Coefficient-Robust A Posteriori Error Estimation for H(curl)-elliptic Problems

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1 Introduction

Adaptive mesh refinement (AMR) is a popular tool in numerical simulations as it is able to resolve singularity from nonsmooth data and irregular space domains. A building block of AMR is a posteriori error estimation, see, e.g., [10] for a classical introduction. On the other hand, preconditioners are discrete operators used to accelerate Krylov subspace methods for solving sparse linear systems (cf. [11]). Recently, [7, 6] introduced a novel framework linking posteriori error estimation and *preconditioning* in the Hilbert space. Such an approach yields many old and new error estimators for boundary value problems posed on de Rham complexes.

In particular, for the positive-definite H(curl) problem, [6] presents a new residual estimator robust w.r.t. high-contrast constant coefficients. In this paper, we extend the idea in [6] to the H(curl) interface problem and derive new a posteriori error estimates robust w.r.t. *both* extreme coefficient magnitude as well as large coefficient jump. The analysis avoids regularity assumptions used in existing works. We numerically compare the performance of the estimator in [6] with the one analyzed in [9].

1.1 H(curl)-Elliptic Problems

Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ be a bounded Lipschitz domain, and n be a unit vector normal to $\partial\Omega$. Let $\nabla\times$ be the usual curl in $\mathbb{R}^3$, $\nabla\times = (\partial_{x_2}, -\partial_{x_1})\cdot$ in $\mathbb{R}^2$. We define

$$V = \left\{ v \in [L^2(\Omega)]^d : \nabla \times v \in [L^2(\Omega)]^{\frac{d(d-1)}{2}},\ v \wedge n = 0 \text{ on } \partial\Omega \right\},$$

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where $v \wedge n = v \times n$ in $\mathbb{R}^3$, $v \wedge n = v \cdot n^\perp$ in $\mathbb{R}^2$ with $n^\perp$ the counter-clockwise rotation of n by $\frac{\pi}{2}$, and $[X]^d$ the Cartesian product of d copies of X. Let $(\cdot,\cdot)_{\Omega_0}$ denote the $L^2(\Omega_0)$ inner product and $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$. Given $f \in L^2(\Omega)$ and positive $\varepsilon, \kappa \in L^\infty(\Omega)$, the H(curl)-elliptic boundary value problem seeks $u \in V$ s.t.

$$(\varepsilon\nabla \times u, \nabla \times v) + (\kappa u, v) = (f, v), \quad \forall v \in V. \tag{1}$$

The space V is equipped with the V-norm and energy inner product based on

$$(v, w)_V = (\varepsilon\nabla \times v, \nabla \times w) + (\kappa v, w), \quad \forall v, w \in V.$$

Let $\mathcal{T}_h$ be a conforming tetrahedral or hexahedral partition of Ω. Problem (1) is often discretized using the Nédélec edge element space $V_h \subset V$. The discrete problem is to find $u_h \in V_h$ s.t.

$$(\varepsilon\nabla \times u_h, \nabla \times v) + (\kappa u_h, v) = (f, v), \quad \forall v \in V_h. \tag{2}$$

The semi-discrete Maxwell equation is an important example of (1). In this case, ε is the reciprocal of the magnetic permeability and κ is proportional to $1/\tau^2$, where τ is the time stepsize. Therefore, we are interested in ε with large jump and potentially huge κ. In particular, we assume $\kappa > 0$ is a constant, $\Omega_1 \subset \Omega$, $\Omega_2 \subset \Omega$ are non-overlapping and simply-connected polyhedrons aligned with $\mathcal{T}_h$, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, and

$$\varepsilon|_{\Omega_1} = \varepsilon_1, \quad \varepsilon|_{\Omega_2} = \varepsilon_2, \tag{3}$$

where $\varepsilon_1 \geq \varepsilon_2 > 0$ are constants. The interface is $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2$. A posteriori error analysis for more general ε, κ is beyond the scope of this work but is possible by making monotonicity-type assumptions on distributions of ε and κ, cf. [2, 3].

Throughout the rest of this paper, we say $\alpha \preccurlyeq \beta$ provided $\alpha \leq C\beta$, where C is an absolute constant depending solely on Ω, the aspect ratio of elements in $\mathcal{T}_h$, and the polynomial degree used in V_h. We say $\alpha \simeq \beta$ if $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha$. Given a Lipschitz manifold $\Sigma \subset \Omega$, by $\|\cdot\|_\Sigma$ we denote the $L^2(\Sigma)$ norm.

2 Nodal Auxiliary Space Preconditioning

The key idea in [6] is *nodal auxiliary space preconditioning*, originally proposed in [4] for solving discrete H(curl) and H(div) problems. The auxiliary H^1 space here is

$$W = \left\{w \in L^2(\Omega) : \nabla w \in [L^2(\Omega)]^d,\ w|_{\partial\Omega} = 0\right\},$$

endowed with the inner product

$$(w_1, w_2)_W = (\varepsilon\nabla w_1, \nabla w_2) + (\kappa w_1, w_2)$$

and the induced W-norm. The next regular decomposition (with *mixed* boundary condition, cf. [6, 4]) is widely used in the analysis of H(curl) problems.

Theorem 1 *Given $v \in V|_{\Omega_1}$, there exist $\varphi \in W|_{\Omega_1}$, $z \in [W|_{\Omega_1}]^d$, s.t. $v = \nabla\varphi + z$,*

$$\|z\|_{H^1(\Omega_1)} \le C_0\|\nabla \times v\|,$$
$$\|\varphi\|_{H^1(\Omega_1)} \le C_0(\|v\| + \|\nabla \times v\|),$$

where C_0 is a constant depending only on Ω_1.

To derive a posteriori error bounds for (2) uniform w.r.t. constant $\varepsilon \ll \kappa$, the work [6] utilizes the following modified regular decomposition.

Theorem 2 *Given $v \in V|_{\Omega_1}$, there exist $\varphi \in W|_{\Omega_1}$, $z \in [W|_{\Omega_1}]^d$, s.t. $v = \nabla\varphi + z$ and*

$$\|\varphi\|_{H^1(\Omega_1)} + \|z\| \le C_1\|v\|,$$
$$|z|_{H^1(\Omega_1)} \le C_1(\|v\| + \|\nabla \times v\|),$$

where C_1 is a constant depending only on Ω_1.

In the following, we give a new regular decomposition robust w.r.t. constant κ and piecewise constant ε. See also [5] for a weighted Helmholtz decomposition.

Theorem 3 *Given $v \in V$, there exist $\varphi \in W$ and $z \in [W]^d$, s.t. $v = \nabla\varphi + z$ and*

$$\|\kappa^{\frac{1}{2}}\varphi\|_{H^1(\Omega)} + \|z\|_W \le C_2\|v\|_V,$$

where C_2 is a constant depending solely on Ω, Ω_1, Ω_2.

Proof The proof is divided into two cases. When $\varepsilon_1 \ge \kappa$, we use Theorem 1 on Ω_1 to obtain $\varphi_1 \in H^1(\Omega_1)$, $z_1 \in [H^1(\Omega_1)]^d$ both vanishing on $\partial\Omega_1\backslash\Gamma$ s.t.

$$\begin{aligned} v|_{\Omega_1} &= \nabla\varphi_1 + z_1, \\ \|z_1\|_{H^1(\Omega_1)} &\preccurlyeq \|\nabla \times v\|_{\Omega_1}, \\ \|\varphi_1\|_{H^1(\Omega_1)} &\preccurlyeq \|v\|_{\Omega_1} + \|\nabla \times v\|_{\Omega_1}. \end{aligned} \tag{4}$$

When $\varepsilon_1 < \kappa$, applying Theorem 2 to $v|_{\Omega_1}$ yields $\varphi_1 \in H^1(\Omega_1)$, $z_1 \in [H^1(\Omega_1)]^d$ s.t.

$$\begin{aligned} &v|_{\Omega_1} = \nabla\varphi_1 + z_1, \quad \varphi_1|_{\partial\Omega_1\backslash\Gamma} = 0,\ z_1|_{\partial\Omega_1\backslash\Gamma} = 0, \\ &\|\varphi_1\|_{H^1(\Omega_1)} + \|z_1\|_{\Omega_1} \preccurlyeq \|v\|_{\Omega_1}, \\ &|z_1|_{H^1(\Omega_1)} \preccurlyeq \|v\|_{\Omega_1} + \|\nabla \times v\|_{\Omega_1}, \end{aligned} \tag{5}$$

In either case, it holds that

$$\|\kappa^{\frac{1}{2}}\varphi_1\|_{H^1(\Omega_1)} + \|z_1\|_{W|_{\Omega_1}} \preccurlyeq \|v\|_{V|_{\Omega_1}}. \tag{6}$$

First let $\hat{\varphi}_1 \in H^1(\mathbb{R}^d\backslash\Omega_2)$ and $\hat{z}_1 \in [H^1(\mathbb{R}^d\backslash\Omega_2)]^d$ be zero extensions of φ_1 and z_1 to $\mathbb{R}^d\backslash\Omega_2$, respectively. Then we take $\tilde{\varphi}_1 \in H^1(\Omega)$, $\tilde{z}_1 \in H^1(\Omega)$ to be the Stein universal extensions of $\hat{\varphi}_1$, $\hat{z}_1$ to $\mathbb{R}^d$ satisfying

$$\|\tilde{\varphi}_1\|_{\Omega_2} \leqslant \|\varphi_1\|_{\Omega_1}, \quad \|\tilde{\varphi}_1\|_{H^1(\Omega_2)} \leqslant \|\varphi_1\|_{H^1(\Omega_1)},$$
$$\|\tilde{z}_1\|_{\Omega_2} \leqslant \|z_1\|_{\Omega_1}, \quad \|\tilde{z}_1\|_{H^1(\Omega_2)} \leqslant \|z_1\|_{H^1(\Omega_1)}. \tag{7}$$

On Ω_2, applying Theorem 1 (if $\varepsilon_2 \geq \kappa$) or Theorem 2 (if $\varepsilon_2 < \kappa$) to $w = v|_{\Omega_2} - \nabla\tilde{\varphi}_1|_{\Omega_2} - \tilde{z}_1|_{\Omega_2}$ ($w \wedge n = 0$ on $\partial\Omega_2$), we have $\varphi_2 \in H_0^1(\Omega_2)$, $z_2 \in [H_0^1(\Omega_2)]^d$ s.t.

$$v|_{\Omega_2} - \nabla\tilde{\varphi}_1|_{\Omega_2} - \tilde{z}_1|_{\Omega_2} = \nabla\varphi_2 + z_2, \tag{8a}$$

$$\|\kappa^{\frac{1}{2}}\varphi_2\|_{H^1(\Omega_2)} + \|z_2\|_{W|_{\Omega_2}} \leqslant \|v\|_{V|_{\Omega_2}} + \|\kappa^{\frac{1}{2}}\nabla\tilde{\varphi}_1\|_{\Omega_2} + \|\tilde{z}_1\|_{V|_{\Omega_2}}. \tag{8b}$$

Here (8b) follows from similar reasons for (6). Define $\varphi \in H_0^1(\Omega)$, $z \in [H_0^1(\Omega)]^d$ as

$$\varphi := \begin{cases} \varphi_1 & \text{on } \Omega_1 \\ \tilde{\varphi}_1 + \varphi_2 & \text{on } \Omega_2 \end{cases}, \quad z := \begin{cases} z_1 & \text{on } \Omega_1 \\ \tilde{z}_1 + z_2 & \text{on } \Omega_2 \end{cases},$$

and obtain $v = \nabla\varphi + z$ on Ω. If $\varepsilon_1 \geq \kappa$, it follows from (8b), (7), (4), $\varepsilon_2 \leq \varepsilon_1$ that

$$\begin{aligned} &\|\kappa^{\frac{1}{2}}\varphi\|_{H^1(\Omega_2)} + \|z\|_{W|_{\Omega_2}} \\ &\leqslant \|v\|_{V|_{\Omega_2}} + \kappa^{\frac{1}{2}}\|\varphi_1\|_{H^1(\Omega_1)} + (\kappa^{\frac{1}{2}} + \varepsilon_2^{\frac{1}{2}})\|z_1\|_{\Omega_1} + \varepsilon_2^{\frac{1}{2}}|z_1|_{H^1(\Omega_1)} \\ &\leqslant \|v\|_{V|_{\Omega_2}} + \kappa^{\frac{1}{2}}\|v\| + \varepsilon_1^{\frac{1}{2}}\|\nabla \times v\|_{\Omega_1}. \end{aligned} \tag{9}$$

Similarly when $\varepsilon_1 < \kappa$, it follows from (8b), (7), (5), $\varepsilon_2 \leq \varepsilon_1 < \kappa$ that

$$\|\kappa^{\frac{1}{2}}\varphi\|_{H^1(\Omega_2)} + \|z\|_{W|_{\Omega_2}} \leqslant \|v\|_{V|_{\Omega_2}} + \kappa^{\frac{1}{2}}\|v\|_{\Omega_1}. \tag{10}$$

Combining (6), (9), (10) completes the proof. □

Remark 1 The work [12] gives a robust regular decomposition for the H(curl) interface problem with $\kappa = s\varepsilon$, $s \in (0, 1]$. In contrast, Theorem 3 is able to deal with large jump of ε as well as large $\kappa \gg \varepsilon$.

Given a Hilbert space X, let X' denote its dual space, and $\langle\cdot,\cdot\rangle$ the action of X' on X. We introduce bounded linear operators $A : V \to V'$, $A_\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$, $A_W : W^d \to ([W]^d)'$ as

$$\begin{aligned} \langle Av, w\rangle &= (\varepsilon\nabla \times v, \nabla \times w) + (\kappa v, w), \quad v, w \in V, \\ \langle A_\Delta v, w\rangle &= (\nabla v, \nabla w) + (v, w), \quad v, w \in H_0^1(\Omega), \\ \langle A_W v, w\rangle &= (\varepsilon\nabla v, \nabla w) + (\kappa v, w), \quad v, w \in [W]^d. \end{aligned}$$

Let $r \in V'$ be the residual given by

$$\langle r, v\rangle = (f, v) - (\varepsilon\nabla \times u_h, \nabla \times v) - (\kappa u_h, v), \quad v \in V. \tag{11}$$

Clearly the inclusion $I : [W]^d \hookrightarrow V$ and the gradient operator $\nabla : W \to V$ are uniformly bounded w.r.t. ε and κ. Then using such boundedness, Theorem 3, and the *fictitious space lemma* (cf. [8, 4] and Corollary 5.1 in [6]), we obtain the uniform spectral equivalence of two continuous operators

$$A^{-1} \simeq B := \nabla(\kappa A_\Delta)^{-1}\nabla' + I A_W^{-1} I', \tag{12}$$

where $I' : V' \to ([W]^d)'$ and $\nabla' : V' \to W'$ are adjoint operators. By $A^{-1} \simeq B$ from V' to V in (12) we mean $\langle R, A^{-1}R\rangle \simeq \langle R, BR\rangle$, $\forall R \in V'$. It is noted that $A(u - u_h) = r \in V'$. Therefore a direct consequence of (12) is

$$\begin{aligned}\|u - u_h\|_V^2 &= \langle A(u - u_h), u - u_h\rangle = \langle r, A^{-1}r\rangle \simeq \langle r, Br\rangle \\ &= \langle \nabla' r, (\kappa A_\Delta)^{-1}\nabla' r\rangle + \langle I' r, A_W^{-1} I' r\rangle = \kappa^{-1}\|\nabla' r\|_{H^{-1}(\Omega)}^2 + \|I' r\|_{([W]^d)'}^2.\end{aligned} \tag{13}$$

3 A Posteriori Error Estimates

The goal of this paper is to derive a robust two-sided bound $\|u - u_h\|_V \simeq \eta_h$. The quantity η_h is computed from u_h and split into element-wise error indicators for AMR. Such local error indicators are used to predict element errors in the current grid and mark those tetrahedra/hexahedra with large errors for subdivision.

When deriving the error estimator, we assume that the source f is piecewise H^1-regular w.r.t. $\mathcal{T}_h$. By $\mathcal{S}_h$ we denote the collection of $(d-1)$-simplexes in $\mathcal{T}_h$ that are not contained in $\partial\Omega$. Each $S \in \mathcal{S}_h$ shared by $T_S^+, T_S^- \in \mathcal{T}_h$ is assigned with a unit normal n_S pointing from T_S^+ to T_S^-. Let h, h_s be the mesh size functions s.t. $h|_T = h_T := \mathrm{diam}(T)\ \forall T \in \mathcal{T}_h$, $h_s|_S = h_S := \mathrm{diam}(S)\ \forall S \in \mathcal{S}_h$. The weighted mesh size functions are

$$\bar{h} := \min\left\{\frac{h}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\kappa}}\right\}, \quad \bar{h}_s := \min\left\{\frac{h_s}{\sqrt{\varepsilon_s}}, \frac{1}{\sqrt{\kappa}}\right\},$$

where $\varepsilon_s|_S = \max\{\varepsilon_{T_S^+}, \varepsilon_{T_S^-}\}\ \forall S \in \mathcal{S}_h$. For each $T \in \mathcal{T}_h$, $S \in \mathcal{S}_h$, let Ω_T denote the union of elements in $\mathcal{T}_h$ sharing an edge with T, and $\Omega_S = \cup_{S \in \mathcal{S}_h, S \subset \partial T}\Omega_T$. For each $S \in \mathcal{S}_h$, let $[\![\omega]\!]_S = \omega|_{T_S^+} - \omega|_{T_S^-}$ be the jump of ω across S. We define

$$\begin{aligned}&R_1|_T = -\nabla\cdot(f - \kappa u_h)|_T, \quad J_1|_S = [\![f - \kappa u_h]\!]_S \cdot n_S, \\ &R_2|_T = (f - (\nabla\times)^*(\varepsilon\nabla\times u_h) - \kappa u_h)|_T, \quad J_2|_S = -[\![\varepsilon\nabla\times u_h]\!]_S \wedge n_S,\end{aligned}$$

where $(\nabla\times)^* = \nabla\times$ in $\mathbb{R}^3$ and $(\nabla\times)^* = (-\partial_{x_2}, \partial_{x_1})$ in $\mathbb{R}^2$. By the element-wise Stokes' (in $\mathbb{R}^3$) or Green's (in $\mathbb{R}^2$) formula, we have

$$\langle \nabla' r, \psi\rangle = \langle r, \nabla\psi\rangle = \sum_{T\in\mathcal{T}_h}(R_1, \psi)_T + \sum_{S\in\mathcal{S}_h}(J_1, \psi)_S, \quad \psi \in H_0^1(\Omega), \tag{14}$$

$$\langle I' r, \varphi\rangle = \langle r, \varphi\rangle = \sum_{T\in\mathcal{T}_h}(R_2, \varphi)_T + \sum_{S\in\mathcal{S}_h}(J_2, \varphi)_S, \quad \varphi \in [W]^d. \tag{15}$$

In view of (13), it remains to estimate $\|\nabla' r\|_{H^{-1}(\Omega)}$ and $\|I' r\|_{([W]^d)'}$. Let $(\cdot,\cdot)_{\mathcal{S}_h}$ denote the inner product $\sum_{S\in\mathcal{S}_h}(\cdot,\cdot)_S$ and $\|\cdot\|_{\mathcal{S}_h}$ the corresponding norm. Let Q_h (resp. Q_h^s) be the L^2 projection onto the space of discontinuous and piecewise polynomials of fixed degrees on $\mathcal{T}_h$ (resp. S_h). The estimation of $\|\nabla' r\|_{H^{-1}(\Omega)}$ is standard (cf. [6]) and given as

$$\|hR_1\| + \|h_s^{\frac{1}{2}} J_1\|_{\mathcal{S}_h} - \mathrm{osc}_{h,1} \preccurlyeq \|\nabla' r\|_{H^{-1}(\Omega)} \preccurlyeq \|hR_1\| + \|h_s^{\frac{1}{2}} J_1\|_{\mathcal{S}_h}, \tag{16}$$

where $\mathrm{osc}_{h,1} := \|h(R_1 - Q_h R_1)\| + \|h_s^{\frac{1}{2}}(J_1 - Q_h^s J_1)\|_{\mathcal{S}_h}$ is the data oscillation. We also need the second data oscillation $\mathrm{osc}_{h,2} := \|\bar{h}(R_2 - Q_h R_2)\| + \|\bar{h}_s^{\frac{1}{2}}(J_2 - Q_h^s J_2)\|_{\mathcal{S}_h}$. In the next lemma, we derive two-sided bounds for $\|I' r\|_{([W]^d)'}$.

Lemma 1 *It holds that*

$$\|\bar{h}R_2\| + \|\varepsilon^{-\frac{1}{4}}\bar{h}_s^{\frac{1}{2}} J_2\|_{\mathcal{S}_h} - \mathrm{osc}_{h,2} \preccurlyeq \|I' r\|_{([W]^d)'} \preccurlyeq \|\bar{h}R_2\| + \|\varepsilon^{-\frac{1}{4}}\bar{h}_s^{\frac{1}{2}} J_2\|_{\mathcal{S}_h}.$$

Proof The proof is similar to Lemma 4.4 of [6] except the use of the modified Clément-type interpolation $\widetilde{\Pi}_h : [L^2(\Omega)]^d \to V_h^0$ proposed in [3] for dealing with huge jump of ε. Here $V_h^0 \subseteq V_h$ is the lowest order edge element space. For any $v \in [W]^d$ and $T \in \mathcal{T}_h$, the analysis in Theorem 4.6 of [3] implies that

$$\|v - \widetilde{\Pi}_h v\|_T \preccurlyeq h_T \varepsilon|_T^{-\frac{1}{2}} \|\varepsilon^{\frac{1}{2}}\nabla v\|_{\Omega_T} \le h_T \varepsilon|_T^{-\frac{1}{2}} \|v\|_{W|_{\Omega_T}}, \tag{17}$$

$$\|\nabla(v - \widetilde{\Pi}_h v)\|_T \preccurlyeq \varepsilon|_T^{-\frac{1}{2}} \|\varepsilon^{\frac{1}{2}}\nabla v\|_{\Omega_T} \le \varepsilon|_T^{-\frac{1}{2}} \|v\|_{W|_{\Omega_T}}. \tag{18}$$

The L^2-boundedness of $\widetilde{\Pi}_h$ implies that

$$\|v - \widetilde{\Pi}_h v\|_T \preccurlyeq \|v\|_{\Omega_T} \le \kappa^{-\frac{1}{2}} \|v\|_{W|_{\Omega_T}}. \tag{19}$$

A direct consequence of (17) and (19) is

$$\|v - \widetilde{\Pi}_h v\|_T \preccurlyeq \bar{h}_T \|v\|_{W|_{\Omega_T}}. \tag{20}$$

Given a face/edge $S \in \mathcal{S}_h$, let T be the element containing S over which ε is maximal. Using a trace inequality, (20), $h_S^{-1} \le \bar{h}_S^{-1}\varepsilon_S^{-\frac{1}{2}}$, (18), $\bar{h}_S \simeq \bar{h}_T$, we have

$$\begin{aligned}\|v - \widetilde{\Pi}_h v\|_S^2 &\preccurlyeq h_S^{-1}\|v - \widetilde{\Pi}_h v\|_T^2 + \|v - \widetilde{\Pi}_h v\|_T \|\nabla(v - \widetilde{\Pi}_h v)\|_T \\ &\preccurlyeq h_S^{-1}\bar{h}_T^2 \|v\|_{W|_{\Omega_T}}^2 + \bar{h}_T \varepsilon|_T^{-\frac{1}{2}} \|v\|_{W|_{\Omega_T}}^2 \preccurlyeq \varepsilon|_T^{-\frac{1}{2}} \bar{h}_S \|v\|_{W|_{\Omega_T}}^2.\end{aligned} \tag{21}$$

It follows from $r|_{V_h} = 0$, (15), the Cauchy–Schwarz inequality that

$$\|I' r\|_{([W]^d)'} = \sup_{v\in[W]^d, \|v\|_W = 1} \langle r, v\rangle = \sup_{v\in[W]^d, \|v\|_W = 1} \langle r, v - \widetilde{\Pi}_h v\rangle$$

$$\preccurlyeq \big(\|\bar{h}R_2\| + \|\varepsilon_s^{-\frac{1}{4}}\bar{h}_s^{\frac{1}{2}}J_2\|_{\mathcal{S}_h}\big) \sup_{\substack{v\in[W]^d\\ \|v\|_W=1}} \big(\|\bar{h}^{-1}(v-\widetilde{\Pi}_h v)\| + \|\varepsilon_s^{\frac{1}{4}}\bar{h}_s^{-\frac{1}{2}}(v-\widetilde{\Pi}_h v)\|_{\mathcal{S}_h}\big).$$

Then the upper bound of $\|I'r\|_{([W]^d)'}$ is a consequence of the above inequality and (20), (21). The uniform lower bound of $\|I'r\|_{([W]^d)'}$ w.r.t. ε, κ follows from the bubble function technique explained in [10] and extremal definitions of $\bar{h}$, $\bar{h}_s$, ε_s. □

For each $T \in \mathcal{T}_h$, we define the error indicator

$$\eta_h(T) = \kappa^{-1}h_T^2\|R_1\|_T^2 + \bar{h}|_T^2\|R_2\|_T^2 + \sum_{S\in\mathcal{S}_h, S\subset\partial T}\left\{\kappa^{-1}h_S\|J_1\|_S^2 + \bar{h}_s|_S\|\varepsilon^{-\frac{1}{4}}J_2\|_S^2\right\}.$$

Combining (13), (16) and Lemma 1 leads to the robust a posteriori error estimate

$$\sum_{T\in\mathcal{T}_h}\eta_h(T) - \mathrm{osc}_{h,1} - \mathrm{osc}_{h,2} \preccurlyeq \|u-u_h\|_V^2 \preccurlyeq \sum_{T\in\mathcal{T}_h}\eta_h(T). \tag{22}$$

Remark 2 Our analysis for (22) is based on regular decomposition and minimal regularity while the theoretical analysis of *recovery* estimators in [3] hinges on Helmholtz decomposition and full elliptic regularity of the underlying domain. Our estimator $\eta_h(T)$ is robust w.r.t. both large jump of ε and extreme magnitude of ε, κ.

4 Numerical Demonstration of Robustness

In the end, we focus on (1) with *constant* and *positive* ε and κ, which is a special case of the interface problem considered before. In this case, the error indicator $\eta_h(T)$ reduces to the one derived in [6]. For constant ε and κ, the classical a posteriori error estimator for (2) (cf. [1, 9]) reads

$$\tilde{\eta}_h(T) = \kappa^{-1}h_T^2\|R_1\|_T^2 + \varepsilon^{-1}h_T^2\|R_2\|_T^2 + \sum_{S\in\mathcal{S}_h, S\subset\partial T}\left\{\kappa^{-1}h_S\|J_1\|_S^2 + \varepsilon^{-1}h_S\|J_2\|_S^2\right\}.$$

Although weighted with ε, κ, this estimator is not fully robust w.r.t. ε and κ. In fact, the ratio $\|u-u_h\|_V/(\sum_{T\in\mathcal{T}_h}\tilde{\eta}_h(T))^{\frac{1}{2}}$ may tend to zero as $\varepsilon \ll \kappa$, i.e., the constant $\underline{C}$ in the lower bound $\underline{C}(\sum_{T\in\mathcal{T}_h}\tilde{\eta}_h(T))^{\frac{1}{2}} \leq \|u-u_h\|_V$ + h.o.t. is not uniform.

To validate the result, we test $\eta_h(T)$ and $\tilde{\eta}_h(T)$ by the lowest order edge element discretization of (1) defined on $\Omega = [0,1]^2$ with the exact solution $u(x_1,x_2) = \big(\cos(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\cos(\pi x_2)\big)$. The initial partition of Ω is a 4×4 uniform triangular mesh. A sequence of nested grids is computed by uniform quad-refinement. Let $e = \|u-u_h\|_V$, $\eta = (\sum_{T\in\mathcal{T}_h}\eta_h(T))^{\frac{1}{2}}$ and $\tilde{\eta} = (\sum_{T\in\mathcal{T}_h}\tilde{\eta}_h(T))^{\frac{1}{2}}$. Numerical results are shown in Table 1. In its last row, we compute effectivity index "eff" of η (resp. $\tilde{\eta}$), which is the algorithmic mean of e/η (resp. $e/\tilde{\eta}$) over all grid levels. It is observed that the performance of η is uniformly effective for all ε, κ, while the efficiency of $\tilde{\eta}$ deteriorates for small ε and large κ.

Table 1: Convergence history of the lowest order edge element and error estimators

number of elements	e $\varepsilon = 0.1$ $\kappa = 10$	η $\varepsilon = 0.1$ $\kappa = 10$	$\tilde{\eta}$ $\varepsilon = 0.1$ $\kappa = 10$	e $\varepsilon = 10^{-3}$ $\kappa = 10^3$	η $\varepsilon = 10^{-3}$ $\kappa = 10^3$	$\tilde{\eta}$ $\varepsilon = 10^{-3}$ $\kappa = 10^3$	e $\varepsilon = 10^{-5}$ $\kappa = 10^5$	η $\varepsilon = 10^{-5}$ $\kappa = 10^5$	$\tilde{\eta}$ $\varepsilon = 10^{-5}$ $\kappa = 10^5$
32	8.42e-1	3.72	3.94	8.24	3.72e+1	1.46e+3	8.24e+1	3.72e+2	1.46e+6
128	4.35e-1	2.04	2.04	4.30	2.04e+1	3.80e+2	4.30e+1	2.04e+2	3.80e+5
512	2.19e-1	1.04	1.04	2.18	1.06e+1	9.70e+1	2.18e+1	1.06e+2	9.64e+4
2048	1.10e-1	5.26e-1	5.26e-1	1.10	5.36	2.48e+1	1.10e+1	5.36e+1	2.42e+4
8192	5.49e-2	2.64e-1	2.64e-1	5.49e-1	2.69	6.61	5.49	2.69e+1	6.06e+3
eff	N/A	2.13e-1	2.11e-1	N/A	2.09e-1	3.33e-2	N/A	2.09e-1	3.51e-4

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