Consistent and Asymptotic-Preserving Finite-Volume Robin Transmission Conditions for Singularly Perturbed Elliptic Equations

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1 Introduction

Adaptive Dirichlet-Neumann and Robin-Neumann algorithms for singularly-perturbed advection-diffusion equations were introduced in [2], accounting for transport along characteristics, see also [6] for the discrete setting and damped versions using a modified quadrature rule to recover the hyperbolic limit. Non-overlapping Schwarz DDMs with Robin transmission conditions (TCs) applied to advection-diffusion equations were analyzed in [10, 1] and a stabilized finite-element method for singularly perturbed problems was discussed in [9], see also [3, 4] and references therein for heterogeneous couplings. However, the behavior of these DDMs in the limit of vanishing diffusion has not been addressed.

Our goal is to develop finite volume Robin TCs such that the associated non-overlapping DDM is consistent *and* asymptotic-preserving (AP). Consistent here means that, for fixed mesh size, the discrete DDM iterates converge to the discrete solution on the entire domain, and AP means that the singular limit in the DDM yields a convergent limit DDM (for more on AP, see e.g. [7]). We first show that the continuous DDM is only AP under a strict condition on the Robin transmission parameter, see Theorem 1. In contrast, our new discrete DDM is AP without restriction on this parameter, see Theorem 3, and fast convergence is automatically recovered in the hyperbolic limit. While our analysis is in 1D, we show numerical experiments also in 2D; for the nonlinear space-time case with triangular meshes, see [5].

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2 The continuous problem and non-overlapping DDM

We consider for $\nu \geq 0$, $a > 0$ and $f \in L^2(-1,1)$ the stationary advection-diffusion equation with homogeneous Dirichlet boundary conditions, i.e.,

$$\mathcal{L}(u) := \nu\partial_{xx}u - a\partial_x u = f \text{ in } \Omega := (-1,1), \quad u(-1) = 0\,, \quad \nu u(1) = 0\,. \tag{1}$$

In the singular limit $\nu = 0$, the PDE in (1) becomes (trivially) advective, and the boundary condition collapses into the inflow condition $u(-1) = 0$ only. It is easy to see that there exists a unique weak solution $u \in H^1(-1,1)$ of (1) for $\nu \geq 0$.

We apply a non-overlapping DDM with two sub-domains $\Omega_1 = (-1,0)$ and $\Omega_2 = (0,1)$ to (1). The problem (1) is then rewritten using at $x = 0$ the Robin TCs

$$\mathcal{B}_1(u) = \nu\partial_x u - au + \lambda u\,, \quad \mathcal{B}_2(u) = -\nu\partial_x u + au + \lambda u\,, \quad \lambda > 0\,. \tag{2}$$

Definition 1 (Continuous DDM) Let $u_2^0 \in H^1(\Omega_2)$. For $n \in \mathbb{N}$, the n-th (continuous) DDM-iterate $(u_1^n, u_2^n) \in H^1(\Omega_1) \times H^1(\Omega_2)$ is given as solution of

$$\nu\partial_{xx}u_j^n - a\partial_x u_j^n = f \qquad \text{in} \quad \Omega_j,\ j = 1,2\,, \tag{3}$$

$$u_1^n(-1) = 0\,, \qquad \nu u_2^n(1) = 0\,, \tag{4}$$

$$\nu\mathcal{B}_1(u_1^n) = \nu\mathcal{B}_1(u_2^{n-1})\,, \quad \mathcal{B}_2(u_2^n) = \mathcal{B}_2(u_1^n) \qquad \text{at} \quad x = 0\,. \tag{5}$$

Note that (3)-(5) is equivalent to (1) in the limit $n \to \infty$. In the limit when $\nu \to 0$, we get the stationary advection equation, and the two Robin TCs (5) degenerate into one Dirichlet TC. Note that the multiplication of $\mathcal{B}_1$ by ν is necessary to remove the TC in the limit $\nu \to 0$. The errors $e_j^n := u|_{\Omega_j} - u_j^n$ satisfy (3)-(5) with $f \equiv 0$ due to linearity. Therefore, we have by direct solution

$$\begin{aligned} e_1^n(x) &= A_1^n(\mathrm{e}^{ax/\nu} - \mathrm{e}^{-a/\nu})\,, & e_2^n(x) &= A_2^n(1 - \mathrm{e}^{a(x-1)/\nu}) & &\text{if } \nu > 0\,,\\ e_1^n &\equiv 0\,, & e_2^n &\equiv 0 & &\text{if } \nu = 0\,, \end{aligned}$$

where $A_1^n, A_2^n \in \mathbb{R}$ satisfy the recurrence relations

$$A_1^n = \tfrac{-a+\lambda(1-\mathrm{e}^{-a/\nu})}{a\mathrm{e}^{-a/\nu}+\lambda(1-\mathrm{e}^{-a/\nu})} A_2^{n-1}\,, \qquad A_2^n = \tfrac{-a\mathrm{e}^{-a/\nu}+\lambda(1-\mathrm{e}^{-a/\nu})}{a+\lambda(1-\mathrm{e}^{-a/\nu})} A_1^n\,.$$

This yields the following convergence result.

Theorem 1 (Convergence and AP property of the continuous DDM)
The sequence of continuous DDM-iterates $\{(u_1^n, u_2^n)\}_{n\in\mathbb{N}}$ converges pointwise to $(u|_{\Omega_1}, u|_{\Omega_2})$. For $\nu > 0$, the convergence is linear with convergence factor

$$\rho = \left|\frac{(a-\lambda) + \lambda\mathrm{e}^{-a/\nu}}{(a+\lambda) - \lambda\mathrm{e}^{-a/\nu}}\right| \left|\frac{\lambda - (a+\lambda)\mathrm{e}^{-a/\nu}}{\lambda + (a-\lambda)\mathrm{e}^{-a/\nu}}\right| < 1\,. \tag{6}$$

Convergence in one iteration is achieved iff $\lambda = \frac{a}{1-\mathrm{e}^{-a/\nu}}$ or in the case $\nu = 0$.
The continuous DDM (3)-(5) is AP if $\lambda = \lambda(\nu)$ satisfies $|\lambda - a| = o(1)$ as $\nu \to 0$.

3 Cell-centered finite volume discretization

We discretize (1) and (3)-(5) by a cell-centered finite volume method. For given $I \in \mathbb{N}$, let the step-width be $h := 1/I$ and the volumes $V_i := [ih, (i+1)h]$ for $-I \leq i < I$ be given. Furthermore, define $f_i := \int_{V_i} f(x)\, dx$. We denote the constant, cell-centered approximation of u in V_i by u_i, and encapsulate these for all V_i in the vector $\boldsymbol{u} := (u_i)_{i=-I}^{I-1} \in \mathbb{R}^{2I}$. Using centered differences for the diffusion and upwind fluxes for the advection, the discrete version of problem (1) reads

$$\tfrac{\nu}{h}(u_{i-1} - 2u_i + u_{i+1}) + a(u_{i-1} - u_i) = f_i \quad \text{for } -I < i < I-1, \tag{7}$$

$$\tfrac{\nu}{h}(-3u_{-I} + u_{-I+1}) - 2au_{-I} = f_{-I}\ , \tag{8}$$

$$\tfrac{\nu}{h}(u_{I-2} - 3u_{I-1}) + a(u_{I-2} - u_{I-1}) = f_{I-1}\ . \tag{9}$$

Here, we eliminated the ghost values u_{-I-1} and u_I using a linear interpolation of the boundary conditions. Analogously, one obtains the discrete version of (3) and (4), while (5) becomes

$$B_1(\boldsymbol{u}_1^n) = B_1(\boldsymbol{u}_2^{n-1})\ , \qquad\qquad B_2(\boldsymbol{u}_2^n) = B_2(\boldsymbol{u}_1^n)\ . \tag{10}$$

It remains to discretize the TC (2) to obtain B_1, B_2, and then to eliminate the ghost values $u_{1,0}$ and $u_{2,-1}$. For this, we use centered differences for the diffusion and linear combinations of the values in V_{-1} and V_0 for the other terms to obtain

$$B_1(\boldsymbol{u}) = \tfrac{\nu}{h}(u_0 - u_{-1}) - a((1-\alpha_1)u_{-1} + \alpha_1 u_0) + \lambda((1-\beta_1)u_{-1} + \beta_1 u_0)\ , \tag{11}$$

$$B_2(\boldsymbol{u}) = -\tfrac{\nu}{h}(u_0 - u_{-1}) + a((1-\alpha_2)u_{-1} + \alpha_2 u_0) + \lambda((1-\beta_2)u_{-1} + \beta_2 u_0)\ , \tag{12}$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$. Note that $\alpha_j = \beta_j = 0$, $j = 1,2$, is an upwind discretization, while the centered choice $\alpha_j = \beta_j = 1/2$, $j = 1,2$, is typically used in the diffusion-dominated case $\nu \gg a$ to obtain second-order convergence in h.

To eliminate the ghost values $u_{1,0}$ and $u_{2,-1}$ in (7), we solve (11) for u_0 and (12) for u_{-1}. To eliminate $u_{2,-1}$ in (11) and $u_{1,0}$ in (12), we solve (7) for $u_{1,0}$ and $u_{2,-1}$. Inserting the resulting expressions and using (10), we obtain the following discrete DDM iteration.

Definition 2 (Discrete DDM)

For given $\boldsymbol{u}_2^0 \in \mathbb{R}^I$, let $\tilde{B}_1(\boldsymbol{u}_2^0) := \frac{\nu B_1(\boldsymbol{u}_2^0)}{\nu - ah\alpha_1 + \lambda h\beta_1}$. For $n \in \mathbb{N}$, the n-th discrete DDM-iterate $(\boldsymbol{u}_1^n, \boldsymbol{u}_2^n) \in (\mathbb{R}^I)^2$ satisfies

$$\tfrac{\nu}{h}(u_{j,i-1}^n - 2u_{j,i}^n + u_{j,i+1}^n) + a(u_{j,i-1}^n - u_{j,i}^n) = f_i\ , \tag{13}$$

for $j = 1$, $-I < i < -1$ and for $j = 2$, $0 < i < I-1$,

$$\tfrac{\nu}{h}(-3u_{1,-I}^n + u_{1,-I+1}^n) - 2au_{1,-I}^n = f_{-I}\ , \tag{14}$$

$$\tfrac{\nu}{h}(u_{2,I-2}^n - 3u_{2,I-1}^n) + a(u_{2,I-2}^n - u_{2,I-1}^n) = f_{I-1}\ , \tag{15}$$

$$\tfrac{\nu}{h}\big(u^n_{1,-2}-2u^n_{1,-1}\big)+a\big(u^n_{1,-2}-u^n_{1,-1}\big)+\tfrac{\nu}{h}c_1u^n_{1,-1}=f_{-1}-\tilde{B}_1(\boldsymbol{u}^{n-1}_2)\,, \tag{16}$$

$$\tfrac{\nu}{h}\big(-2u^n_{2,0}+u^n_{2,1}\big)-au^n_{2,0}+\big(\tfrac{\nu}{h}+a\big)c_2u^n_{2,0}=f_0-\tilde{B}_2(\boldsymbol{u}^n_1)\,, \tag{17}$$

where

$$\tilde{B}_1(\boldsymbol{u}^n_2)=\tfrac{\nu}{h}u^n_{2,0}-\tfrac{\nu}{\nu+ah}c_1\big(f_0-\tfrac{\nu}{h}(-2u^n_{2,0}+u^n_{2,1})+au^n_{2,0}\big)\,, \tag{18}$$

$$\tilde{B}_2(\boldsymbol{u}^n_1)=\big(\tfrac{\nu}{h}+a\big)u^n_{1,-1}-\tfrac{\nu+ah}{\nu}c_2\big(f_{-1}-\tfrac{\nu}{h}(u^n_{1,-2}-2u^n_{1,-1})-a(u^n_{1,-2}-u^n_{1,-1})\big)\,, \tag{19}$$

$$c_1=\frac{\frac{\nu}{h}+a(1-\alpha_1)-\lambda(1-\beta_1)}{\frac{\nu}{h}-a\alpha_1+\lambda\beta_1}\,,\quad c_2=\frac{\frac{\nu}{h}-a\alpha_2-\lambda\beta_2}{\frac{\nu}{h}+a(1-\alpha_2)+\lambda(1-\beta_2)}\,. \tag{20}$$

Note that (13)-(19) is uniquely solvable for all $\nu\geq 0$ iff $c_1=O(1/\nu)$ and $c_2=O(\nu)$ as $\nu\to 0$. The resulting system matrix for $\boldsymbol{u}^n_2$ is weakly chained diagonally dominant, and thus non-singular. The same holds for $\boldsymbol{u}^n_1$ if $c_1\leq 1$. Further note that $\tilde{B}_1$ and $\tilde{B}_2$ in (16)-(19) are discrete Robin-to-Dirichlet operators, so that $c_1=c_2=0$ corresponds to Dirichlet TCs, which do not lead to convergence without overlap.

We next investigate how the coefficients α_j,β_j, $j=1,2$, must be chosen to obtain a discrete DDM that is consistent with (7)-(9). Since the discretization (13)-(15) is the same as (7)-(9), consistency follows iff the solution to (16)-(19) in the limit when $n\to\infty$ satisfies (7) and vice versa. The solution $\boldsymbol{u}$ of (7)-(9) solves (16)-(19), as can be directly seen when inserting it into (16)-(19) using (7) for $i=-1,0$. This only requires that νc_1 and c_2/ν are well-defined for all $\nu\geq 0$ and all $\lambda>0$. On the other hand, combining (16) and (18) as well as (17) and (19) yields

$$\begin{aligned}
&\tfrac{\nu}{h}(u_{1,-2}-2u_{1,-1}+u_{2,0})+a(u_{1,-2}-u_{1,-1})\\
&\qquad=f_{-1}+\tfrac{\nu}{\nu+ah}c_1\big(f_0-\tfrac{\nu}{h}(u_{1,-1}-2u_{2,0}+u_{2,1})-a(u_{1,-1}-u_{2,0})\big)\,,\\
&\tfrac{\nu}{h}(u_{1,-1}-2u_{2,0}+u_{2,1})+a(u_{1,-1}-u_{2,0})\\
&\qquad=f_0+\tfrac{\nu+ah}{\nu}c_2\big(f_{-1}-\tfrac{\nu}{h}(u_{1,-2}-2u_{1,-1}+u_{2,0})-a(u_{1,-2}-u_{1,-1})\big)\,.
\end{aligned}$$

We obtain equivalence with (7) iff $1\neq c_1c_2$. Hence, we have proved the following theorem which provides choices for the TC parameters $\alpha_1,\alpha_2,\beta_1,\beta_2$ that ensure consistency for all $\lambda>0$ and $\nu\geq 0$.

Theorem 2 (Consistency of the discrete DDM)

The limit of the discrete DDM iterates (13)-(19) *as $n\to\infty$ is equal to the solution of* (7)-(9) *for all $\lambda>0$ if the following conditions hold:*

(A1) $\alpha_1<\frac{\nu}{ah}$ *(or equal if $\beta_1>0$), and*

(A2) $\nu c_1=O(1)$ *as $\nu\to 0$, i.e. by* (A1), $\nu=O(\nu-ah\alpha_1+\lambda h\beta_1)$ *, and*

(A3) $c_2=O(\nu)$ *as $\nu\to 0$, i.e., $\alpha_2+\beta_2=O(\nu)$, and*

(A4) $c_1c_2\neq 1$, *i.e.,*

$$0\neq a^2(\alpha_2-\alpha_1)+\lambda\big(\tfrac{2\nu}{h}+a(\beta_1+\beta_2-\alpha_1-\alpha_2)\big)+\lambda^2(\beta_1-\beta_2)\,.$$

Remark 1 Note that the simplest choice of the coefficients, which satisfies Theorem 2 is $\alpha_1=\alpha_2=\beta_2=0$ and $\beta_1=1/2$. As shown below, this also yields convergence for any positive discrete Peclet number $\mathrm{Pe} := ah/\nu > 0$. Furthermore, this choice ensures that the discrete DDM is AP as $\nu \to 0$ for any $\lambda > 0$, as we show next.

We split the convergence analysis of the discrete DDM into two regimes due to the different types of solutions: the elliptic case $\nu > 0$ and the singular limit $\nu = 0$. For this, let $\boldsymbol{e}^n := \boldsymbol{u} - (\boldsymbol{u}_1^n, \boldsymbol{u}_2^n)$ be the error of the discrete DDM at iteration n. By linearity, $\boldsymbol{e}^n$ satisfies the discrete DDM (13)-(19) with $\boldsymbol{f} = \boldsymbol{0}$.

The elliptic case $\nu > 0$ **:** Then, (13)-(15) for $\boldsymbol{e}^n$ yield the solution

$$\boldsymbol{e}^n = \left(A_1^n \big(\xi^{(i+1)h} - \big(1 + \tfrac{\mathrm{Pe}}{2}\big)\xi^{-1}\big)_{i=-I}^{-1} \,,\, A_2^n \big(1 + \tfrac{\mathrm{Pe}}{2} - \xi^{(i+1)h-1}\big)_{i=0}^{I-1} \right) ,$$

where we defined $\xi := (1 + \mathrm{Pe})^I$. The constants $A_1^n, A_2^n \in \mathbb{R}$ are determined by (16)-(19), which yield the recurrence relations

$$A_1^n = -\frac{\lambda - a + \big(a\alpha_1 - \lambda(\mathrm{Pe}^{-1} + \beta_1)\big)\frac{2\mathrm{Pe}}{2+\mathrm{Pe}}\xi^{-1}}{\big(a\alpha_1 - \lambda(\mathrm{Pe}^{-1} + \beta_1)\big)\frac{2\mathrm{Pe}}{2+\mathrm{Pe}} + (\lambda - a)\,\xi^{-1}} A_2^{n-1}, \;\; A_2^n = \frac{a\alpha_2 + \lambda(\mathrm{Pe}^{-1} + \beta_2) - (\lambda + a)\frac{2+\mathrm{Pe}}{2\mathrm{Pe}}\xi^{-1}}{(\lambda + a)\frac{2+\mathrm{Pe}}{2\mathrm{Pe}} - \big(a\alpha_2 + \lambda(\mathrm{Pe}^{-1} + \beta_2)\big)\xi^{-1}} A_1^n \,.$$

Therefore, the iteration is linearly convergent iff

$$\rho = \left| \frac{\lambda - a + \big(a\alpha_1 - \lambda(\mathrm{Pe}^{-1} + \beta_1)\big)\frac{2\mathrm{Pe}}{2+\mathrm{Pe}}\xi^{-1}}{\lambda + a - \big(a\alpha_2 + \lambda(\mathrm{Pe}^{-1} + \beta_2)\big)\frac{2\mathrm{Pe}}{2+\mathrm{Pe}}\xi^{-1}} \right| \left| \frac{a\alpha_2 + \lambda(\mathrm{Pe}^{-1} + \beta_2) - (\lambda + a)\frac{2+\mathrm{Pe}}{2\mathrm{Pe}}\xi^{-1}}{a\alpha_1 - \lambda(\mathrm{Pe}^{-1} + \beta_1) + (\lambda - a)\frac{2+\mathrm{Pe}}{2\mathrm{Pe}}\xi^{-1}} \right| < 1 \,. \quad (21)$$

Note that convergence in one iteration is possible for the choice

$$\lambda = \lambda_{\mathrm{opt}} := \frac{2\nu + ah - 2\alpha_1 ah\xi^{-1}}{2\nu + ah - 2\,(\nu + \beta_1 ah)\,\xi^{-1}} a \quad \xrightarrow{h \to 0} \quad \frac{a}{1 - e^{-a/\nu}} \,, \qquad (22)$$

which is almost mesh independent when $\alpha_1 = 0$ and $\beta_1 = 1/2$. This is consistent with the continuous DDM and also yields $\lambda_{\mathrm{opt}} \to a$ as $\nu \to 0$.

Furthermore, note that (21) for $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1/2$ is satisfied for all $\lambda > 0$. But $\beta_2 = 1/2$ does not satisfy (A3) of Theorem 2, so that $\tilde{B}_2$ (and thus ρ) degenerate when $\nu \to 0$. However, choosing $\alpha_1 = \alpha_2 = \beta_2 = 0$ and $\beta_1 = 1/2$, Theorem 2 is satisfied for all $\nu > 0$, and (21) is satisfied for all $\lambda > 0$ due to $\mathrm{Pe} > 0$.

The singular limit $\nu = 0$ **:** Then, (13)-(15) for $\boldsymbol{e}^n$ yields

$$\boldsymbol{e}^n = \left((0)_{i=-I}^{-2}, A_1^n,\, (A_2^n)_{i=0}^{I-1} \right) ,$$

with $A_1^n, A_2^n \in \mathbb{R}$ determined by (16)-(19). To obtain $A_1^1 = 0$, i.e., the correct solution in Ω_1, this requires by (16)

$$0 = A_1^1 = \frac{-\tilde{B}_1(\boldsymbol{e}^0)}{\frac{\nu}{h}c_1 - a} \,, \qquad\qquad \tilde{B}_1(\boldsymbol{e}^0) = \frac{\nu B_1(\boldsymbol{e}^0)}{\nu - ah\alpha_1 + \lambda h\beta_1} \,.$$

Since $\nu c_1 = O(1)$ as $\nu \to 0$ by (A2), this holds iff $\lim_{\nu \to 0} \nu c_1 \neq ah$ and $\lim_{\nu \to 0} \nu/(\nu - ah\alpha_1 + \lambda h\beta_1) = 0$. Using (A1) of Theorem 2, this simplifies to $\nu/\beta_1 = o(1)$ as $\nu \to 0$ and implies $c_1 = o(1)$. For A_2^1, we then obtain by (17)-(19)

and (A3) that $A_2^1 = 0$, i.e., convergence in one iteration. Then, $A_1^n = A_2^n = 0$ for all $n > 2$ follows by induction using (16)-(19).

Summarizing the above analysis, we obtain the following result.

Theorem 3 (Convergence and AP property of the discrete DDM)

Let (A1)-(A4) *from Theorem 2 be satisfied. The sequence of discrete DDM iterates* $\{(\boldsymbol{u}_1^n, \boldsymbol{u}_2^n)\}_{n\in\mathbb{N}}$ *from* (13)-(19) *converges linearly to the solution of* (7)-(9) *for* $\nu > 0$ *iff* (21) *is satisfied.*
Convergence in one iteration is achieved if λ *satisfies* (22) *or for* $\nu = 0$ *if the limit discrete DDM for* $\nu/\beta_1 = o(1)$ *as* $\nu \to 0$ *is used.*
The discrete DDM (13)-(19) *is AP if* $|\lambda - a| = o(1)$ *or* $\nu/\beta_1 = o(1)$ *as* $\nu \to 0$.

Note that as shown above, the choice $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1/2$ yields linear convergence for $\nu > 0$, but the convergence rate degenerates for $\nu \to 0$. The choice $\alpha_1 = \alpha_2 = \beta_2 = 0$ and $\beta_1 = 1/2$ leads to linear convergence for $\nu > 0$ uniformly in ν with 1-step convergence for $\nu = 0$, and thus is AP.

Remark 2 (Convergence order and mass conservation) As the iterates of the discrete DDM converge to the solution of (7)-(9), which is a first-order convergent finite volume method (uniform in ν and a), the same holds for the discrete DDM at convergence (and before as soon as $\boldsymbol{e}^n = O(h)$). Furthermore, the finite volume method is locally mass conservative, such that mass conservation holds in each subdomain of the discrete DDM. At the interface between the subdomains, mass conservation is ensured at convergence, since the discrete DDM recovers the (implicit) monodomain finite volume formulation. In contrast, methods based on an explicit splitting at the interface (see e.g. [11, 8]) directly ensure mass conservation, but require the usual time-step restriction of CFL-type when the diffusion vanishes ($\nu \to 0$).

4 Numerical examples

We now study numerically the convergence properties of the discrete DDM as $\nu \to 0$ for various choices of the parameters in the discrete Robin TCs. Since $\alpha_j = O(\nu)$, $j = 1, 2$, is required for convergence, we restrict our study to $\alpha_1 = \alpha_2 = 0$ and vary only β_1, β_2 and λ. We consider (1) for $f(x) = -\nu(k\pi)^2 \sin(k\pi x) - ak\pi\cos(k\pi x)$, which leads to the exact solution $u(x) = \sin(k\pi x)$. We fix $a = 1$, $k = 3$, $B_1(u_2^0) = 1$ and $I = 100$, and study the number of iterations required to reach an error of $\|\boldsymbol{e}^n\|_\infty < 10^{-12}$, see Fig. 1, both for experiments in 1D and 2D. As discussed above, the choice $\beta_1 = \beta_2 = 1/2$ leads to a degeneration as $\nu \to 0$, while the choice $\beta_1 = \beta_2 = \min(1/2, \nu/(ah))$ yields linear convergence, but is only AP for $\lambda \to a$. As predicted by Theorem 3, the convergence improves for all choices such that $\nu/\beta_1 = o(1)$ and $\beta_2 = O(\nu)$ as $\nu \to 0$. In particular, the number of iterations decreases faster when β_1 is large, which illustrates well the convergence factor ρ in (21), which satisfies $\rho = \frac{|\lambda-a|}{\lambda+a} O\left(\frac{\nu}{\nu+\beta_1}\right) + O\left(\nu^{I-1}\right)$. Note that the finite volume method permits a straightforward extension of the discrete DDM to higher dimensions. For our 2D

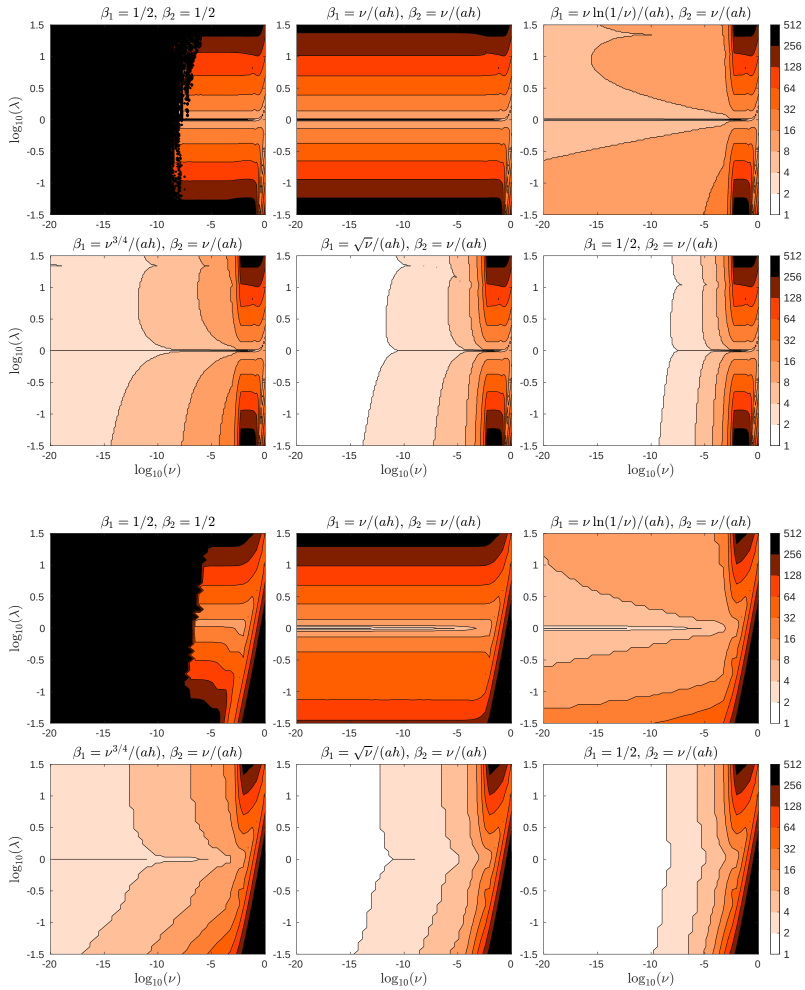

Fig. 1: Number of iterations for various β_1 and β_2 in 1D (top 6 panels) and 2D (bottom 6 panels).

example with equidistant rectangular mesh, the two-point fluxes across the edges on the interface between the subdomains can be constructed exactly as in 1D based on the TCs and ghost values. This leads to the 2D results in Fig. 1 for $\nu\Delta u - \nabla \cdot u = f$ in $(-1, 1) \times (0, 1)$, $u(-1, y) = u(x, 0) = 0$, $\nu u(1, y) = \nu u(x, 1) = 0$ for f chosen such that the exact solution is $u(x, y) = \sin(3\pi x)\sin(3\pi y)$. The technique developped here also works for non-linear time dependent advection-diffusion problems on triangular meshes, see [5].

5 Conclusion

The continuous non-overlapping DDM with Robin TCs applied to singularly-perturbed advection-diffusion problems is AP only when the transmission parameter λ tends to the advection speed as $\nu \to 0$. We showed that a much better result can be obtained for a discrete DDM based on a cell-centered finite volume method: in contrast to the continuous algorithm, a proper, but asymmetric choice of the discrete parameters $(\alpha_j, \beta_j, j = 1, 2)$ in the Robin TCs yields the AP property without any restriction on the transmission parameter λ. We illustrated the theoretical results by numerical examples in one and two spatial dimensions, see also the forthcoming work [5] where we show how the present techniques can be used for robust DDMs for nonlinear advection-diffusion equations in space-time on triangular meshes.

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References

1. Bennequin, D., Gander, M.J., Gouarin, L., Halpern, L.: Optimized Schwarz waveform relaxation for advection reaction diffusion equations in two dimensions. Numerische Mathematik **134**, 513–567 (2016)
2. Carlenzoli, C., Quarteroni, A.: Adaptive domain decomposition methods for advection-diffusion problems. In: Modeling, Mesh Generation, and Adaptive Numerical Methods for Partial Differential Equations, *IMA Vol. Math. Appl.*, vol. 75. Springer (1995)
3. Gander, M.J., Halpern, L., Martin, V.: A new algorithm based on factorization for heterogeneous domain decomposition. Numerical Algorithms **73**, 167–195 (2016)
4. Gander, M.J., Halpern, L., Martin, V.: Multiscale analysis of heterogeneous domain decomposition methods for time-dependent advection reaction diffusion problems. J. Comp. Appl. Math. **344**, 904–924 (2018)
5. Gander, M.J., Lunowa, S.B., Rohde, C.: Non-overlapping Schwarz waveform-relaxation for nonlinear advection-diffusion equations (2021). Preprint available at `uhasselt.be/Documents/CMAT/Preprints/2021/UP2103.pdf`
6. Gastaldi, F., Gastaldi, L., Quarteroni, A.: ADN and ARN domain decomposition methods for advection-diffusion equations. In: Ninth international conference on domain decomposition methods, pp. 334–341 (1998)
7. Jin, S.: Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review. Riv. Math. Univ. Parma (N.S.) **3**, 177–216 (2012)
8. Liang, D., Zhou, Z.: The conservative splitting domain decomposition method for multicomponent contamination flows in porous media. J. Comput. Phys. **400**, 1–27 (2020)
9. Lube, G., Müller, L., Otto, F.C.: A non-overlapping domain decomposition method for the advection-diffusion problem. Computing **64**, 49–68 (2000)
10. Nataf, F., Rogier, F.: Factorization of the convection-diffusion operator and the Schwarz algorithm. M^3AS **5**, 67–93 (1995)
11. Zhou, Z., Liang, D.: The mass-preserving and modified-upwind splitting DDM scheme for time-dependent convection-diffusion equations. J. Comput. Appl. Math. **317**, 247–273 (2017)