Space-Time Finite Element Tearing and Interconnecting Domain Decomposition Methods

Douglas R. Q. Pacheco and Olaf Steinbach

1 Introduction

Finite element tearing and interconnecting (FETI) domain decomposition methods [4] are well-established techniques for the parallel solution of elliptic problems. This is mainly due to their simple implementation and the availability of efficient and robust preconditioning strategies. Among other variants to deal with floating subdomains, total FETI [2] or all-floating FETI [8] methods handle all subdomains as *floating*, incorporating also Dirichlet boundary conditions via Lagrange multipliers. This can simplify the implementation, in particular when considering systems of partial differential equations. While the original derivation of the FETI method was based on a constrained minimization problem, related methods can be formulated for the Helmholtz [12] and Maxwell [13] equations as well, using tearing and interconnecting on the discrete level only. Nonetheless, domain decomposition and FETI methods have been so far mainly restricted to elliptic problems, or to time-dependent problems which are discretized through tensor-product ansatz spaces. Parallelization in time is in most cases based on the parareal algorithm [7] combing coarse and fine temporal grids.

In recent years, space-time discretization methods have become very popular, see, e.g., the review article [14] and the references given therein. These methods consider time as just another spatial coordinate, using a finite element discretization in the whole space-time domain [10]. As this allows an adaptive resolution in space and time simultaneously, solving the resulting algebraic system requires efficient solution strategies in parallel. Domain decomposition methods are a natural choice to provide efficient, robust preconditioning and allow parallelization when considering one subdomain per processor.

Douglas R. Q. Pacheco and Olaf Steinbach
Institut für Angewandte Mathematik, TU Graz, Steyrergasse 30, 8010 Graz, Austria
e-mail: pacheco@math.tugraz.at,o.steinbach@tugraz.at

While the work presented in Ref. [11] considers standard domain decomposition methods [1, 5] for the heat equation, the focus of the present contribution is on FETI methods applied to the Stokes system and the heat equation. In Section 2 we describe the space-time finite element discretization of the related model problems. For the solution of the resulting linear systems we present in Section 3 a FETI method, including a discussion on floating subdomains. When considering all subdomains as floating, we end up with an all-floating FETI method. First numerical results in Section 4 indicate the great potential of space-time FETI domain decomposition methods, including parallel-in-time algorithms.

2 Space-time finite element methods

We start with the homogeneous Dirichlet problem for the transient heat equation:

$$\begin{aligned} \partial_t u - \Delta_x u &= f \quad \text{in } Q, \\ u &= 0 \quad \text{on } \Sigma \cup \Sigma_0, \end{aligned} \tag{1}$$

where for a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, and a finite time horizon T we have the space-time domain $Q := \Omega \times (0, T) \subset \mathbb{R}^{d+1}$ with lateral and bottom boundaries $\Sigma := \partial\Omega \times (0, T)$ and $\Sigma_0 := \Omega \times \{0\}$, respectively. For simplicity, we only consider homogeneous boundary and initial conditions, but inhomogeneous data and other types of boundary conditions can be handled as well. The space-time variational formulation of (1) reads to find $u \in X := L^2(0, T; H^1_0(\Omega)) \cap H^1_{0,}(0, T; H^{-1}(\Omega))$ such that

$$\int_0^T \int_\Omega \Big[v \partial_t u + \nabla_x u \cdot \nabla_x v \Big] \, dx \, dt = \int_0^T \int_\Omega f \, v \, dx \, dt \tag{2}$$

is satisfied for all $v \in Y := L^2(0, T; H^1_0(\Omega))$. Note that the ansatz space X covers zero boundary and initial conditions. For a space-time finite element discretization of (2), we introduce conforming finite element spaces $X_h \subset X$ and $Y_h \subset Y$, assuming $X_h \subset Y_h$. In particular, we use the finite element spaces $X_h = Y_h$ of continuous, piecewise linear basis functions, defined with respect to some admissible decomposition of the space-time domain Q into shape-regular simplicial finite elements. Detailed stability and error analysis of this space-time finite element method can be found in Refs. [10, 11]. The space-time finite element discretization of (2) results in a large linear system of algebraic equations which we shall solve using an appropriate tearing and interconnecting domain decomposition method.

As a second model problem, we consider the time-dependent Stokes system

$$\begin{aligned} \partial_t u - \mu \Delta_x u + \nabla_x p &= f \quad \text{in } Q, \\ \nabla_x \cdot u &= 0 \quad \text{in } Q, \\ u &= 0 \quad \text{on } \Sigma \cup \Sigma_0, \end{aligned} \tag{3}$$

once again assuming homogeneous boundary and initial conditions, for simplicity. The variational formulation of (3) seeks $u \in X^d$ and $p \in L^2(Q)$ such that

$$\int_0^T \int_\Omega \Big[\partial_t u \cdot v + \mu \nabla_x u : \nabla_x v - p \, \nabla_x \cdot v\Big] dx\, dt = \int_0^T \int_\Omega f \cdot v \, dx\, dt, \tag{4}$$

$$\int_0^T \int_\Omega q \, \nabla_x \cdot u \, dx\, dt + \int_0^T \left(\int_\Omega p \, dx \int_\Omega q \, dx \right) dt = 0 \tag{5}$$

is satisfied for all $v \in Y^d$ and $q \in L^2(Q)$. Note that the additional term in (5) ensures the scaling condition $p \in L^2_0(\Omega)$ for all $t \in (0, T)$. The space-time variational formulation (4)–(5) can be analyzed similarly to what was done in Ref. [10] in the case of the heat equation, extending to the space-time setting the spatial inf-sup stability condition for the divergence. Note that inhomogeneous essential boundary and initial conditions g and u_0 can be handled through homogenization by using suitable extensions of such data into the space-time domain. For the space-time finite element discretization of (4) and (5) we use inf-sup stable pairs to approximate u_h and p_h. In particular, we extend the well established Taylor–Hood elements to the space-time setting using simplicial finite elements. As an alternative we may also use prismatic space-time Taylor–Hood elements, see Ref. [9] for first numerical results. A more detailed stability and error analysis will be published elsewhere.

3 Tearing and interconnecting domain decomposition methods

The space-time finite element discretization of the heat equation (1) and of the Stokes system (3) results in very large systems of algebraic equations which must be solved in parallel and, if possible, simultaneously in space and time. One possible approach is to use space-time finite element tearing and interconnecting methods, which are well established for elliptic problems. Here we generalize this approach to parabolic time-dependent problems. The space-time domain $Q = \Omega \times (0, T)$ is decomposed into s non-overlapping space-time subdomains Q_i which can be rather general, see Fig. 1 for a selection of possible simple decompositions. With respect to this space-time domain decomposition we consider the localized problems, where the continuity of the primal unknowns along the interface is enforced via discrete Lagrange multipliers. This results in the global linear system

$$\begin{pmatrix} K_1 & & & B_1^\top \\ & \ddots & & \vdots \\ & & K_s & B_s^\top \\ B_1 & \cdots & B_s & \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_s \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_s \\ \underline{0} \end{pmatrix}, \tag{6}$$

where the K_i are the local space-time finite element stiffness matrices and the B_i are Boolean matrices. While (6) corresponds directly to the heat equation (1), it formally also includes the Stokes problem (3) with all quantities defined accordingly. Although we have chosen to enforce the interface continuity of the pressure field, this is in principle not necessary since the variational problem allows $p \in L^2(Q)$.

At this time, we assume that all local matrices K_i are invertible, so that when using direct solvers locally we end up with the Schur complement system

$$\sum_{i=1}^{s} B_i K_i^{-1} B_i^\top \underline{\lambda} = \sum_{i=1}^{s} B_i K_i^{-1} \underline{f}_i \,. \tag{7}$$

The heat equation can be seen as a diffusion equation with convection in the temporal direction. Since there is no difference between the spatial and temporal mesh size h, we conclude a spectral condition number of $O(h^{-2})$ for (6), and of $O(h^{-1})$ for the Schur complement system (7). The global linear system (7) is solved here by a GMRES method, either without preconditioning or with a simple diagonal preconditioner. More advanced preconditioning strategies also including some coarse grid contributions seem to be mandatory for more complex problems, being a topic of further research.

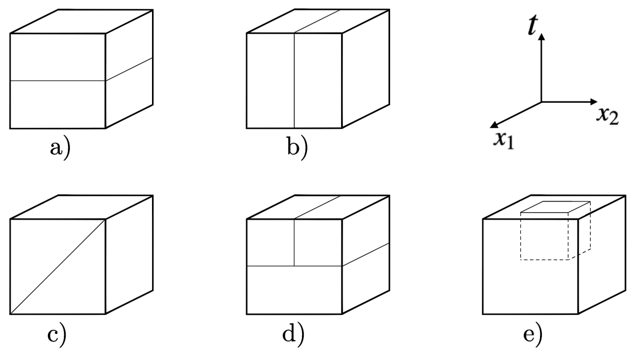

Fig. 1: Different decompositions for the space-time domain $Q = \Omega \times (0, T) \subset \mathbb{R}^3$.

In what follows, we discuss the more general situation in which a local matrix K_i is not invertible, i.e., when the subdomain Q_i is *floating*. Using a pseudo-inverse K_i^+ of K_i, we can describe the solutions of the local subproblems as

$$\underline{u}_i = K_i^+(\underline{f}_i - B_i^\top \underline{\lambda}) + R_i \underline{\alpha}_i, \tag{8}$$

where the local matrices R_i describe the kernels $\mathcal{N}(K_i)$ of K_i, and $\underline{\alpha}_i$ are coefficients to be determined. The application of the pseudo-inverse K_i^+ also requires the solvability condition $\underline{f}_i - B_i^\top \underline{\lambda} \in \mathcal{R}(K_i)$, which is equivalent to

$$\widetilde{R}_i^\top (\underline{f}_i - B_i^\top \underline{\lambda}) = \underline{0}\,,$$

where the local matrices $\widetilde{R}_i$ describe the kernels $\mathcal{N}(K_i^\top)$. In the case of floating subdomains we therefore end up with the Schur complement system

$$\begin{pmatrix} S & -G \\ \widetilde{G}^\top & 0 \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \underline{\alpha} \end{pmatrix} = \begin{pmatrix} \underline{d} \\ \underline{e} \end{pmatrix}, \tag{9}$$

where

$$S = \sum_{i=1}^{s} B_i K_i^+ B_i^\top, \quad G = \Big(B_1 R_1, \cdots, B_s R_s\Big), \quad \widetilde{G} = \Big(B_1 \widetilde{R}_1, \cdots, B_s \widetilde{R}_s\Big),$$

$$\underline{d} = \sum_{i=1}^{s} B_i K_i^+ \underline{f}_i, \quad \underline{e} = \begin{pmatrix} \widetilde{R}_1^\top \underline{f}_1 \\ \vdots \\ \widetilde{R}_s^\top \underline{f}_s \end{pmatrix}.$$

Similarly as in FETI methods for elliptic problems, we introduce a projection

$$P := I - G(\widetilde{G}^\top G)^{-1} \widetilde{G}^\top,$$

and it remains to solve the constrained linear system

$$PS\underline{\lambda} = P\underline{d}, \quad \widetilde{G}^\top \underline{\lambda} = \underline{e}\,, \tag{10}$$

which can be done via a GMRES method [6]. Afterwards we can compute

$$\underline{\alpha} = (\widetilde{G}^\top G)^{-1} \widetilde{G}^\top (S\underline{\lambda} - \underline{d})\,.$$

Notice that the square matrix $\widetilde{G}^\top G$ is small, since it does not depend on the finite element mesh but only on the number s of subdomains. In fact, its dimension is simply s for the heat equation, or sd for the Stokes problem. Therefore, the inverse $(\widetilde{G}^\top G)^{-1}$ can be computed directly and works as a coarse-grid solver.

It remains to characterize the kernels $\mathcal{N}(K_i)$ and $\mathcal{N}(K_i^\top)$ of the local stiffness matrices K_i and their transposed matrices, respectively. For this we consider the heat equation in $Q_i = \Omega_i \times (t_{i-1}, t_i)$, where K_i corresponds to the space-time discretization with zero Neumann boundary conditions and *without* initial or terminal conditions at t_{i-1} or t_i, respectively. In the continuous case, the solution in Q_i is given by

$$u_i(x,t) = \sum_{k=0}^{\infty} u_{i,k} e^{-\lambda_{i,k} t} v_{i,k}(x) \quad \text{for } (x,t) \in Q_i, \tag{11}$$

where $v_{i,k}$ are the eigenfunctions of the Neumann eigenvalue problem for the spatial Laplacian in Ω_i, with eigenvalues $\lambda_{i,k} \geq 0$. For the space-time finite element discretization we use continuous, piecewise linear basis functions as partition of unity in Q_i, i.e., $v_{i,0} \in X_{h|Q_i}$ for $\lambda_{i,0} = 0$. Due to the exponential decay in the solution (11) for $k \geq 1$, no more eigenfunctions are represented in the local finite element space $X_{h|Q_i}$, and hence we conclude $\mathcal{N}(K_i) = \big\{\underline{1}\big\}$ in the case of the heat equation

(1). Similarly, for the Stokes problem (3) we have d constant eigenfunctions for the velocity, and additionally null pressure [15]. In both cases, the constant eigenfunctions remain true for general space-time subdomains Q_i. While the kernel $\mathcal{N}(K_i)$ is trivially constructed, the basis for $\mathcal{N}(K_i^\top)$ is in general mesh-dependent. Such bases are however easily obtained as subproducts of numerical techniques for computing pseudo-inverses K_i^+, see Ref. [3].

To simplify the implementation and to include all subdomains in the coarse-grid matrix $\widetilde{G}^\top G$, we may consider all subdomains as floating, incorporating Dirichlet boundary conditions via Lagrange multipliers as well. This results in the all-floating [8] or total [2] FETI approach.

4 Numerical results

As a first numerical example we consider the Stokes system (3) in the spatial domain $\Omega = (0,1)^2$ for $T = 1$, i.e., $Q = (0,1)^3$. To check the expected order of convergence we consider for $\mu = 1$ the manufactured solution

$$\begin{aligned} u_1(x,t) &= 2(1-e^{-t})(x_2 - 3x_2^2 + 2x_2^3)[x_1(1-x_1)]^2, \\ u_2(x,t) &= 2(1-e^{-t})(3x_1^2 - x_1 - 2x_1^3)[x_2(1-x_2)]^2, \\ p(x,t) &= (1 + x_1 - e^{-x_1 x_2 t})t^2, \end{aligned}$$

with the right-hand side f computed accordingly. In this first example we consider decompositions of the space-time domain Q into only a few subdomains, see Fig. 1. Our particular interest is in the effect of the interface orientation on the number of required GMRES iterations to reach a given relative accuracy of $\varepsilon = 10^{-6}$, see also the discussion in Ref. [11] in the case of a standard domain decomposition approach for the heat equation. We solve the global Schur complement system without any preconditioning (I), or with a simple diagonal preconditioner (D). In all cases we observe a significant reduction in the number of iterations, with the best results appearing when considering a decomposition in time (a) or space (b) only, and for the diagonal decomposition (c). The results are not as good when considering the decomposition (d) and the inclusion (e). In general, some coarse-grid preconditioner should be used to further reduce the number of iterations.

In the second example we have the heat equation (1) in the spatially one-dimensional domain $\Omega = (0,1)$ and with the final time $T = 1$, i.e., $Q = (0,1)^2$. As solution we have chosen $u(x,t) = \sin\frac{1}{2}\pi t \, \sin \pi x$. Here we consider a decomposition of the space-time domain Q into up to 64 time slabs, applying both the space-time FETI approach and the all-floating (AF) formulation. The results are given in Table 2, where we observe a reasonable number of iterations in all cases. Note that the number of degrees of freedom is significantly larger when using the all-floating approach instead of the standard FETI method. Although the latter requires fewer iterations in most examples, this is not always the case (*cf.* Table 2). Based on previous experiences [2, 8], we expect that this behaviour can be further

Table 1: Space-time FETI domain decomposition method for the time-dependent Stokes system in $Q = (0, 1)^3$. Number of GMRES iterations for the Schur complement system without (I) and with diagonal (D) preconditioning, for different numbers N_e of elements.

					Domain decomposition									
					a)		b)		c)		d)		e)	
N_e	$\lVert\nabla_x(u-u_h)\rVert_{L^2(Q)}$		$\lVert p-p_h\rVert_{L^2(Q)}$		I	D	I	D	I	D	I	D	I	D
192	6.86e-3		2.63e-2		15	11	26	13	31	15	36	19		
1536	2.19e-3	1.64	6.53e-3	2.01	25	13	54	17	57	20	79	29	72	28
12288	5.82e-4	1.92	1.57e-3	2.05	36	17	94	22	105	27	165	44	181	50
98304	1.47e-4	1.98	3.81e-4	2.04	55	22	180	34	206	39	374	66	325	83

improved by using appropriate preconditioners for the all-floating scheme. Also note that this approach is strongly related to the parareal algorithm [7] where the coarse grid corresponds to the time slabs of the domain decomposition, see also the results in Ref. [11].

Table 2: Classical and all-floating (AF) space-time FETI methods for the heat equation. Number of GMRES iterations for a sequence of time slabs and meshes.

	$s=2$		$s=4$		$s=8$		$s=16$		$s=32$		$s=64$	
N_e	FETI	AF	FETI	AF	FETI	AF	FETI	AF	FETI	AF	FETI	AF
128	5	12	7	12	9	12						
512	7	12	8	14	12	18	17	17				
2048	8	13	10	15	14	21	23	29	34	27		
8192	9	15	11	18	16	24	26	36	40	53	69	49
32768	9	18	12	23	17	29	28	44	47	68	79	104

5 Conclusions

In this contribution, we have presented and described first results for space-time finite element tearing and interconnecting domain decomposition methods, including also the all-floating approach. Model problems include the heat equation and the Stokes system, but more complex partial differential equations can be considered as well. The space-time finite element discretization and the tearing and interconnecting approach follow the lines of the FETI method for elliptic problems, considering time as just an additional spatial coordinate. The main distinction here stems from the asymmetry of the space-time stiffness matrix, which requires a modified projection operator and also a numerical procedure to construct local kernels. First numerical results show the potential of the proposed method, in particular when using state-of-the-art parallel computing facilities for time-dependent problems. It is clear that a more detailed numerical analysis, in particular with respect to suitable preconditioning strategies for general space-time domain decompositions, is required. Related results will be investigated and published elsewhere.

Acknowledgements The authors acknowledge Graz University of Technology for the financial support of the Lead-project: Mechanics, Modeling and Simulation of Aortic Dissection.

References

1. Bramble, J. H., Pasciak, J. E., Schatz, A. H.: The construction of preconditioners for elliptic problems by substructuring. I. Math. Comp. **47**, 103–134 (1986).
2. Dostál, Z., Horák, D., Kučera, R.: Total FETI – an easier implementable variant of the FETI method for numerical solution of elliptic PDE. Comm. Numer. Methods Engrg. **22**, 1155–1162 (2006).
3. Farhat, C., Géradin, M.: On the general solution by a direct method of a large-scale singular system of linear equations: application to the analysis of floating structures. Int. J. Numer. Meth. Engrg. **41**, 675–696 (1998).
4. Farhat, C., Roux, F.-X.: A method of finite element tearing and interconnecting and its parallel solution algorithm. Int. J. Numer. Meth. Engrg. **32**, 1205–1227 (1991).
5. Haase, G., Langer, U., Meyer, A.: The approximate Dirichlet domain decomposition method. Part I: An algebraic approach. Computing **47**, 137–151 (1991).
6. Kučera, R., Kozubek, T., Markopoulus, A., Haslinger, J., Mocek, L.: Projected Krylov methods for solving non-symmetric two-by-two block linear systems arising from fictitious domain formulations. Adv. Electr. Electron. Eng. **12**, 131–143 (2014).
7. Lions, J.-L., Maday, Y., Turinici, G.: Résolution d'EDP par un schéma en temps "pararéel". C. R. Acad. Sci. Paris Sér. I Math. **332**, 661–668 (2001).
8. Of, G., Steinbach, O.: The all-floating boundary element tearing and interconnecting method. J. Numer. Math. **17**, 277–298 (2009).
9. Pacheco, D.R.Q., Steinbach, O.: Space-time Taylor–Hood elements for incompressible flows. Computer Meth. Material Sci. **19**, 64–69 (2019).
10. Steinbach, O.: Space-time finite element methods for parabolic problems. Comput. Meth. Appl. Math. **15**, 551–566 (2015).
11. Steinbach, O., Gaulhofer, P.: On space-time finite element domain decomposition methods for the heat equation. In: Brenner, S., Chung, E., Klawonn, A., Kwok, F., Xu, J., Zou, J. (eds.) Domain Decomposition Methods in Science and Engineering XXVI, Lecture Notes in Computational Science and Engineering, Springer, Cham, pp. 515–522, 2022.
12. Steinbach, O., Windisch, M.: Stable boundary element domain decomposition methods for the Helmholtz equation. Numer. Math. **118**, 171–195 (2011).
13. Steinbach, O., Windisch, M.: Stable BETI methods in electromagnetics. In: Bank, R., Holst, M., Widlund, O., Xu, J. (eds.) Domain Decomposition Methods in Science and Engineering XX, Lecture Notes in Computational Science and Engineering, vol. 91, pp. 223–230, Springer, Heidelberg (2013).
14. Steinbach, O., Yang, H.: Space-time finite element methods for parabolic evolution equations: Discretization, a posteriori error estimation, adaptivity and solution. In: Langer, U., Steinbach, O. (eds.) Space-Time Methods. Applications to Partial Differential Equations, Radon Series on Computational and Applied Mathematics, vol. 25, pp. 207–248, de Gruyter, Berlin (2019).
15. Vereecke, B., Bavestrello, H., Dureisseix, D.: An extension of the FETI domain decomposition method for incompressible and nearly incompressible problems. Comput. Methods Appl. Mech. Engrg. **192**, 3409–3429 (2003).