IETI-DP for Conforming Multi-Patch Isogeometric Analysis in Three Dimensions

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1 Introduction

We are interested in fast domain decomposition solvers for multi-patch Isogeometric Analysis (IgA; [4]). We focus on variants of FETI-DP solvers, see [2, 10] and references therein. Such methods have been adapted to IgA in [5], where the individual patches of the multi-patch discretization are used as subdomains for the solver. This method is sometimes referred to as the dual-primal isogeometric tearing and interconnecting (IETI-DP) method. These methods are similar to Balancing Domain Decomposition by Constraints (BDDC) methods, which have also been adapted for IgA, see [1, 11] and references therein. That the spectra of FETI-DP and BDDC are almost identical is established in [6].

Much progress for the IETI-DP methods has been made in the PhD-thesis by C. Hofer, including the extension to various discontinuous Galerkin formulations, see [3]. Recently, the authors of this paper have extended the condition number bounds for the preconditioned Schur complement system to be explicit not only in the grid size but also in the spline degree, see [8] for the conforming case and [9] for an extension to the discontinuous Galerkin case. The analysis follows the framework from [6]. One key ingredient for the analysis in [8] has been the construction of a bounded harmonic extension operator for splines, which follows the ideas of [7]. The analysis in [8] treats the two-dimensional case. As usual for FETI-like methods, the extension of the analysis to three dimensions is not effortless. The goal of this paper

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is to demonstrate that the proposed method also performs well for higher spline degrees in three dimensions.

The remainder of this paper is organized as follows. In Section 2, we introduce the model problem, discuss its discretization and the proposed IETI-DP algorithm. In Section 3, numerical experiments for a three-dimensional example are presented.

2 Model problem and its solution

We consider a standard Poisson model problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For given $f \in L_2(\Omega)$, we are interested in solving for $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\Omega$$

holds in a weak sense. We assume that the closure of the computational domain Ω is the union of the closure of K non-overlapping patches $\Omega^{(k)}$ that are parametrized with geometry functions

$$G_k : \widehat{\Omega} := (0,1)^d \to \Omega^{(k)} := G_k(\widehat{\Omega})$$

such that for any $k \neq \ell$, the intersection $\overline{\Omega^{(k)}} \cap \overline{\Omega^{(\ell)}}$ is empty, a common vertex, a common edge, or (in three dimensions) a common face (cf. [8, Ass. 2]). We assume that both, ∇G_k and $(\nabla G_k)^{-1}$, are in $L_\infty(\widehat{\Omega})$ for all patches. For the analysis, we need a uniform bound on the L_∞-norm and a uniform bound on the number of neighbors of each patch, cf. [8, Ass. 1 and 3].

For each of the patches, we introduce a tensor B-spline discretization on the parameter domain $\widehat{\Omega}$. The discretization is then mapped to the physical patch $\Omega^{(k)}$ using the pull-back principle. We use a standard basis as obtained by the Cox-de Boor formula. We need a fully matching discretization, which means that for each basis function that has a non-vanishing trace on one of the interfaces, there is exactly one basis function on each of the patches sharing this interface with the traces of the basis functions agreeing (cf. [8, Ass. 5]). This is a standard assumption for any multi-patch setting that is not treated using discontinuous Galerkin methods. For the analysis, we assume quasi-uniformity of grids within each patch, cf. [8, Ass. 4].

In the following, we explain how the IETI-DP solver is set up. Here, we loosely follow the notation used in the IETI-DP solution framework that was recently included in the public part of the G+Smo library. We choose the patches as IETI subdomains. We obtain patch-local stiffness matrices $A^{(k)}$ by evaluating the bilinear forms $a^{(k)}(u,v) = \int_{\Omega^{(k)}} \nabla^\top u(x) \nabla v(x) \mathrm{d}x$ using the basis functions of the corresponding patch. We set up matrices $C^{(k)}$ such that their null spaces are the coefficient vectors of the patch-local functions that vanish at the primal degrees of freedom. In [8], we have considered corner values, edge averages, and the combination of both. In the three dimensional case, we can choose corner values, edge averages, face averages, and any combination thereof. We set up fully redundant jump matrices $B^{(k)}$. We

omit the corner values if and only if the corners are chosen as primal degrees of freedom. We setup the primal problem in the usual way, i.e., we first, for $k = 1, \dots, K$, compute a basis by

$$\Psi^{(k)} := \begin{pmatrix} I & 0 \end{pmatrix} (\widetilde{A}^{(k)})^{-1} \begin{pmatrix} 0 \\ R_c^{(k)} \end{pmatrix}, \quad \text{where} \quad \widetilde{A}^{(k)} := \begin{pmatrix} A^{(k)} & (C^{(k)})^\top \\ C^{(k)} & \end{pmatrix}$$

and $R_c^{(k)}$ is a binary matrix that relates the primal constraints (with their patch-local indices) to the degrees of freedom of the primal problem (with their global indices) and set then

$$\widetilde{A}^{(K+1)} := \sum_{k=1}^{K} (\Psi^{(k)})^\top A^{(k)} \Psi^{(k)}, \quad \text{and} \quad \widetilde{B}^{(K+1)} := \sum_{k=1}^{K} B^{(k)} \Psi^{(k)}.$$

We consider the Schur complement problem $F\underline{\lambda} = \underline{g}$, where

$$F := \sum_{k=1}^{K+1} \widetilde{B}^{(k)} (\widetilde{A}^{(k)})^{-1} (\widetilde{B}^{(k)})^\top \quad \text{and} \quad \widetilde{B}^{(k)} := \begin{pmatrix} B^{(k)} & 0 \end{pmatrix} \text{ for } k = 1, \dots, K.$$

The derivation of $\underline{g}$ is a patch-local preprocessing step. We solve the Schur complement problem using a preconditioned conjugate gradient (PCG) solver with the scaled Dirichlet preconditioner

$$M_{\mathrm{sD}} := \sum_{k=1}^{K} B_\Gamma D_k^{-1} \left(A_{\Gamma\Gamma}^{(k)} - A_{\Gamma I}^{(k)} (A_{II}^{(k)})^{-1} A_{I\Gamma}^{(k)} \right) D_k^{-1} (B_\Gamma)^\top,$$

where the index Γ refers to the rows/columns of $A^{(k)}$ and the columns of $B^{(k)}$ that refer to basis functions with non-vanishing trace, the index I refers to the remaining rows/columns, and the matrix D_k is a diagonal matrix defined based on the principle of multiplicity scaling. For the analysis, it is important that its coefficients are constant within each interface. The solution u itself is obtained from $\underline{\lambda}$ using the usual patch-local steps, cf. [8].

Under the presented assumptions, the condition number of the preconditioned Schur complement system is in the two-dimensional case bounded by

$$C\, p \left(1 + \log p + \max_{k=1,\dots,K} \log \frac{H_k}{h_k} \right)^2, \tag{1}$$

where p is the spline degree, H_k is the patch size, and h_k the grid size, see [8]. The constant C is independent of these quantities, the number of patches, and the smoothness of the splines within the patches. The authors conjecture that such a condition number estimate also holds for the three-dimensional case, except when only the vertex values are chosen as primal degrees of freedom.

3 Numerical results

In the following, we present numerical results for a three-dimensional domain and refer to the original paper [8] for the two-dimensional case. The computational domain Ω is a twisted version of a Fichera corner, see Fig. 1. The original geometry consists of 7 patches. We subdivide each patch uniformly into $4 \times 4 \times 4$ patches to obtain a decomposition into 448 patches.

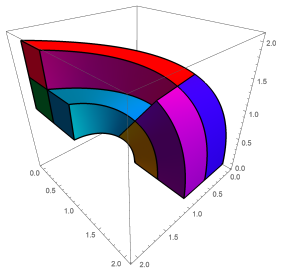

Fig. 1: Computational domain

We solve the model problem $-\Delta u(x, y, z) = 3\pi^2 \sin(\pi x) \sin(\pi y) \sin(\pi z)$ for $(x, y, z) \in \Omega$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$ by means of the IETI-DP solver outlined in the previous sections. Within the patches, we consider tensor-product B-spline discretizations of degree p and maximum smoothness C^{p-1}. We consider several grid sizes, the refinement level $r = 0$ corresponds to a discretization of each patch with polynomials. The next refinement levels $r = 1, 2, \ldots$ are obtained by uniform refinement. All experiments have been carried out in the C++ library G+Smo[1] and have been executed on the Radon1 cluster[2] in Linz. All computations have been performed with a single core.

Concerning the choice of the primal degrees of freedom, we consider all possibilities. For the two-dimensional case, the common choices are the corner values, the edge averages, and a combination of both. We have seen in [8] that all approaches work, typically the corner values are better than when using the edge averages. As expected, the combination of both yields the best results. For the three dimensional case, we have more possibilities. We report on these approaches in the Tables 1 (vertex values = V), 2 (edge averages = E) and 3 (face averages = F). The combinations V+E, V+F, E+F and V+E+F are only included in the diagrams. In any case, we report on the number of iterations (it) required by the PCG solver to reduce the residual with a random starting vector by a factor of 10^{-6} compared to the right-hand side. Moreover, we report on the condition numbers (κ) of the preconditioned system as estimated by the PCG solver.

[1] `https://github.com/gismo/gismo`, example file `examples/ieti_example.cpp`.

[2] `https://www.ricam.oeaw.ac.at/hpc/`

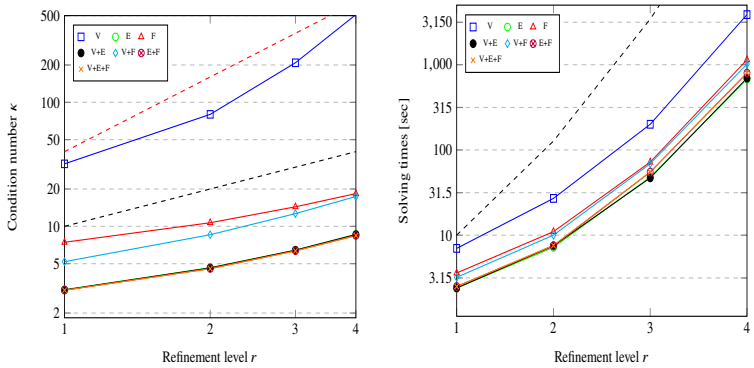

Fig. 2: Condition numbers and solving times for $p = 3$

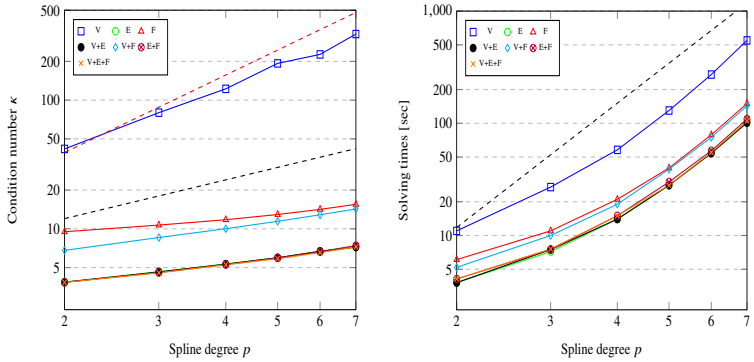

Fig. 3: Condition numbers and solving times for $r = 2$

	$p = 2$		$p = 3$		$p = 4$		$p = 5$		$p = 6$		$p = 7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	33	14	51	32	64	45	89	84	108	109	136	178
2	57	42	79	80	98	122	124	193	148	227	176	326
3	94	116	123	208	149	315	175	439	199	566	OoM	
4	146	275	176	509	OoM		OoM		OoM		OoM	

Table 1: Iterations (it) and condition number (κ); Vertex (V)

In Figure 2, the dependence on the refinement level is depicted. Here, we have chosen the spline degree $p = 3$ and have considered all of the possibilities for primal degrees of freedom. Here, we have 44 965 ($r = 1$), 133 629 ($r = 2$), 549 037 ($r = 3$), and 2 934 285 ($r = 4$) degrees of freedom (dofs). We observe that choosing only

	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	14	2.5	17	3.1	20	3.8	23	4.4	27	5.1	29	5.5
2	18	3.9	21	4.6	23	5.3	26	6.0	29	6.7	32	7.3
3	23	5.6	25	6.4	28	7.3	30	8.0	33	8.8	OoM	
4	27	7.5	30	8.6	OoM		OoM		OoM		OoM	

Table 2: Iterations (it) and condition number (κ); Edges (E)

	$p=2$		$p=3$		$p=4$		$p=5$		$p=6$		$p=7$	
r	it	κ	it	κ	it	κ	it	κ	it	κ	it	κ
1	22	6.1	26	7.4	29	8.3	33	9.5	37	10.4	41	11.5
2	29	9.5	31	10.7	34	11.8	37	12.9	42	14.2	46	15.5
3	35	13.1	38	14.4	41	15.9	43	17.0	47	18.3	OoM	
4	41	17.1	44	18.4	OoM		OoM		OoM		OoM	

Table 3: Iterations (it) and condition number (κ); Faces (F)

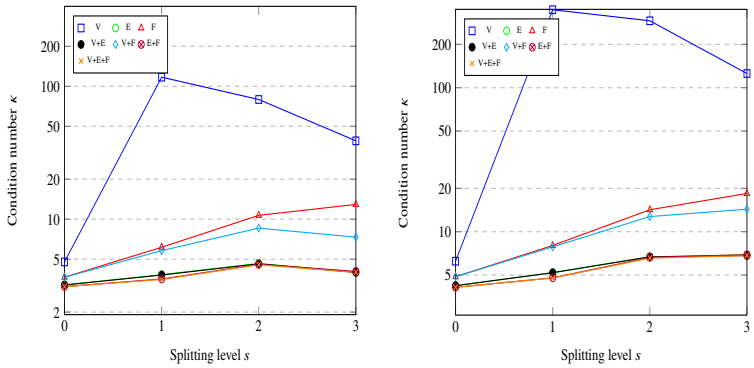

Fig. 4: Condition numbers for $p=3$ (left) and $p=6$ (right)

vertex values as primal degrees of freedom leads to the largest condition numbers. We observe that in this case the condition number grows like r^2 (the dashed red line indicates the slope of such a growth). This corresponds to a growth like $(1+\log H/h)^2$, as predicted by the theory for the two-dimensional case. All other options yield significantly better results, particularly those that include edge averages. In these cases, the growth seems to be less than linear in $r \approx \log H/h$ (the dashed black like shows such a slope). In the right diagram, we can see that choosing a strategy with smaller condition numbers also yields a faster method. Since the dimensions and the bandwidths of the local stiffness matrices grow like $(H_k/h_k)^3$ and $(H_k/h_k)^2$,

respectively, the complexity of the LU decompositions grows like $\sum_{k=1}^{K}(H_k/h_k)^7$. The complexity analysis indicates that they are the dominant factor. The dashed black line indicates such a growth.

In Figure 3, the dependence on the spline degree is presented, where we have chosen $r = 2$. Here, the number of dofs ranges from 66 989 ($p = 2$) to 549 037 ($p = 7$). Also in this picture, we see that the vertex values perform worst and the edge averages best. Again, we obtain a different asymptotic behavior for the corner values. For those primal degrees of freedom, the condition number grows like p^2 (the dashed red line indicates the corresponding slope). All the other primal degrees of freedom seem to lead to a growth that is smaller than linear in p (the dashed black line indicates the slope of a linear growth). Note that for the two-dimensional case, the theory predicts a growth like $p(1 + \log p)^2$. In the right diagram, we can see that the solving times grow like p^4 (the dashed line shows the corresponding slope). This seems to be realistic since the number of non-zero entries of the stiffness matrix grows like Np^d, where N is the number of unknowns. For $d = 3$, this yields in combination with the condition number bound the observed rates.

In Figure 4, we present results for the case that the number of patches is increased. We split each of the 7 patches, depicted in Fig. 1, uniformly into 8^s sub-patches. Within each patch, we consider a grid obtained by $r = 4 - s$ uniform refinement steps. The condition numbers grow slightly in s, where the curve seems to flatten for large values of s. If only the vertex values are primal degrees of freedom, the condition numbers seem to decline for $s > 1$.

Concluding, in this paper we have seen that the IETI method as described in [8] can indeed be extended to the three-dimensional case. If not only the vertex values are chosen as primal degrees of freedom, the condition number of the preconditioned system seems to obey the bound (1).

The extension of the proposed solver to more general elliptic differential equations, like problems with heterogeneous diffusion coefficients, is possible. Robustness in such coefficients, is a possible future research direction.

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