Adaptive Schwarz Method for Crouzeix-Raviart Multiscale Problems in 2D

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1 Introduction

In modeling real physical phenomena, we quite often see a heterogeneity of coefficients, e.g., in some ground flow problems in heterogeneous media. After applying a discretization method to the differential equations which model our physical phenomenon, e.g., a finite element method, we obtain a discrete problem which is usually very hard to solve by the standard preconditioned iterative methods, like, e.g., preconditioned CG (PCG) or preconditioned GMRES methods. A popular way of constructing parallel preconditioners is to use the Domain Decomposition Methods (DDMs) approach, in particular Schwarz methods, cf. e.g., [14]. In DDMs, it is very important to construct carefully coarse spaces. The overlapping and non-overlapping Schwarz methods were proposed over thirty years ago, and are extensively developed and analyzed, cf. [14] for overviews. The average Schwarz method was proposed in [2], cf. also [1, 12, 6, 10]. It is a non-overlapping Schwarz method with a very simple coarse space. This class of DDMs, along with other 'classical' DDMs constructed in the 1990s and 2000s, are well suited for the problems with coefficients that are constant or slightly varying in subdomains. However, when the coefficients may be highly varying and discontinuous almost everywhere, those 'classical' methods are not efficient. That's why many researchers start to look for new adaptive coarse spaces which are independent or robust for the jumps of the coefficients, i.e., the convergence of the constructed DDM is independent of the distribution and the magnitude of the coefficients of the original problem. We refer to [8], [13] and the references therein for similar earlier works on domain decomposition methods that

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used adaptivity in the construction of the coarse spaces. In recent years there are many novel works in this direction cf. e.g., [5, 7, 9, 8, 11, 4] and many others.

In our paper, we consider the nonconforming Crouzeix-Raviart element discretization, also called the nonconforming P_1 element discretizationand then construct an average Schwarz method with an adaptive coarse space. We extend the results from [10] when the conforming P_1 element is considered to the case of the average Schwarz method for CR non-conforming discretization applied to highly heterogeneous coefficients.

2 Discrete Problem

Let consider the following elliptic second order boundary value problem in 2D: Find $u^* \in H_0^1(\Omega)$

$$\int_\Omega \alpha(x)\nabla u^* \nabla v\, dx = \int_\Omega f v\, dx, \qquad \forall v \in H_0^1(\Omega), \tag{1}$$

where Ω is a polygonal domain in $\mathbb{R}^2$, $\alpha(x) \geq \alpha_0 > 0$ is a coefficient, α_0 is a positive constant, and $f \in L^2(\Omega)$.

We introduce $\mathcal{T}_h = \{K\}$ as the quasi-uniform triangulation of Ω consisting of opened triangles such that $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$. Further, h_K denotes the diameter of K, and let $h = \max_{K\in\mathcal{T}_h} h_K$ be the mesh parameter for the triangulation.

Let consider a coarse non-overlapping partitioning of Ω into the open, connected Lipschitz polygonal subdomains Ω_i, called substructures or subdomains, such that $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$.

We also assume that those substructures are aligned with the fine triangulation, i.e., any fine triangle K of $\mathcal{T}_h$ is contained in one substructure. Thus each substructure

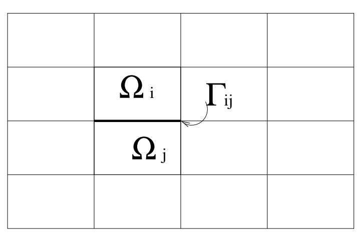

Fig. 1: An example of a coarse partition of Ω, where Γ_{ij} is an interface.

Ω_j has its local triangulation $T_h(\Omega_j)$ of triangles from T_h which are contained in $\overline{\Omega}_j$. For the simplicity of presentation, we further assume that these substructures form a coarse triangulation of the domain which is shape regular in the sense of [3] and let $H = \max_j \operatorname{diam}(\Omega_j)$ be its coarse parameter.

We denote Ω_h^{CR}, $\partial\Omega_h^{CR}$, $\Omega_{i,h}^{CR}$, $\partial\Omega_{i,h}^{CR}$, and $\Gamma_{ij,h}^{CR}$ the sets of midpoints of fine edges of the elements of $\mathcal{T}_h$, contained in Ω, $\partial\Omega$, Ω_i, $\partial\Omega_i$, and Γ_{ij} (the interface between Ω_i and Ω_j, see e.g., Figure 1), respectively. We call those sets the CR (Crouzeix-Raviart) nodal points of the respective sets.

Further, let us define the discrete space $S_h = S_h(\Omega)$ as the standard non-conforming Crouzeix-Raviart linear finite element space defined on the triangulation $\mathcal{T}_h$,

$$S_h(\Omega) := \{u \in L^2(\Omega) : \ u_{|K} \in P_1, \ K \in \mathcal{T}_h, u - \text{continuous} \\ \text{at CR nodal points and } u(x) = 0, \ x \in \partial\Omega_h^{CR} \ \}.$$

The degrees of freedom of a CR function on a fine triangle K are the values at the midpoints of its edges, cf. Figure 2.

Note that a function in S_h is multivalued on boundaries of all fine triangles of $\mathcal{T}_h$ except the midpoints of the edges (CR nodal points). Thus $S_h \not\subset H_0^1(\Omega)$ as a space of discontinuous functions. S_h is only a subspace of $L^2(\Omega)$.

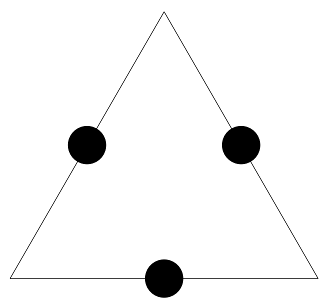

Fig. 2: The CR nodal points, i.e., the degrees of freedom of the Crouzeix-Raviart finite element space on a fine triangle.

We also introduce the local discrete space S_i as the subspace of S_h formed by all functions of S_h which are zeros at all CR nodal points which are NOT in Ω_i^{CR}, or equivalently, formed by functions which are restricted to Ω_i, are zero on $\partial\Omega_{i,h}^{CR}$, and extended by zero elsewhere. Naturally, formally S_i is a subspace of S_h but in practice, it is a local space of functions defined by the values at Ω_i^{CR}.

We consider the following Crouzeix-Raviart discrete problems: We want to find $u_h^* \in S_h$:

$$a_h(u_h^*.v) = f(v) \qquad \forall v \in S_h, \tag{2}$$

where $a_h(u, v) = \sum_{K \in T_h} \int_K \alpha_{|K}(x) \nabla u \nabla v \, dx$ is the so called broken bilinear form. Note that ∇u_h for $u_h \in S_h$ is a piecewise constant over the fine triangles of T_h. We further assume that α is piecewise constant function over the elements of $\mathcal{T}_h$ since $\int_K \alpha \nabla u \nabla v \, dx = (\nabla u)_{|K} (\nabla v)_{|K} \int_K \alpha(x) \, dx$. Since the broken form is S_h-elliptic, the discrete problem has a unique solution.

3 Additive Schwarz Method

In this section, we present our non-overlapping average Schwarz method for solving (2). Our method is based on the abstract Additive Schwarz Method framework, cf. e.g., [14].

Space S_h is decomposed into local sub-spaces and a global average Schwarz "spectrally enriched" coarse space. For the local spaces, we take $\{S_i\}_i$. We have that $S_h = \sum_{i=1}^N S_i$.

Coarse space

We introduce our spectrally enriched coarse space in this section.

First, we define the classical average Schwarz coarse space, see e.g. [2]. Let $I_{AS} : S_h \to S_h$ be the linear interpolating operator defined as follows:

$$I_{AS}u(x) = \begin{cases} u(x) & x \in \bigcup_{k=1}^N \partial\Omega_{i,h}^{CR}, \\ \overline{u}_i & x \in \Omega_{i,h}^{CR} \quad i = 1, \ldots, N, \end{cases} \tag{3}$$

where $\overline{u}_i = \frac{1}{M_i} \sum_{x \in \partial\Omega_{i,h}^{CR}} u(x)$ with $M_i = \#\partial\Omega_{i,h}^{CR}$, i.e., $\overline{u}_i$ is the CR discrete average of u over $\partial\Omega_i$. The standard coarse space of the average Schwarz method is the image of this interpolating operator:

$$V_{AS} = I_{AS} S_h. \tag{4}$$

We introduce two types of the local generalized eigenvalue problem, which is to find the eigenvalue and its associated eigenfunction: $(\lambda_i^j, \psi_i^j) \in \mathbb{R}_+ \times S_j$ such that

$$a_h(\psi_i^j, v) = \lambda_i^j b_j^{type}(\psi_i^j, v), \qquad \forall v \in S_j, \quad type \in \{\mathrm{I}, \mathrm{II}\}, \tag{5}$$

where

$$b_j^{type}(u, v) = \begin{cases} \sum_{K \in T_h(\Omega_j)} \int_K \underline{\alpha}_j \nabla u \nabla v \, dx & type = \mathrm{I} \\ \sum_{K \subset \Omega_j^\delta} \int_K \underline{\alpha}_j \nabla u \nabla v \, dx + & \\ \qquad + \sum_{K \subset \Omega_j \setminus \Omega_j^\delta} \int_K \alpha_j \nabla u \nabla v \, dx & type = \mathrm{II} \end{cases}$$

where $\underline{\alpha}_j := \inf_{x \in \overline{\Omega}_j} \alpha(x)$ and Ω_j^δ is the discrete boundary layer in Ω_j comprising those fine triangles of the local triangulation of Ω_j which have a fine edge on $\partial\Omega_j$.

Naturally, ψ_i^j should be denoted $\psi_i^{j,type}$ as it depends on the type of the RHS form but we try to have the notation as simple as possible, and we keep in mind this dependence.

Note that it follows from the definition $a_h(u, u) \geq b_j^{type}(u, u)$ for any $u \in S_j$, thus all eigenvalues $\lambda_i^j \geq 1$ for the both types of the form $b_j(\cdot, \cdot)$.

We order the eigenvalues in the decreasing way as follows

$$\lambda_1^j \geq \lambda_2^j \geq \ldots \geq \lambda_{M_j}^j \geq 1$$

for $M_j = \dim(S_j)$. Next we introduce the local spectral component of the coarse space for all Ω_j and further the enriched coarse space V_0:

$$S_j^{eig} = \mathrm{Span}(\psi_i^j)_{i=1}^{n_j}, \tag{6}$$

where $0 \leq n_j \leq M_j$ is the number of eigenfunctions ψ_i^j selected by an user, e.g. in such a way that the eigenvalue $\lambda_{n_j}^j \geq \lambda$, where $\lambda \geq 1$ is a pre-selected threshold. Finally, the coarse space S_0 is introduced as:

$$S_0 = V_{AS} + \sum_{j=1}^{N} S_j^{eig}. \tag{7}$$

There are two types of this coarse space but the difference is not significant, and below S_0 means one of the described coarse spaces.

Average Schwarz operator *T*

Next we define the projection operators $T_i : S_h \to S_i$ as

$$a_h(T_i u, v) = a_h(u, v), \quad \forall v \in S_i, \qquad i = 0, \ldots, N. \tag{8}$$

Note that to compute $T_i u$, $i = 1, \ldots, N$ we have to solve N independent local problems.

Let $T := \sum_{i=0}^{N} T_i$, be the average Schwarz operator. We further replace (2) by the following equivalent problem: Find $u_h^* \in S_h$ such that

$$T u_h^* = g, \tag{9}$$

where $g = \sum_{i=0}^{N} g_i$ and $g_i = T_i u_h^*$. The functions g_i may be computed without knowing the solution u_h^* of (2), cf. e.g., [14].

The following theoretical estimated of the condition number can be obtained:

Theorem 1 *For all $u \in S_h$, the following holds,*

$$c \left(1 + \max_j \lambda_{n_j+1}^j\right)^{-1} \frac{h}{H} a_h(u, u) \leq a_h(T u, u) \leq C\, a_h(u, u),$$

where C and c are positive constants independent of the coefficient α, the mesh parameter h and the subdomain size H, and $\lambda_{n_j+1}^j$ is defined in (5) for both types of the coarse space.

The proof is based on the standard abstract ASM Method framework, cf. e.g. [14]. We have to prove three key assumptions, the most technical is the stable splitting ass., namely we can show that for any $u \in S_h$ there exists: $u_j \in S_j$ $j = 0, \dots, N$ such that $\sum_{j=0}^{N} a_h(u_j, u_j) \leq c^{-1} \left(1 + \max_j \lambda^j_{n_j+1}\right) a(u,u)$. The two others assumptions are easy to verify. Namely, the stability constant is equal to one since the broken form is used as local forms. The third ass., the bound of the spectral radius of the matrix of the constants of the strengthened Cauchy-Schwarz inequalities is also equal to one, since the local subspaces are a_h orthogonal subspaces to each other.

4 Numerical tests

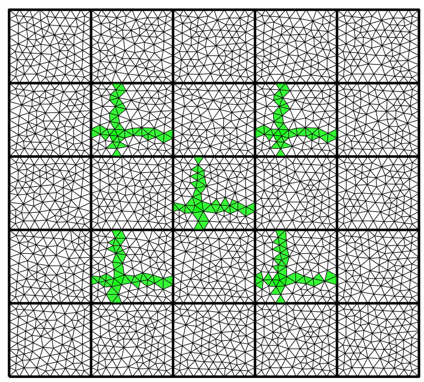

Fig. 3: The location of all jumps in $\alpha(x)$, where $\Omega = [0,1] \times [0,1]$ is partitioned into 5×5 subdomains. The values of jumps on the white and green triangles are 1 and 1.0e4, respectively. To get numerical results, we use these green channels as the periodic patterns for different number of subdomains.

In this section, we consider the right-hand side function

$$f(x,y) = 2\pi^2 \sin(\pi x) \sin(\pi y),$$

where $(x,y) \in \Omega = [0,1] \times [0,1]$. To confirm the validity of the theoretical result numerically, we also divide all jumps in $\alpha(x)$ into $\alpha_b = 1$ and $\alpha_i = 1.0e4$ corresponding to the coefficients defined on the background and green channels, respectively, cf. Figure 3.

h	$H = 1/3$	$H = 1/6$	$H = 1/9$
1/18	7.2601e6	7.5289e6	1.3895e7
1/36	3.0394e7	2.8114e7	2.8563e7
1/54	7.4566e7	6.2314e7	5.8596e7

Table 1: The condition numbers of the non-preconditioned system for different values of H and h.

h	$H = 1/3$	$H = 1/6$	$H = 1/9$
1/18	57.2051 (47)	33.9312 (42)	20.7272 (38)
1/36	120.7130 (67)	54.8475 (62)	40.6837 (55)
1/54	177.2259 (85)	83.2981 (77)	56.8240 (65)

Table 2: The condition numbers of the additive average Schwarz preconditioner $type$ = I, and the number of iterations of preconditioned CG method (in parentheses). Further, the given threshold to construct the enrichment coarse space is 100.

h	$H = 1/3$	$H = 1/6$	$H = 1/9$
1/18	57.2051 (44)	33.9312 (42)	20.7272 (38)
1/36	120.7130 (70)	54.8474 (60)	40.6747 (52)
1/54	177.2259 (93)	83.2881 (74)	56.8249 (67)

Table 3: The condition numbers of the additive average Schwarz preconditioner $type$ = II, and the number of iterations of preconditioned CG method (in parentheses). Further, the given threshold to construct the enrichment coarse space is 100.

	$type$ = I			$type$ = II		
h	$H = 1/3$	$H = 1/6$	$H = 1/9$	$H = 1/3$	$H = 1/6$	$H = 1/9$
1/18	53	148	165	11	92	147
1/36	188	442	609	19	93	255
1/54	440	947	1317	26	122	298

Table 4: The number of eigenfunctions associated with the eigenvalues greater than 100 used in the construction of the enrichment part of the coarse space, where $type \in \{\text{I}, \text{II}\}$, $H \in \{1/3, 1/6, 1/9\}$ and $h \in \{1/18, 1/36, 1/54\}$.

Table 1 presents the condition number of the non-preconditioned system. To see the efficiency of the enriched additive average Schwarz preconditioners for both types I and II, we refer to Tables 2 and 3. Those tables also present the numbers of iteration of the preconditioned CG method with the tolerance 1e − 6. For different values of H and h, the first observation is that there is a slight difference between the two types of enrichment in terms of the condition numbers and iteration numbers. The second observation is that the ratio of the condition numbers is proportional to the ratio of H/h, for instance, the condition numbers represented by purple color are very close together, where the ratio of H/h is identical. This means that the validity of Theorem 1 is confirmed numerically. Finally, Table 4 includes the number of eigenfunctions used in the construction of the enriched coarse space and shows that the second type of enrichment has a good performance throughout the implementation in comparison to the first type.

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