Numerical Results for an Unconditionally Stable Space-Time Finite Element Method for the Wave Equation

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1 Introduction

As a model problem, we consider the Dirichlet boundary value problem for the wave equation,

$$\left.\begin{aligned} \partial_{tt} u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma := \partial\Omega \times [0,T], \\ u(x,0) = \partial_t u(x,t)_{|t=0} &= 0 && \text{for } x \in \Omega, \end{aligned}\right\} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is some bounded Lipschitz domain, $T > 0$ is a finite time horizon, and f is some given source. For simplicity, we only consider homogeneous boundary and initial conditions, but inhomogeneous data or other types of boundary conditions can be handled as well. To compute an approximate solution of the wave equation (1), different numerical methods are available. Classical approaches are time-stepping schemes together with finite element methods in space, see [1] for an overview. An alternative is to discretize the time-dependent problem without separating the temporal and spatial variables. However, on the one hand, most space-time approaches are based on discontinuous Galerkin methods, see, e.g., [3, 6]. On the other hand, conforming tensor-product space-time discretizations with piecewise polynomial, continuous ansatz and test functions are of Petrov–Galerkin type, see,

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e.g., [7, 8, 12], where a stabilization is needed to avoid a CFL condition, i.e., a relation between the time mesh size and the spatial mesh size.

In this work, we use a modified Hilbert transformation to introduce a new space-time variational formulation of the wave equation (1), where ansatz and test spaces are equal. Conforming discretizations of this new variational setting, using polynomial, globally continuous ansatz and test functions, lead to space-time Galerkin–Bubnov finite element methods, which are unconditionally stable and provide optimal convergence rates in $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$, respectively. The rest of the paper is organized as follows: In Section 2, a modified Hilbert transformation and its main properties are given. Section 3 states the space-time variational setting for the wave equation and introduces the new space-time Galerkin–Bubnov finite element method. Numerical examples for a one- and a two-dimensional spatial domain are presented in Section 4. Finally, we draw some conclusions in Section 5.

2 A modified Hilbert transformation

In this section, we summarize the definition and some of the most important properties of the modified Hilbert transformation $\mathcal{H}_T$ as introduced in [8], see also [9, 11]. Since the modified Hilbert transformation covers the dependency in time only, in this section, we consider functions $u(t)$ for $t \in (0,T)$, where a generalization to functions in (x,t) is straightforward.

For $u \in L^2(0,T)$, we consider the Fourier series expansion

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad u_k := \frac{2}{T}\int_0^T u(t)\, \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)\, \mathrm{d}t,$$

and we define the modified Hilbert transformation $\mathcal{H}_T$ as

$$(\mathcal{H}_T u)(t) = \sum_{k=0}^{\infty} u_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad t \in (0,T). \tag{2}$$

By interpolation, we introduce $H^s_{0,}(0,T) := [H^1_{0,}(0,T), L^2(0,T)]_s$ for $s \in [0,1]$, where the space $H^1_{0,}(0,T)$ covers the initial condition $u(0) = 0$ for $u \in H^1(0,T)$. Analogously, we define $H^s_{,0}(0,T)$ for $s \in [0,1]$. With these notations, the mapping $\mathcal{H}_T\colon\ H^s_{0,}(0,T) \to H^s_{,0}(0,T)$ is an isomorphism for $s \in [0,1]$, where the inverse is the $L^2(0,T)$ adjoint, i.e., $\langle \mathcal{H}_T u, w\rangle_{L^2(0,T)} = \langle u, \mathcal{H}_T^{-1} w\rangle_{L^2(0,T)}$ for all $u, w \in L^2(0,T)$. In addition, the relations

$$\begin{aligned} \langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} &> 0 && \text{for } 0 \neq v \in H^s_{0,}(0,T), 0 < s \leq 1, \\ \langle \partial_t \mathcal{H}_T u, v\rangle_{L^2(0,T)} &= -\langle \mathcal{H}_T^{-1} \partial_t u, v\rangle_{L^2(0,T)} && \text{for } u \in H^1_{0,}(0,T), v \in L^2(0,T) \end{aligned}$$

hold true. For the proofs of these aforementioned properties, we refer to [8, 9, 11]. Furthermore, the modified Hilbert transformation (2) allows a closed representation [8, Lemma 2.8] as Cauchy principal value integral, i.e., for $u \in L^2(0,T)$,

$$(\mathcal{H}_T u)(t) = \text{v.p.} \int_0^T \frac{1}{2T} \left(\frac{1}{\sin \frac{\pi(s+t)}{2T}} + \frac{1}{\sin \frac{\pi(s-t)}{2T}} \right) u(s)\,\mathrm{d}s, \quad t \in (0,T).$$

This representation can be used for an efficient realization, also using low-rank approximations of related discrete matrix representations, see [9] for a more detailed discussion.

3 Space-time variational formulations

A possible space-time variational formulation for the Dirichlet boundary value problem (1) is to find $u \in H^{1,1}_{0;0,}(Q) := L^2(0,T; H^1_0(\Omega)) \cap H^1_{0,}(0,T; L^2(\Omega))$ such that

$$-\langle \partial_t u, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} = \langle f, v \rangle_{L^2(Q)} \tag{3}$$

is satisfied for all $v \in H^{1,1}_{0;,0}(Q) := L^2(0,T; H^1_0(\Omega)) \cap H^1_{,0}(0,T; L^2(\Omega))$. Note that the space $H^1_{0,}(0,T; L^2(\Omega))$ covers zero initial conditions, while the space $H^1_{,0}(0,T; L^2(\Omega))$ involves zero terminal conditions at $t = T$. For $f \in L^2(Q)$, there exists a unique solution u of (3), satisfying the stability estimate

$$\|u\|_{H^{1,1}_{0;0,}(Q)} := |u|_{H^1(Q)} := \sqrt{\|\partial_t u\|^2_{L^2(Q)} + \|\nabla_x u\|^2_{L^2(Q)}} \leq \frac{1}{\sqrt{2}} T \, \|f\|_{L^2(Q)},$$

see [4, 8, 12]. Note that the solution operator $\mathcal{L}\colon\ L^2(Q) \to H^{1,1}_{0;0,}(Q)$, $\mathcal{L}f := u$, is not an isomorphism, i.e., $\mathcal{L}$ is not surjective, see [10] for more details.

A direct numerical discretization of the variational formulation (3) would result in a Galerkin–Petrov scheme with different ansatz and test spaces, being zero at the initial and the terminal time, respectively. Hence, introducing some bijective operator $A\colon\ H^{1,1}_{0;0,}(Q) \to H^{1,1}_{0;,0}(Q)$, we can express the test function v in (3) as $v = Aw$ for $w \in H^{1,1}_{0;0,}(Q)$ to end up with a Galerkin–Bubnov scheme. While the time reversal map $\kappa_T w(x,t) := w(x, T-t)$ as used, e.g., in [2], is rather of theoretical interest, in the case of a tensor-product space-time finite element discretization, one may use the transformation $Aw_h(x,t) := w_h(x,T) - w_h(x,t)$, see [8]. However, the resulting numerical scheme is only stable when a CFL condition is satisfied, e.g., $h_t < h_x/\sqrt{d}$ when using piecewise linear basis functions and a tensor-product structure also in space. Although it is possible to derive an unconditionally stable scheme by using some stabilization approach, see [7, 12], our particular interest is in using an appropriate transformation A to conclude an unconditionally stable scheme without any further stabilization. A possible choice is the use of the modified Hilbert transformation $\mathcal{H}_T$ as introduced in Section 2. So, with the properties of $\mathcal{H}_T$, given in Section 2, we conclude that

$$-\langle \partial_t u, \partial_t \mathcal{H}_T w \rangle_{L^2(Q)} = \langle \partial_t u, \mathcal{H}_T^{-1} \partial_t w \rangle_{L^2(Q)} = \langle \mathcal{H}_T \partial_t u, \partial_t w \rangle_{L^2(Q)}$$

for all $u, w \in H^{1,1}_{0;0,}(Q)$, which leads to the variational formulation to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$\langle \mathcal{H}_T \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \mathcal{H}_T w \rangle_{L^2(Q)} = \langle f, \mathcal{H}_T w \rangle_{L^2(Q)} \tag{4}$$

is satisfied for all $w \in H^{1,1}_{0;0,}(Q)$. Since the mapping $\mathcal{H}_T\colon\ H^{1,1}_{0;0,}(Q) \to H^{1,1}_{0;,0}(Q)$ is an isomorphism, unique solvability of the new variational formulation (4) follows from the unique solvability of the variational formulation (3).

Let $V_h = \operatorname{span}\{\phi_i\}_{i=1}^M \subset H^{1,1}_{0;0,}(Q)$ be some conforming space-time finite element space. The Galerkin–Bubnov formulation of the variational formulation (4) is to find $u_h \in V_h$ such that

$$\langle \mathcal{H}_T \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x \mathcal{H}_T w_h \rangle_{L^2(Q)} = \langle f, \mathcal{H}_T w_h \rangle_{L^2(Q)} \tag{5}$$

is satisfied for all $w_h \in V_h$. Note that for any conforming space-time finite element space $V_h \subset H^{1,1}_{0;0,}(Q)$, the related bilinear form in (5) is positive definite, since both summands are discretizations of second-order differential operators, which lead, together with the properties of $\mathcal{H}_T$, to two positive definite bilinear forms. Further details on the numerical analysis of this new Galerkin–Bubnov variational formulation (5) are far beyond the scope of this contribution, we refer to [5]. The discrete variational formulation (5) corresponds to the linear system $K_h \underline{u} = \underline{f}$ with the stiffness matrix $K_h = A_h + B_h$, and

$$\begin{aligned}
A_h[i,j] &= \int_0^T \int_\Omega \mathcal{H}_T \partial_t \phi_j(x,t)\, \partial_t \phi_i(x,t)\, \mathrm{d}x\, \mathrm{d}t, \\
B_h[i,j] &= \int_0^T \int_\Omega \nabla_x \phi_j(x,t) \cdot \nabla_x \mathcal{H}_T \phi_i(x,t)\, \mathrm{d}x\, \mathrm{d}t
\end{aligned}$$

for $i, j = 1, \ldots, M$. Since the realization of the modified Hilbert transformation $\mathcal{H}_T$ is much easier for solely time-dependent functions, see [9, 11], here we choose as a special case a tensor-product ansatz. For this purpose, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0, L)$ for $d = 1$, polygonal for $d = 2$, or polyhedral for $d = 3$. We consider admissible decompositions

$$\overline{Q} = \overline{\Omega} \times [0,T] = \bigcup_{i=1}^{N_x} \overline{\omega_i} \times \bigcup_{\ell=1}^{N_t} [t_{\ell-1}, t_\ell]$$

with $N := N_x \cdot N_t$ space-time elements, where the time intervals $(t_{\ell-1}, t_\ell)$ with mesh sizes $h_{t,\ell} = t_\ell - t_{\ell-1}$ are defined via the decomposition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N_t - 1} < t_{N_t} = T$$

of the time interval $(0,T)$. The maximal and the minimal time mesh sizes are denoted by $h_t := h_{t,\max} := \max_\ell h_{t,\ell}$, and $h_{t,\min} := \min_\ell h_{t,\ell}$, respectively. For the spatial domain Ω, we consider a shape-regular sequence $(\mathcal{T}_\eta)_{\eta\in\mathbb{N}}$ of admissible decompositions $\mathcal{T}_\eta := \{\omega_i \subset \mathbb{R}^d : i = 1, \ldots, N_x\}$ of Ω into finite elements $\omega_i \subset \mathbb{R}^d$ with mesh sizes $h_{x,i}$ and the maximal mesh size $h_x := \max_i h_{x,i}$. The spatial elements ω_i are intervals for $d = 1$, triangles for $d = 2$, and tetrahedra for $d = 3$. Next, we introduce the finite element space $Q^1_{h,0}(Q) := S^1_{h_x,0}(\Omega) \otimes S^1_{h_t,0,}(0,T)$ of piecewise multilinear, continuous functions, i.e.,

$$S^1_{h_x,0}(\Omega) := S^1_{h_x}(\Omega) \cap H^1_0(\Omega) = \operatorname{span}\{\psi^1_j\}_{j=1}^{M_x},$$
$$S^1_{h_t,0,}(0,T) := S^1_{h_t}(0,T) \cap H^1_{0,}(0,T) = \operatorname{span}\{\varphi^1_\ell\}_{\ell=1}^{N_t},$$

where ψ^1_j, $j = 1, \ldots, M_x$, are the spatial nodal basis functions, and φ^1_ℓ, $\ell = 1, \ldots, N_t$, are the temporal nodal basis functions. In fact, $S^1_{h_t}(0,T)$ is the space of piecewise linear, continuous functions on intervals, and $S^1_{h_x}(\Omega)$ is the space of piecewise linear, continuous functions on intervals ($d = 1$), triangles ($d = 2$), and tetrahedra ($d = 3$).

Choosing $V_h = Q^1_{h,0}(Q)$ in (5) leads to the space-time Galerkin–Bubnov variational formulation to find $u_h \in Q^1_{h,0}(Q)$ such that

$$\langle \mathcal{H}_T \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x \mathcal{H}_T w_h \rangle_{L^2(Q)} = \langle Q^0_h f, \mathcal{H}_T w_h \rangle_{L^2(Q)} \tag{6}$$

for all $w_h \in Q^1_{h,0}(Q)$. Here, for an easier implementation, we approximate the right-hand side $f \in L^2(Q)$ by

$$f \approx Q^0_h f \in S^0_{h_x}(\Omega) \otimes S^0_{h_t}(0,T), \tag{7}$$

where $Q^0_h \colon L^2(Q) \to S^0_{h_x}(\Omega) \otimes S^0_{h_t}(0,T)$ is the $L^2(Q)$ projection on the space $S^0_{h_x}(\Omega) \otimes S^0_{h_t}(0,T)$ of piecewise constant functions. The discrete variational formulation (6) is equivalent to the global linear system

$$K_h \underline{u} = \underline{\widetilde{f}} \tag{8}$$

with the system matrix

$$K_h = A^{\mathcal{H}_T}_{h_t} \otimes M_{h_x} + M^{\mathcal{H}_T}_{h_t} \otimes A_{h_x} \in \mathbb{R}^{N_t \cdot M_x \times N_t \cdot M_x},$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote spatial mass and stiffness matrices given by

$$M_{h_x}[i,j] = \langle \psi^1_j, \psi^1_i \rangle_{L^2(\Omega)}, \quad A_{h_x}[i,j] = \langle \nabla_x \psi^1_j, \nabla_x \psi^1_i \rangle_{L^2(\Omega)}, \quad i,j = 1, \ldots, M_x,$$

and $M^{\mathcal{H}_T}_{h_t} \in \mathbb{R}^{N_t \times N_t}$ and $A^{\mathcal{H}_T}_{h_t} \in \mathbb{R}^{N_t \times N_t}$ are defined by

$$M^{\mathcal{H}_T}_{h_t}[\ell,k] := \langle \varphi^1_k, \mathcal{H}_T \varphi^1_\ell \rangle_{L^2(0,T)}, \quad A^{\mathcal{H}_T}_{h_t}[\ell,k] := \langle \mathcal{H}_T \partial_t \varphi^1_k, \partial_t \varphi^1_\ell \rangle_{L^2(0,T)}$$

for $\ell, k = 1, \ldots, N_t$. The matrices $M_{h_t}^{\mathcal{H}_T}$, $A_{h_t}^{\mathcal{H}_T}$ are nonsymmetric, but positive definite, which follows from the properties of $\mathcal{H}_T$, given in Section 2. Additionally, the matrices M_{h_x}, A_{h_x} are positive definite. Thus, standard properties of the Kronecker product yield that the system matrix K_h is also positive definite. Hence, the global linear system (8) is uniquely solvable.

4 Numerical results

In this section, numerical examples for the Galerkin–Bubnov finite element method (6) for a one- and a two-dimensional spatial domain are given. For both cases, the number of degrees of freedom is given by dof $= N_t \cdot M_x$. The assembling of the matrices $A_{h_t}^{\mathcal{H}_T}$, $M_{h_t}^{\mathcal{H}_T}$ is done as proposed in [11, Subsection 2.2]. Further, to accelerate the computations, data-sparse approximations as known from boundary element methods, e.g., hierarchical matrices, can be used, see [9]. The integrals for computing the projection $Q_h^0 f$ in (7) are calculated by using high-order quadrature rules. The global linear system (8) is solved by a direct solver.

For the first numerical example, we consider the one-dimensional spatial domain $\Omega := (0, 1)$ with the terminal time $T = 10$, i.e., the rectangular space-time domain

$$Q := \Omega \times (0, T) := (0, 1) \times (0, 10). \tag{9}$$

As an exact solution, we choose

$$u_1(x, t) = t^2 \sin(10\pi x) \sin(t\, x), \quad (x, t) \in Q. \tag{10}$$

The spatial domain $\Omega = (0, 1)$ is decomposed into nonuniform elements with the vertices

$$x_0 = 0, \quad x_1 = 1/4, \quad x_2 = 1, \tag{11}$$

whereas the temporal domain $(0, T) = (0, 10)$ is decomposed into nonuniform elements with the vertices

$$t_0 = 0, \quad t_1 = 5/4, \quad t_2 = 5/2, \quad t_3 = 10 = T, \tag{12}$$

see Fig. 1 for the resulting space-time mesh. We apply a uniform refinement strategy for the meshes (11), (12). The numerical results for the smooth solution u_1 in (10) are given in Table 1, where we observe unconditional stability, quadratic convergence in $\|\cdot\|_{L^2(Q)}$, and linear convergence in $|\cdot|_{H^1(Q)}$.

For the second numerical example, the two-dimensional spatial Γ-shaped domain

$$\Omega := (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]) \subset \mathbb{R}^2 \tag{13}$$

and the terminal time $T = 2$ are considered for the solution

$$u_2(x_1, x_2, t) = \sin(\pi x_1) \sin(\pi x_2) (\sin(t x_1 x_2))^2, \quad (x_1, x_2, t) \in Q = \Omega \times (0, T). \tag{14}$$

Table 1: Numerical results of the Galerkin–Bubnov finite element discretization (6) for the space-time cylinder (9) for the function u_1 in (10) for a uniform refinement strategy.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_1 - u_{1,h}\|_{L^2(Q)}$	eoc	$\|u_1 - u_{1,h}\|_{H^1(Q)}$	eoc
3	0.7500	0.2500	7.5000	1.2500	5.0e+02	-	3.2e+03	-
18	0.3750	0.1250	3.7500	0.6250	4.2e+02	0.3	2.7e+03	0.2
84	0.1875	0.0625	1.8750	0.3125	3.2e+02	0.4	2.5e+03	0.1
360	0.0938	0.0312	0.9375	0.1562	8.4e+01	1.9	2.1e+03	0.2
1488	0.0469	0.0156	0.4688	0.0781	2.6e+01	1.7	1.0e+03	1.0
6048	0.0234	0.0078	0.2344	0.0391	7.2e+00	1.9	5.0e+02	1.1
24384	0.0117	0.0039	0.1172	0.0195	1.8e+00	2.0	2.5e+02	1.0
97920	0.0059	0.0020	0.0586	0.0098	4.7e-01	2.0	1.2e+02	1.0
392448	0.0029	0.0010	0.0293	0.0049	1.2e-01	2.0	6.2e+01	1.0
1571328	0.0015	0.0005	0.0146	0.0024	2.9e-02	2.0	3.1e+01	1.0

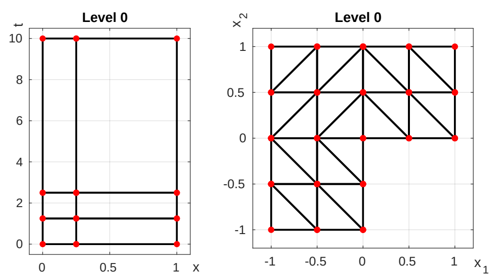

Fig. 1 Starting meshes for the one-dimensional spatial domain (left) and the two-dimensional spatial domain (right).

Table 2: Numerical results of the Galerkin–Bubnov finite element discretization (6) for the Γ-shape (13) and $T = 2$ for the function u_2 in (14) for a uniform refinement strategy.

dof	h_x	$h_{t,\max}$	$h_{t,\min}$	$\|u_2 - u_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}\|_{H^1(Q)}$	eoc
20	0.3536	1.5000	0.1250	1.756e-01	-	1.331e+00	-
264	0.1768	0.7500	0.0625	6.370e-02	1.5	6.882e-01	1.0
2576	0.0884	0.3750	0.0312	1.903e-02	1.7	3.439e-01	1.0
22560	0.0442	0.1875	0.0156	5.206e-03	1.9	1.730e-01	1.0
188480	0.0221	0.0938	0.0078	1.306e-03	2.0	8.555e-02	1.0
1540224	0.0110	0.0469	0.0039	3.284e-04	2.0	4.268e-02	1.0

The spatial domain Ω is decomposed into uniform triangles with uniform mesh size h_x as given in Fig. 1 for the first level. The temporal domain $(0,2) = (0,T)$ is decomposed into nonuniform elements with the vertices

$$t_0 = 0, \quad t_1 = 1/8, \quad t_2 = 1/4, \quad t_3 = 1/2, \quad t_4 = 2 = T. \tag{15}$$

When a uniform refinement strategy is applied for the temporal mesh (15) and for the spatial mesh, the numerical results for the smooth solution u_2 are given in Table 2, where unconditional stability is observed and the convergence rates in $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ are optimal.

5 Conclusions

In this work, we introduced new conforming space-time Galerkin–Bubnov methods for the wave equation. These methods are based on a space-time variational formulation, where ansatz and test spaces are equal, using also integration by parts with respect to the time variable and the modified Hilbert transformation $\mathcal{H}_T$. As discretizations of this variational setting, we considered a conforming tensor-product approach with piecewise multilinear, continuous basis functions. However, a generalization to piecewise polynomials of higher-order degree is straightforward. We gave numerical examples, where the unconditional stability, i.e., no CFL condition is required, and optimal convergence rates in space-time norms were illustrated. For a more detailed stability and error analysis, we refer to our ongoing work [5]. Other topics include the realization for arbitrary space-time meshes, a posteriori error estimates and adaptivity, and the parallel solution including domain decomposition methods.

References

1. Bangerth, W., Geiger, M., Rannacher, R.: Adaptive Galerkin finite element methods for the wave equation. Comput. Methods Appl. Math. **10**(1), 3–48 (2010)
2. Costabel, M.: Boundary integral operators for the heat equation. Integral Equations Operator Theory **13**(4), 498–552 (1990)
3. Dörfler, W., Findeisen, S., Wieners, C.: Space-time discontinuous Galerkin discretizations for linear first-order hyperbolic evolution systems. Comput. Methods Appl. Math. **16**(3), 409–428 (2016)
4. Ladyzhenskaya, O.A.: The boundary value problems of mathematical physics, *Applied Mathematical Sciences*, vol. 49. Springer-Verlag, New York (1985)
5. Löscher, R., Steinbach, O., Zank, M.: An unconditionally stable space-time finite element method for the wave equation (2022). In preparation.
6. Moiola, A., Perugia, I.: A space-time Trefftz discontinuous Galerkin method for the acoustic wave equation in first-order formulation. Numer. Math. **138**(2), 389–435 (2018)
7. Steinbach, O., Zank, M.: A stabilized space-time finite element method for the wave equation. In: Advanced Finite Element Methods with Applications. Selected papers from the 30th Chemnitz FEM Symposium 2017, *Lect. Notes Comput. Sci. Eng.*, vol. 128, pp. 341–370. Springer, Cham (2019)
8. Steinbach, O., Zank, M.: Coercive space-time finite element methods for initial boundary value problems. Electron. Trans. Numer. Anal. **52**, 154–194 (2020)
9. Steinbach, O., Zank, M.: A note on the efficient evaluation of a modified Hilbert transformation. J. Numer. Math. **29**(1), 47–61 (2021)
10. Steinbach, O., Zank, M.: A generalized inf-sup stable variational formulation for the wave equation. J. Math. Anal. Appl. **505**(1), Paper No. 125457, 24 (2022)
11. Zank, M.: An exact realization of a modified Hilbert transformation for space-time methods for parabolic evolution equations. Comput. Methods Appl. Math. **21**(2), 479–496 (2021)
12. Zlotnik, A.A.: Convergence rate estimates of finite-element methods for second-order hyperbolic equations. In: Numerical methods and applications, pp. 155–220. CRC, Boca Raton, FL (1994)