Numerical Results for an Unconditionally Stable Space-Time Finite Element Method for the Wave Equation

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1 Introduction

As a model problem, we consider the Dirichlet boundary value problem for the wave equation,

\[
\begin{aligned}
\partial_{tt} u(x, t) - \Delta_x u(x, t) &= f(x, t) \quad \text{for } (x, t) \in Q := \Omega \times (0, T), \\
u(x, 0) &= 0 \quad \text{for } (x, t) \in \Sigma := \partial \Omega \times [0, T], \\
u(x, 0) = \partial_t u(x, t)|_{t=0} &= 0 \quad \text{for } x \in \Omega,
\end{aligned}
\]

(1)

where \( \Omega \subset \mathbb{R}^d, d = 1, 2, 3 \), is some bounded Lipschitz domain, \( T > 0 \) is a finite time horizon, and \( f \) is some given source. For simplicity, we only consider homogeneous boundary and initial conditions, but inhomogeneous data or other types of boundary conditions can be handled as well. To compute an approximate solution of the wave equation (1), different numerical methods are available. Classical approaches are time-stepping schemes together with finite element methods in space, see [1] for an overview. An alternative is to discretize the time-dependent problem without separating the temporal and spatial variables. However, on the one hand, most space-time approaches are based on discontinuous Galerkin methods, see, e.g., [3, 6]. On the other hand, conforming tensor-product space-time discretizations with piecewise polynomial, continuous ansatz and test functions are of Petrov–Galerkin type, see,

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By interpolation, we introduce \( \mathcal{H}_T \), where the space \( \mathcal{H}_T \) covers the dependency in time only, in Section 2, a modified Hilbert transformation and its main properties are introduced. In this section, we consider functions \( u(t) \) for \( t \in (0, T) \), where a generalization to functions in \((x, t)\) is straightforward.

For \( u \in L^2(0, T) \), we consider the Fourier series expansion

\[
    u(t) = \sum_{k=0}^{\infty} u_k \sin \left( \left( \frac{\pi}{2} + k\pi \right) \frac{t}{T} \right), \quad u_k := \frac{2}{T} \int_{0}^{T} u(t) \sin \left( \left( \frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \, dt,
\]

and we define the modified Hilbert transformation \( \mathcal{H}_T \) as

\[
    (\mathcal{H}_T u)(t) = \sum_{k=0}^{\infty} u_k \cos \left( \left( \frac{\pi}{2} + k\pi \right) \frac{t}{T} \right), \quad t \in (0, T). \tag{2}
\]

By interpolation, we introduce \( H^s_0(0, T) := [H^1_0(0, T), L^2(0, T)]_s \) for \( s \in [0, 1] \), where the space \( H^1_0(0, T) \) covers the initial condition \( u(0) = 0 \) for \( u \in H^1(0, T) \). Analogously, we define \( H^s_0(0, T) \) for \( s \in [0, 1] \). With these notations, the mapping \( \mathcal{H}_T : H^1_0(0, T) \to H^s_0(0, T) \) is an isomorphism for \( s \in [0, 1] \), where the inverse is the \( L^2(0, T) \) adjoint, i.e., \( \langle \mathcal{H}_T u, w \rangle_{L^2(0, T)} = \langle u, \mathcal{H}^{-1}_T w \rangle_{L^2(0, T)} \) for all \( u, w \in L^2(0, T) \). In addition, the relations

\[
    \langle v, \mathcal{H}_T v \rangle_{L^2(0, T)} > 0 \quad \text{for } 0 \neq v \in H^s_0(0, T), 0 < s \leq 1,
\]

\[
    \langle \partial_t \mathcal{H}_T u, v \rangle_{L^2(0, T)} = -\langle \mathcal{H}^{-1}_T \partial_t u, v \rangle_{L^2(0, T)} \quad \text{for } u \in H^1_0(0, T), v \in L^2(0, T)
\]

Finally, we draw some conclusions in Section 5.
hold true. For the proofs of these aforementioned properties, we refer to [8, 9, 11]. Furthermore, the modified Hilbert transformation (2) allows a closed representation [8, Lemma 2.8] as Cauchy principal value integral, i.e., for $u \in L^2(0,T)$,

$$(\mathcal{H}_T u)(t) = \text{v.p.} \int_0^T \frac{1}{2T} \left( \frac{1}{\sin \frac{\pi(s+t)}{2T}} + \frac{1}{\sin \frac{\pi(s-t)}{2T}} \right) u(s) \, ds, \quad t \in (0,T).$$

This representation can be used for an efficient realization, also using low-rank approximations of related discrete matrix representations, see [9] for a more detailed discussion.

### 3 Space-time variational formulations

A possible space-time variational formulation for the Dirichlet boundary value problem (1) is to find $u \in H^{1,1}_{0,0}(Q) := L^2(0,T; H^1_0(\Omega)) \cap H^1_0(0,T; L^2(\Omega))$ such that

$$-(\partial_t u, \partial_t v)_{L^2(Q)} + (\nabla_x u, \nabla_x v)_{L^2(Q)} = \langle f, v \rangle_{L^2(Q)} \quad (3)$$

is satisfied for all $v \in H^{1,1}_{0,0}(Q) := L^2(0,T; H^1_0(\Omega)) \cap H^1_0(0,T; L^2(\Omega))$. Note that the space $H^1_0(0,T; L^2(\Omega))$ covers zero initial conditions, while the space $H^1_0(0,T; L^2(\Omega))$ involves zero terminal conditions at $t = T$. For $f \in L^2(Q)$, there exists a unique solution $u$ of (3), satisfying the stability estimate

$$\|u\|_{H^{1,1}_{0,0}(Q)} := \|u\|_{H^1_0(Q)} := \sqrt{\|\partial_t u\|^2_{L^2(Q)} + \|
abla_x u\|^2_{L^2(Q)}} \leq \frac{1}{\sqrt{2}} T \|f\|_{L^2(Q)},$$

see [4, 8, 12]. Note that the solution operator $L : L^2(Q) \rightarrow H^{1,1}_{0,0}(Q)$, $Lf := u$, is not an isomorphism, i.e., $L$ is not surjective, see [10] for more details.

A direct numerical discretization of the variational formulation (3) would result in a Galerkin–Petrov scheme with different ansatz and test spaces, being zero at the initial and the terminal time, respectively. Hence, introducing some bijective operator $A : H^{1,1}_{0,0}(Q) \rightarrow H^{1,1}_{0,0}(Q)$, we can express the test function $v$ in (3) as $v = Aw$ for $w \in H^{1,1}_{0,0}(Q)$ to end up with a Galerkin–Bubnov scheme. While the time reversal map $\kappa_T w(x,t) := w(x,T-t)$ as used, e.g., in [2], is rather of theoretical interest, in the case of a tensor-product space-time finite element discretization, one may use the transformation $Aw_h(x,t) := w_h(x,T) - w_h(x,t)$, see [8]. However, the resulting numerical scheme is only stable when a CFL condition is satisfied, e.g., $h_t < h_x/\sqrt{d}$ when using piecewise linear basis functions and a tensor-product structure also in space. Although it is possible to derive an unconditionally stable scheme by using some stabilization approach, see [7, 12], our particular interest is in using an appropriate transformation $A$ to conclude an unconditionally stable scheme without any further stabilization. A possible choice is the use of the modified Hilbert transformation $\mathcal{H}_T$ as introduced in Section 2. So, with the properties of $\mathcal{H}_T$, given
in Section 2, we conclude that

\[
- \langle \partial_t u, \partial_t \mathcal{H}_T w \rangle_{L^2(Q)} = \langle \partial_t u, \mathcal{H}_T^{-1} \partial_t w \rangle_{L^2(Q)} = \langle \mathcal{H}_T \partial_t u, \partial_t w \rangle_{L^2(Q)}
\]

for all \( u, w \in H^{1,1}_{0,0}(Q) \), which leads to the variational formulation to find \( u \in H^{1,1}_{0,0}(Q) \) such that

\[
\langle \mathcal{H}_T \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \mathcal{H}_T w \rangle_{L^2(Q)} = \langle f, \mathcal{H}_T w \rangle_{L^2(Q)}
\]

(4)
is satisfied for all \( w \in H^{1,1}_{0,0}(Q) \). Since the mapping \( \mathcal{H}_T : H^{1,1}_{0,0}(Q) \to H^{1,1}_{0,0}(Q) \) is an isomorphism, unique solvability of the new variational formulation (4) follows from the unique solvability of the variational formulation (3).

Let \( V_h = \text{span}\{\phi_i\}_{i=1}^M \subset H^{1,1}_{0,0}(Q) \) be some conforming space-time finite element space. The Galerkin–Bubnov formulation of the variational formulation (4) is to find \( u_h \in V_h \) such that

\[
\langle \mathcal{H}_T \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x \mathcal{H}_T w_h \rangle_{L^2(Q)} = \langle f, \mathcal{H}_T w_h \rangle_{L^2(Q)}
\]

(5)
is satisfied for all \( w_h \in V_h \). Note that for any conforming space-time finite element space \( V_h \subset H^{1,1}_{0,0}(Q) \), the related bilinear form in (5) is positive definite, since both summands are discretizations of second-order differential operators, which lead, together with the properties of \( \mathcal{H}_T \), to two positive definite bilinear forms. Further details on the numerical analysis of this new Galerkin–Bubnov variational formulation (5) are far beyond the scope of this contribution, we refer to [5]. The discrete variational formulation (5) corresponds to the linear system \( K_h u = f \) with the stiffness matrix \( K_h = A_h + B_h \), and

\[
A_h[i, j] = \int_0^T \int_{\Omega} \mathcal{H}_T \partial_t \phi_j(x, t) \partial_t \phi_i(x, t) \, dx \, dt,
\]

\[
B_h[i, j] = \int_0^T \int_{\Omega} \nabla_x \phi_j(x, t) \cdot \nabla_x \mathcal{H}_T \phi_i(x, t) \, dx \, dt
\]

for \( i, j = 1, \ldots, M \). Since the realization of the modified Hilbert transformation \( \mathcal{H}_T \) is much easier for solely time-dependent functions, see [9, 11], here we choose as a special case a tensor-product ansatz. For this purpose, let the bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d \) be an interval \( \Omega = (0, L) \) for \( d = 1 \), polygonal for \( d = 2 \), or polyhedral for \( d = 3 \). We consider admissible decompositions

\[
\overline{Q} = \overline{\Omega} \times [0, T] = \bigcup_{i=1}^{N_s} \overline{\Omega} \times \bigcup_{\ell=1}^{N_t} [t_{\ell-1}, t_\ell]
\]

with \( N := N_s \times N_t \) space-time elements, where the time intervals \( (t_{\ell-1}, t_\ell) \) with mesh sizes \( h_{s, \ell} = t_\ell - t_{\ell-1} \) are defined via the decomposition

\[
0 = t_0 < t_1 < t_2 < \cdots < t_{N_s-1} < t_{N_s} = T
\]
of the time interval \((0, T)\). The maximal and the minimal time mesh sizes are denoted by \(h_t := h_{t, \text{max}} := \max_{\ell} h_{t, \ell}\), and \(h_{t, \text{min}} := \min_{\ell} h_{t, \ell}\), respectively. For the spatial domain \(\Omega\), we consider a shape-regular sequence \((T_\eta)_{\eta \in I}\) of admissible decompositions \(T_\eta := \{\omega_\ell \subset \mathbb{R}^d : \ell = 1, \ldots, N_\ell\}\) of \(\Omega\) into finite elements \(\omega_\ell \subset \mathbb{R}^d\) with mesh sizes \(h_\ell\) and the maximal mesh size \(h_x := \max_\ell h_{x, \ell}\). The spatial elements \(\omega_\ell\) are intervals for \(d = 1\), triangles for \(d = 2\), and tetrahedra for \(d = 3\).

Next, we introduce the finite element space \(Q_{h,0}^1(\Omega) := S_{h,0}^1(\Omega) \otimes S_{h,0}^1(0, T)\) of piecewise multilinear, continuous functions, i.e.,

\[
\begin{align*}
S_{h,0}^1(\Omega) := & \text{span}\{\psi_j^1\}_{j=1}^{M_x}, \\
S_{h,0}^1(0, T) := & \text{span}\{\varphi_\ell^1\}_{\ell=1}^{N_t},
\end{align*}
\]

where \(\psi_j^1, j = 1, \ldots, M_x\), are the spatial nodal basis functions, and \(\varphi_\ell^1, \ell = 1, \ldots, N_t\), are the temporal nodal basis functions. In fact, \(S_{h,0}^1(0, T)\) is the space of piecewise linear, continuous functions on intervals, and \(S_{h,0}^1(\Omega)\) is the space of piecewise linear, continuous functions on intervals \((d = 1)\), triangles \((d = 2)\), and tetrahedra \((d = 3)\).

Choosing \(V_h = Q_{h,0}^1(\Omega)\) in (5) leads to the space-time Galerkin–Bubnov variational formulation to find \(u_h \in Q_{h,0}^1(\Omega)\) such that

\[
\langle \mathcal{H}_T \partial_t u_h, \partial_t w_h \rangle_{L^2(\Omega)} + \langle \nabla_x u_h, \nabla_x \mathcal{H}_T w_h \rangle_{L^2(\Omega)} = \langle Q_h^0 f, \mathcal{H}_T w_h \rangle_{L^2(\Omega)}
\]

for all \(w_h \in Q_{h,0}^1(\Omega)\). Here, for an easier implementation, we approximate the right-hand side \(f \in L^2(\Omega)\) by

\[
f \approx Q_h^0 f \in S_{h,0}^0(\Omega) \otimes S_{h,0}^0(0, T),
\]

where \(Q_h^0 : L^2(\Omega) \rightarrow S_{h,0}^0(\Omega) \otimes S_{h,0}^0(0, T)\) is the \(L^2(\Omega)\) projection on the space \(S_{h,0}^0(\Omega) \otimes S_{h,0}^0(0, T)\) of piecewise constant functions. The discrete variational formulation (6) is equivalent to the global linear system

\[
K_h u = f
\]

with the system matrix

\[
K_h = A_{h_{x}}^\mathcal{H}_T \otimes M_{h_{x}} + M_{h_{x}}^{\mathcal{H}_T} \otimes A_{h_{x}},
\]

where \(M_{h_{x}} \in \mathbb{R}^{M_x \times M_x}\) and \(A_{h_{x}} \in \mathbb{R}^{M_x \times M_x}\) denote spatial mass and stiffness matrices given by

\[
M_{h_{x}}[i, j] = \langle \psi_j^1, \psi_i^1 \rangle_{L^2(\Omega)}, \quad A_{h_{x}}[i, j] = \langle \nabla_x \psi_j^1, \nabla_x \psi_i^1 \rangle_{L^2(\Omega)}, \quad i, j = 1, \ldots, M_x,
\]

and \(M_{h_{x}}^{\mathcal{H}_T} \in \mathbb{R}^{N_t \times N_t}\) and \(A_{h_{x}}^{\mathcal{H}_T} \in \mathbb{R}^{N_t \times N_t}\) are defined by

\[
M_{h_{x}}^{\mathcal{H}_T}[\ell, k] := \langle \varphi_\ell^1, \mathcal{H}_T \varphi_k^1 \rangle_{L^2(0, T)}, \quad A_{h_{x}}^{\mathcal{H}_T}[\ell, k] := \langle \mathcal{H}_T \partial_t \varphi_\ell^1, \partial_t \varphi_k^1 \rangle_{L^2(0, T)}
\]
for $\ell, k = 1, \ldots, N_t$. The matrices $M_{P_h}^{H_T}$, $A_{H_T}^{H_T}$ are nonsymmetric, but positive definite, which follows from the properties of $H_T$, given in Section 2. Additionally, the matrices $M_{H_h}$, $A_{H_h}$ are positive definite. Thus, standard properties of the Kronecker product yield that the system matrix $K_h$ is also positive definite. Hence, the global linear system (8) is uniquely solvable.

4 Numerical results

In this section, numerical examples for the Galerkin–Bubnov finite element method (6) for a one- and a two-dimensional spatial domain are given. For both cases, the number of degrees of freedom is given by $\text{dof} = N_t \cdot M_s$. The assembling of the matrices $A_{P_h}^{H_T}$, $M_{H_T}^{H_T}$ is done as proposed in [11, Subsection 2.2]. Further, to accelerate the computations, data-sparse approximations as known from boundary element methods, e.g., hierarchical matrices, can be used, see [9]. The integrals for computing the projection $Q_{H_h}^f$ in (7) are calculated by using high-order quadrature rules. The global linear system (8) is solved by a direct solver.

For the first numerical example, we consider the one-dimensional spatial domain $\Omega := (0, 1)$ with the terminal time $T = 10$, i.e., the rectangular space-time domain

$$Q := \Omega \times (0, T) := (0, 1) \times (0, 10).$$

As an exact solution, we choose

$$u_1(x, t) = t^2 \sin(10\pi x) \sin(t x), \quad (x, t) \in Q.$$  

The spatial domain $\Omega = (0, 1)$ is decomposed into nonuniform elements with the vertices

$$x_0 = 0, \quad x_1 = 1/4, \quad x_2 = 1,$$

whereas the temporal domain $(0, T) = (0, 10)$ is decomposed into nonuniform elements with the vertices

$$t_0 = 0, \quad t_1 = 5/4, \quad t_2 = 5/2, \quad t_3 = 10 = T,$$

see Fig. 1 for the resulting space-time mesh. We apply a uniform refinement strategy for the meshes (11), (12). The numerical results for the smooth solution $u_1$ in (10) are given in Table 1, where we observe unconditional stability, quadratic convergence in $\| \cdot \|_{L^2(Q)}$, and linear convergence in $\| \cdot \|_{H^1(Q)}$.

For the second numerical example, the two-dimensional spatial $\Gamma$-shaped domain

$$\Omega := (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]) \subset \mathbb{R}^2$$

and the terminal time $T = 2$ are considered for the solution

$$u_2(x_1, x_2, t) = \sin(\pi x_1) \sin(\pi x_2)(\sin(tx_1x_2))^2, \quad (x_1, x_2, t) \in Q = \Omega \times (0, T).$$
Table 1: Numerical results of the Galerkin–Bubnov finite element discretization (6) for the space-time cylinder (9) for the function $u_1$ in (10) for a uniform refinement strategy.

| dof $h_{x,\max}$ $h_{x,\min}$ $h_{t,\max}$ $h_{t,\min}$ $\|u_1 - u_{1,h}\|_{L^2(Q)}$ eoc $\|u_1 - u_{1,h}\|_{H^1(Q)}$ eoc |
|---|---|---|---|---|---|---|---|
| 3 | 0.7500 | 0.2500 | 7.5000 | 1.2500 | 5.0e+02 | - | 3.2e+03 | - |
| 18 | 0.3750 | 0.1250 | 3.7500 | 0.6250 | 4.2e+02 | 0.3 | 2.7e+03 | 0.2 |
| 84 | 0.1875 | 0.0625 | 1.8750 | 0.3125 | 3.2e+02 | 0.4 | 2.5e+03 | 0.1 |
| 360 | 0.0938 | 0.0312 | 0.9375 | 0.0312 | 8.4e+01 | 1.9 | 2.1e+03 | 0.2 |
| 1488 | 0.0469 | 0.0156 | 0.4688 | 0.0156 | 2.0e+01 | 1.7 | 1.0e+03 | 1.0 |
| 6048 | 0.0234 | 0.0078 | 0.2344 | 0.0078 | 7.2e+00 | 1.9 | 5.0e+02 | 1.1 |
| 24384 | 0.0117 | 0.0039 | 0.1172 | 0.0039 | 1.8e+00 | 2.0 | 2.5e+02 | 1.0 |
| 97920 | 0.0059 | 0.0020 | 0.0586 | 0.0020 | 4.7e+00 | 2.0 | 1.2e+02 | 1.0 |
| 392448 | 0.0029 | 0.0010 | 0.0293 | 0.0010 | 1.2e+00 | 2.0 | 6.2e+01 | 1.0 |
| 1571328 | 0.0015 | 0.0005 | 0.0146 | 0.0005 | 4.7e-01 | 2.0 | 3.1e+01 | 1.0 |

The spatial domain $\Omega$ is decomposed into uniform triangles with uniform mesh size $h_x$ as given in Fig. 1 for the first level. The temporal domain $(0, 2) = (0, T)$ is decomposed into nonuniform elements with the vertices

$$
t_0 = 0, \quad t_1 = 1/8, \quad t_2 = 1/4, \quad t_3 = 1/2, \quad t_4 = 2 = T. \quad (15)
$$

When a uniform refinement strategy is applied for the temporal mesh (15) and for the spatial mesh, the numerical results for the smooth solution $u_2$ are given in Table 2, where unconditional stability is observed and the convergence rates in $\| \cdot \|_{L^2(Q)}$ and $\| \cdot \|_{H^1(Q)}$ are optimal.

Table 2: Numerical results of the Galerkin–Bubnov finite element discretization (6) for the $\Gamma$-shape (13) and $T = 2$ for the function $u_2$ in (14) for a uniform refinement strategy.

| dof $h_x$ $h_{t,\max}$ $h_{t,\min}$ $\|u_2 - u_{2,h}\|_{L^2(Q)}$ eoc $\|u_2 - u_{2,h}\|_{H^1(Q)}$ eoc |
|---|---|---|---|---|---|---|---|
| 20 | 0.3536 | 1.5000 | 0.1250 | 1.756e-01 | - | 1.331e+00 | - |
| 264 | 0.1768 | 0.7500 | 0.0625 | 6.370e-02 | 1.5 | 6.882e-01 | 1.0 |
| 2576 | 0.0884 | 0.3750 | 0.0312 | 1.903e-02 | 1.7 | 3.439e-01 | 1.0 |
| 22560 | 0.0442 | 0.1875 | 0.0156 | 5.206e-03 | 1.9 | 1.730e-01 | 1.0 |
| 188480 | 0.0221 | 0.0938 | 0.0078 | 1.306e-03 | 2.0 | 8.555e-02 | 1.0 |
| 1540224 | 0.0110 | 0.0469 | 0.0039 | 3.284e-04 | 2.0 | 4.268e-02 | 1.0 |
5 Conclusions

In this work, we introduced new conforming space-time Galerkin–Bubnov methods for the wave equation. These methods are based on a space-time variational formulation, where ansatz and test spaces are equal, using also integration by parts with respect to the time variable and the modified Hilbert transformation \( \mathcal{H}_T \). As discretizations of this variational setting, we considered a conforming tensor-product approach with piecewise multilinear, continuous basis functions. However, a generalization to piecewise polynomials of higher-order degree is straightforward. We gave numerical examples, where the unconditional stability, i.e., no CFL condition is required, and optimal convergence rates in space-time norms were illustrated. For a more detailed stability and error analysis, we refer to our ongoing work [5]. Other topics include the realization for arbitrary space-time meshes, a posteriori error estimates and adaptivity, and the parallel solution including domain decomposition methods.

References