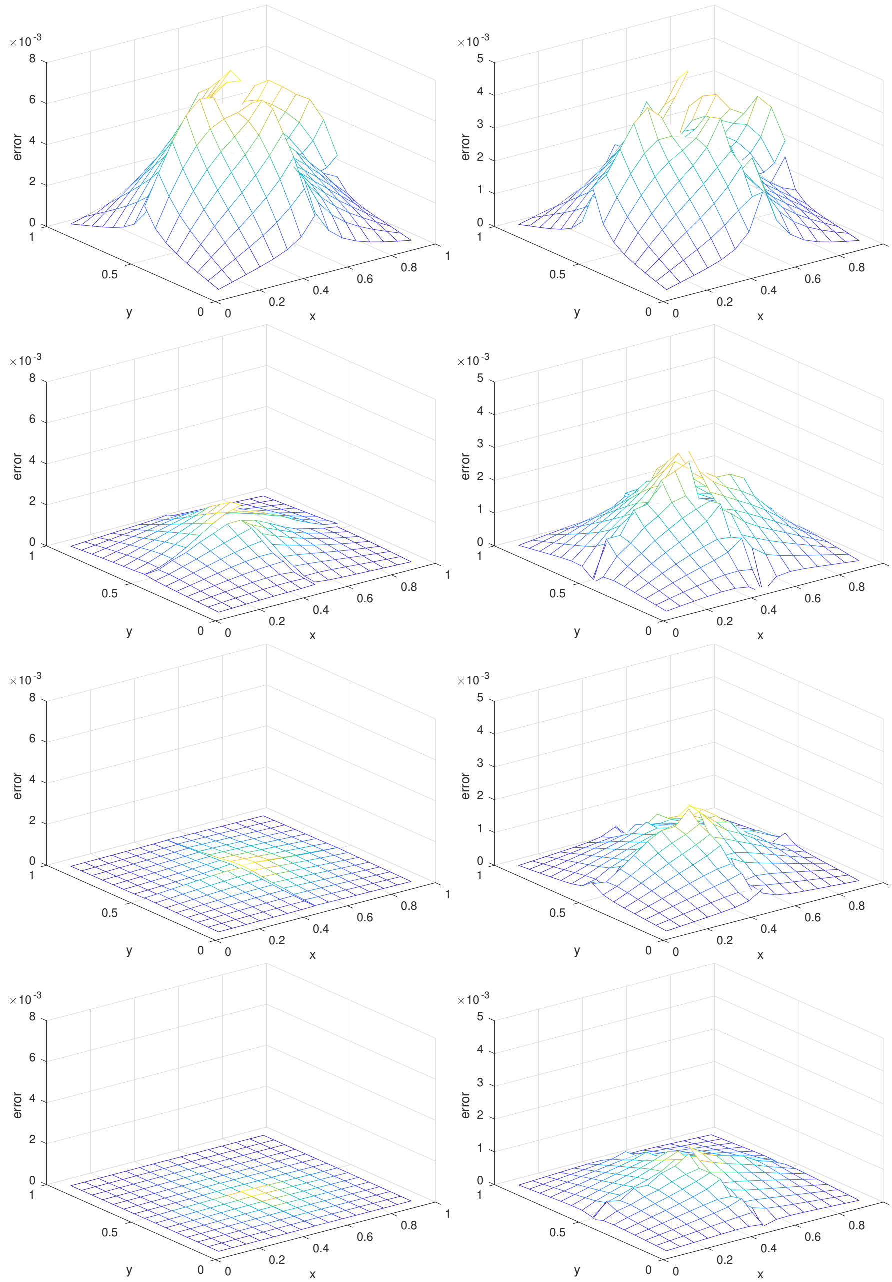


**Fig. 3:** Iterates of OSM with Ventcell transmission conditions on the left, and with Robin transmission conditions on the right.

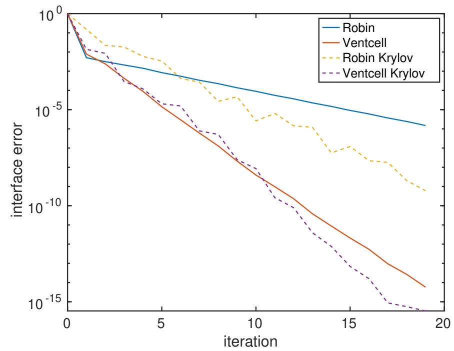



**Fig. 4:** Comparison of the decay of the error in the OSM as function of the iteration index $n$ for optimized Robin and Ventcell transmission conditions.

$$\int_{\Gamma_{12}\cup\Gamma_{14}} \partial_\tau^2 u_1 v = \partial_y u_1(\tfrac{1}{2},\tfrac{1}{2})v(\tfrac{1}{2},\tfrac{1}{2}) - \int_{\Gamma_{12}} \partial_y u_1 \partial_y v + \partial_x u_1(\tfrac{1}{2},\tfrac{1}{2})v(\tfrac{1}{2},\tfrac{1}{2}) - \int_{\Gamma_{14}} \partial_x u_1 \partial_x v.$$

Due to the rectangular nature of the cross point, the two remaining terms there are well defined using the equation and the Ventcell condition at the cross point, as in the finite difference discretization earlier,

$$\partial_x^2 u_1 + \partial_y^2 u_1 = f, \quad (\partial_x + p - q\partial_y^2)u_1 = g, \quad (\partial_y + p - q\partial_x^2)u_1 = \tilde{g},$$

since solving the Ventcell conditions for the second order derivative terms and inserting into the equation evaluated at the cross point leads to

$$\partial_x u_1(\tfrac{1}{2},\tfrac{1}{2}) + \partial_y u_1(\tfrac{1}{2},\tfrac{1}{2}) = -2pu_1(\tfrac{1}{2},\tfrac{1}{2}) + qf(\tfrac{1}{2},\tfrac{1}{2}) + g(\tfrac{1}{2},\tfrac{1}{2}) + \tilde{g}(\tfrac{1}{2},\tfrac{1}{2}),$$

and the variational formulation is complete (for the time dependent case see [1], and for well posedness in the Helmholtz case [16]). Analogously this can be done for the other subdomains, and also the data for the next iteration can be extracted in this way, which leads to a natural finite element discretization for Ventcell transmission conditions at cross points, see also [21] for a similar approach.

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