Décomposition de Domaine et Problème de Helmholtz: Thirty Years After and Still Unique

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1 Introduction

In 1990, Bruno Després published a short note [5] in Comptes rendus de l'Académie des sciences. Série 1, Mathématique. "The aim of this work is, after construction of a domain decomposition method adapted to the Helmholtz problem, to show its convergence." The idea has been further developed in [12], [3], [9], [4] and [2] by employing radiation conditions with a special structure, subdomains without overlap and iterations in parallel or one-sweep. As of today, it seems the *unique* means by which Schwarz iterations (not Krylov-Schwarz) for the Helmholtz equation have been proved to converge in general geometry and variable media; otherwise, e.g., using PML ([1]) as boundary conditions requires the (sub)domain to be convex. This paper is to show that those algorithmic parameters are difficult to perturb even in a rectangle while maintaining convergent Schwarz iterations.

To this end, we consider the Helmholtz equation in $\Omega = (X_0^-, X_N^+) \times (0,1)$:

$$(\Delta + k^2)u = f \text{ in } \Omega,\ \mathcal{B}^{\mp}u = 0 \text{ at } \{X_0^-, X_N^+\} \times (0,1),\ \mathcal{C}u = 0 \text{ at } (X_0^-, X_N^+) \times \{0,1\}, \quad (1)$$

where $k > 0$, and $\mathcal{B}^{\mp}$, $\mathcal{C}$ are some trace operators. In the free space problem $\mathcal{C} = \partial_{\mathbf{n}} - \mathrm{i}k$ and in the waveguide problem $\mathcal{C} = \partial_{\mathbf{n}}$ ($\mathbf{n}$ being the unit outer normal vector). Assume that $\Omega = \cup_{l=1}^N \Omega_l$ with $\Omega_l = (X_{l-1}^-, X_l^+) \times (0,1)$, $X_l^{\pm} := lH \pm \frac{L}{2}$, $H > 0$ and $L \geq 0$. The optimized Schwarz method iteratively solves (1) restricted to Ω_l for $u_l \approx u|_{\Omega_l}$ in parallel or in some order of $l = 1, .., N$ with the transmission conditions $\mathcal{B}^- u_l = \mathcal{B}^- u_{l-1}$ at $\{X_{l-1}^-\} \times (0,1)$, $l > 1$ and $\mathcal{B}^+ u_l = \mathcal{B}^+ u_{l+1}$ at $\{X_l^+\} \times (0,1)$, $l < N$.

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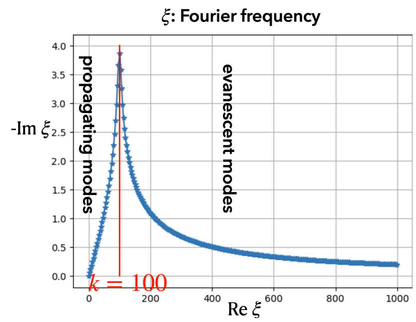

Fig. 1: Fourier frequencies from the Sturm-Liouville problem with $C = \partial_{\mathbf{n}} - \mathrm{i}k$ for $k = 100$.

From the Sturm-Liouville problem $-\varphi'' = \xi^2\varphi$ in $(0,1)$, $C\varphi = 0$ at $\{0,1\}$, the expansion $u(x,y) = \sum_\xi \hat{u}(x,\xi)\varphi(y;\xi)$ transforms (1) to an ODE for $\hat{u}(x,\xi)$ for each ξ, and the iteration operator acting on $\{g_l^\mp := \mathcal{B}^\mp u_l$ at $\{X_{l-1}^-, X_l^+\} \times (0,1)\}$ to a matrix for each ξ; see [8]. The spectral radius of the iteration matrix as a function of ξ or $\operatorname{Re}\xi/k$ is called the convergence factor ρ. The Schwarz iterations converge geometrically if and only if $\sup_\xi \rho < 1$. When $C = \partial_{\mathbf{n}}$, $\xi \in \{0, \pm\pi, \pm 2\pi, \pm 3\pi, ..\}$. When $C = \partial_{\mathbf{n}} - \mathrm{i}k$, ξ's are complex roots of a nonlinear equation; see e.g. Figure 1.

2 What can we change from Després' original method?

In the original method, $\mathcal{B}^\mp = \partial_{\mathbf{n}} - \mathrm{i}k$. From Figure 2 we see that all the curves are below $\rho = 1$, albeit $\rho \to 1$ as $\operatorname{Re}\xi \to \infty$, and the curves seemingly have a limit profile for a fixed N but increasing k, while they move higher up as N grows.

Can we add overlap? The short answer is 'no' for large overlap and 'yes' for small overlap; see Figure 3 and Figure 4. This is in contrast to the Laplace equation for which [10] claimed its proof "also applies ... with overlapping subdomains", though, an actual proof appeared only in [11] by rather different techniques.

Can we reorder subdomain iterations? Després' original method uses parallel iterations between subdomains. The sequential iterations from Ω_1 to Ω_N or one-sweep iterations through other orderings (e.g. red-black) of $\{\Omega_1, .., \Omega_N\}$ behave similarly. In contrast, the double sweep iterations with the forward sweep from Ω_1 to Ω_N followed by a backward sweep from Ω_{N-1} to Ω_1 can diverge; see Figure 5. It is less divergent for larger kH, which suggests the next question.

What if we fix the subdomain size? In this case, the double sweeps converge very well and even better with large overlap; see Figure 6.

Can we add a real part to the Robin coefficient? Yes, if the real parts of the Robin coefficients for two adjacent subdomains are equal in absolute value but opposite in sign, as shown in [9, 4]; otherwise it may diverge. See Figure 7 and Figure 8.

Can we use second-order conditions? Yes, if the imaginary part of the Robin coefficient (i.e., tangential operator) is sign-definite on interfaces and outer boundaries,

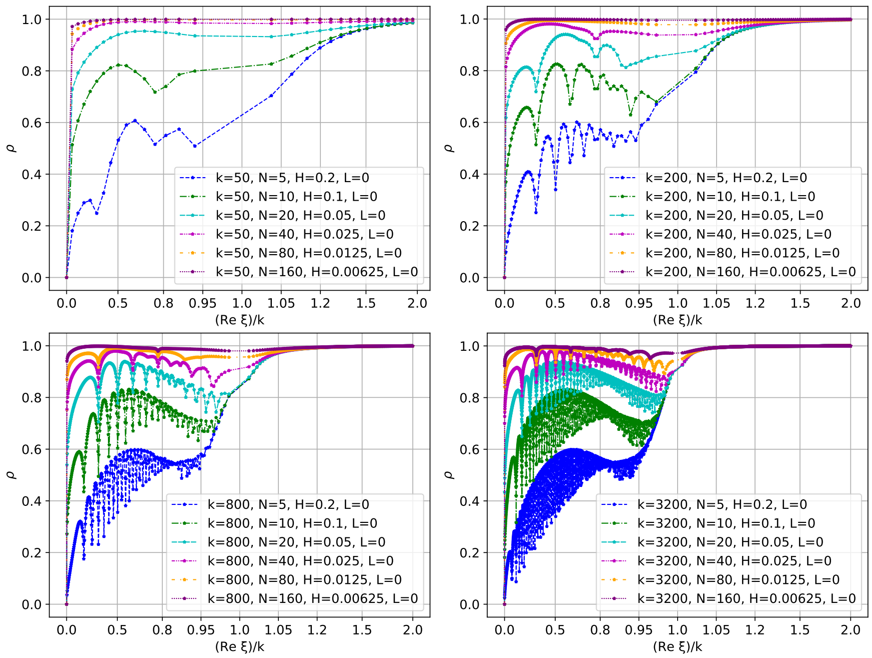

Fig. 2: ρ of Després' method for free space on $[0, 1]^2$.

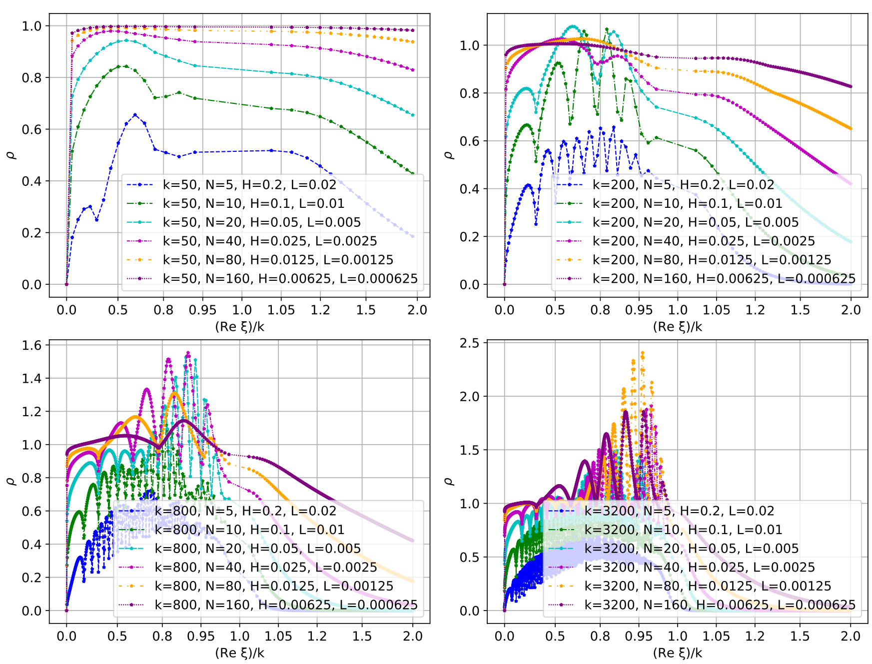

Fig. 3: ρ of Després' method plus $L = \frac{H}{10}$ overlap for free space on $[-\frac{L}{2}, 1 + \frac{L}{2}] \times [0, 1]$.

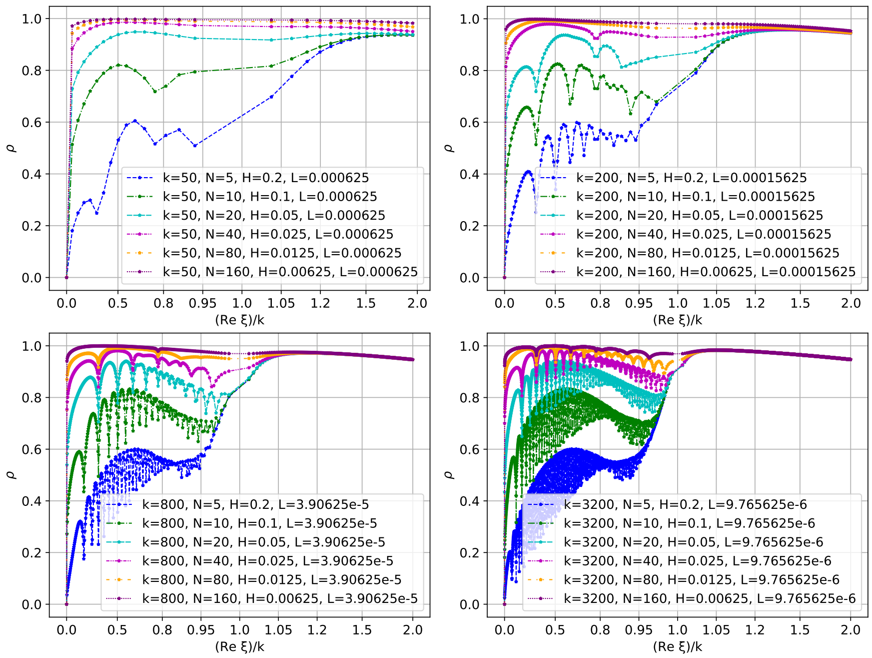

Fig. 4: ρ of Després' method plus $L = \frac{1}{32k}$ overlap for free space on $[-\frac{L}{2}, 1 + \frac{L}{2}] \times [0, 1]$.

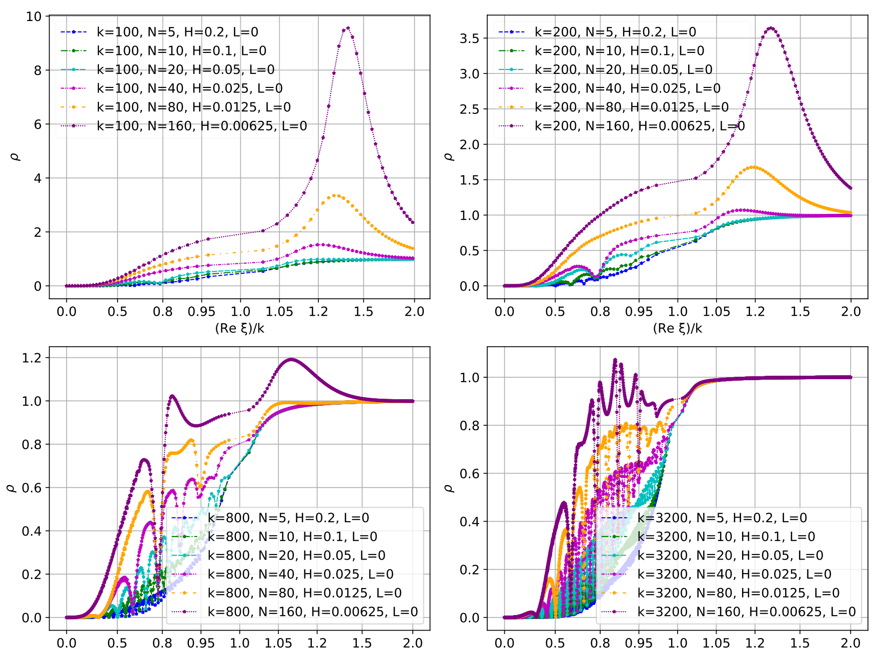

Fig. 5: ρ of Després' method in double sweep for free space on $[0, 1]^2$.

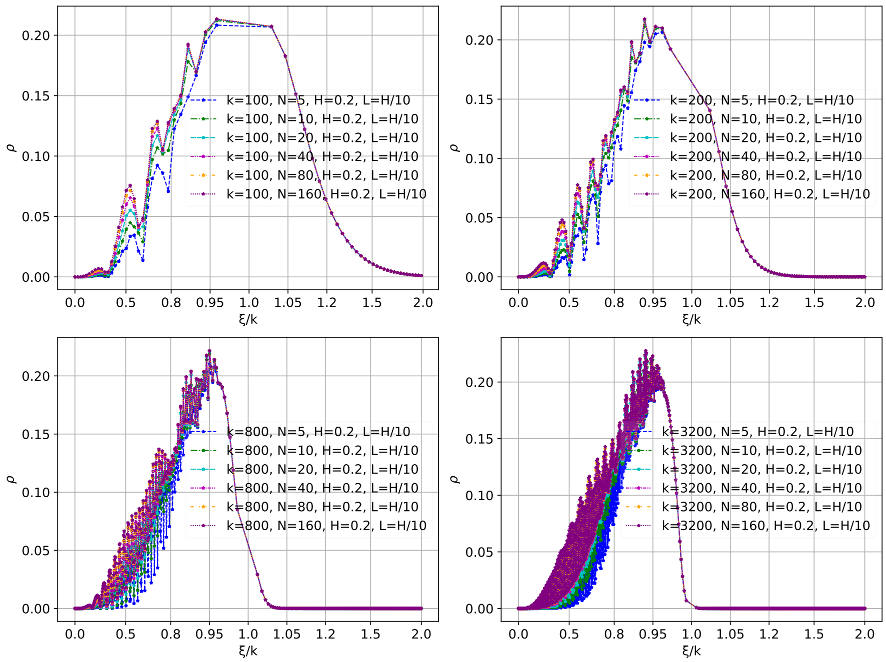

Fig. 6: ρ of Després' method plus $L = \frac{H}{10}$ overlap in double sweep for free space on $[-\frac{L}{2}, \frac{N}{5} + \frac{L}{2}] \times [0, 1]$.

as proved in [12, 3]. The 2nd-order Taylor approximation $\sqrt{k^2 - \xi^2} \approx k(1 - \frac{\xi^2}{2k^2})$ is sign-changing across $\xi^2 = 2k^2$ and thus $\mathcal{B}^{\mp} = \partial_{\mathbf{n}} - \mathrm{i}k(1 + \frac{\Delta_S}{2k^2})$ (Δ_S is the Laplacian on interfaces) falls outside the theory. So, [12] proposed to reverse the sign in front of Δ_S, and [6] uses $\mathcal{B}^{\mp} = \partial_{\mathbf{n}} - \mathrm{i}k(1 - \frac{\Delta_S}{2k^2})^{-1}$. See Figure 9 for comparison.

Can we change the outer boundary conditions? Yes, the proof in [5] and others all work as long as on part of the outer boundary a radiation condition is imposed with imaginary part of the tangential operator being sign-definite. It is thus interesting to check, e.g., with $C = \partial_{\mathbf{n}}$ and $\mathcal{B}^{\mp}$ the 2nd-order Taylor; see Figure 10.

Can we treat variable media? Yes, if the Robin coefficients for any two adjacent subdomains have equal imaginary parts, as proved in [3]. What if the symmetry is broken? For example, the optimal Schwarz method (see, e.g., the review [7]) uses the Dirichlet-to-Neumann maps from the two sides of an interface, which are generally not equal for propagating modes in variable media. In Després' method, to keep the symmetry one can use an average wavenumber on the interfaces, e.g., $k_{ij} = \sqrt{(k_i^2 + k_j^2)/2}$ for $\mathcal{B}^{\mp} = \partial_{\mathbf{n}} - \mathrm{i}k_{ij}$ on $\partial\Omega_i \cap \partial\Omega_j$. To mimic the optimal Schwarz method, one can use the wavenumber from the other side, e.g., $\mathcal{B}^{\mp} = \partial_{\mathbf{n}} - \mathrm{i}k_{i\mp1}$ for u_i on $\partial\Omega_i \cap \partial\Omega_{i\mp1}$. In our example, we split the first dimension into five equal layers on $[0, 1]^2$ and assume the wavenumber k in $\mathbb{R}^2 \backslash [0, 1]^2$ is a constant; see Figure 11.

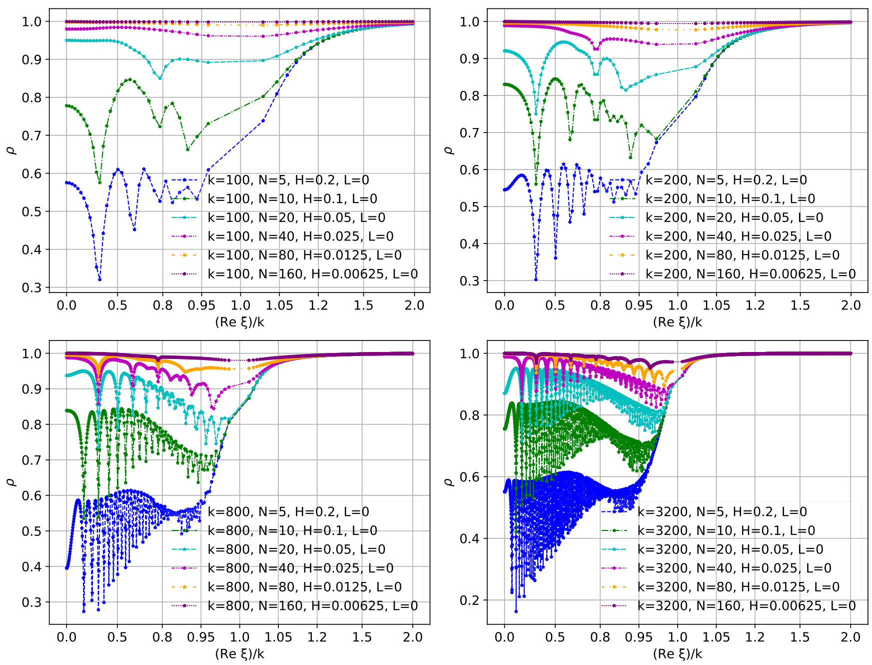

Fig. 7: ρ of Després' method with left/right Robin coeff. $-\mathrm{i}k(1 \pm 0.1\mathrm{i})$ for free space on $[0, 1]^2$.

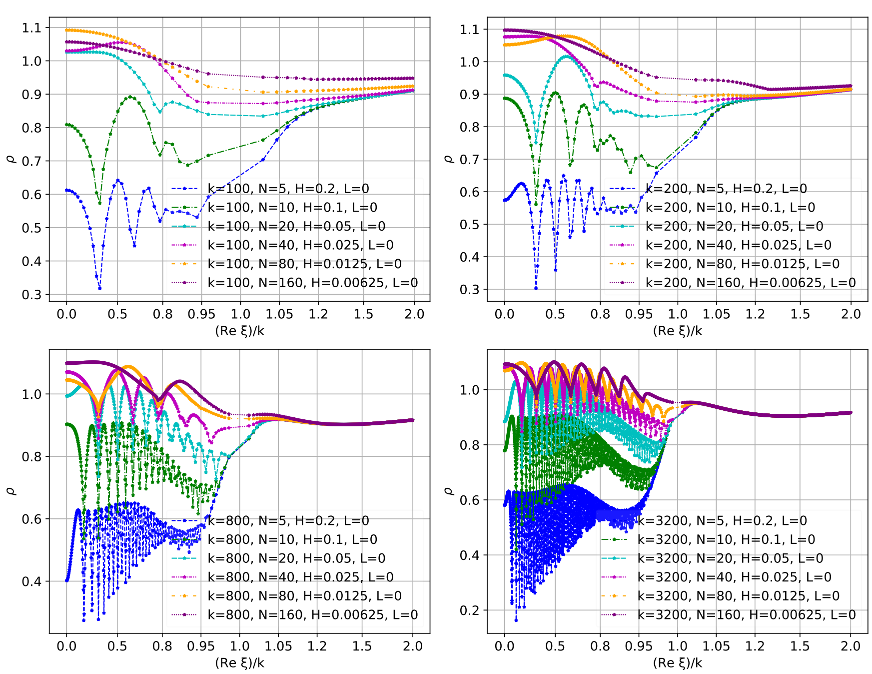

Fig. 8: ρ of Després' method with interface Robin coefficient $-\mathrm{i}k(1{+}0.1\mathrm{i})$ for free space on $[0, 1]^2$.

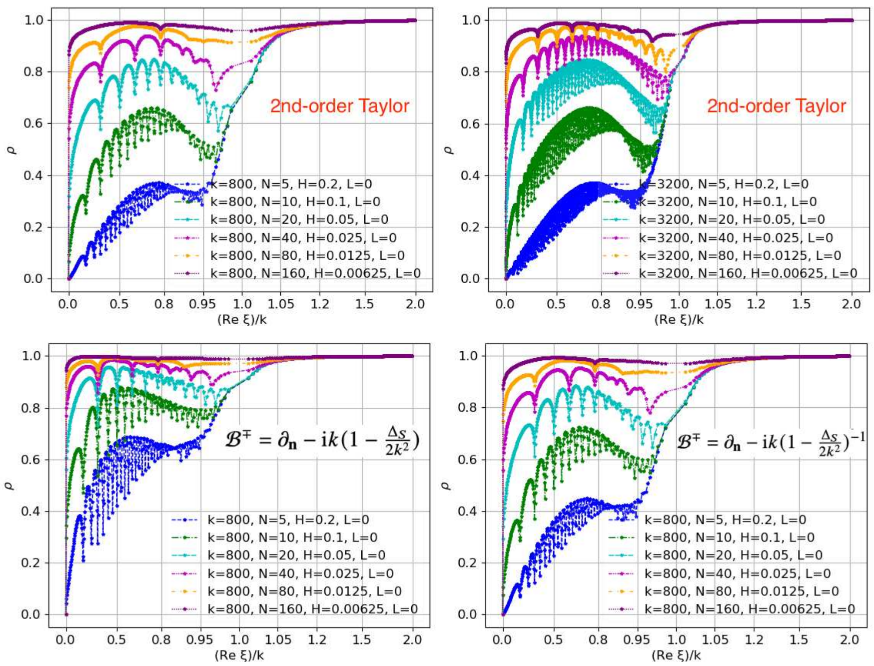

Fig. 9: ρ of Després' method plus a 2nd-order term for free space on $[0, 1]^2$.

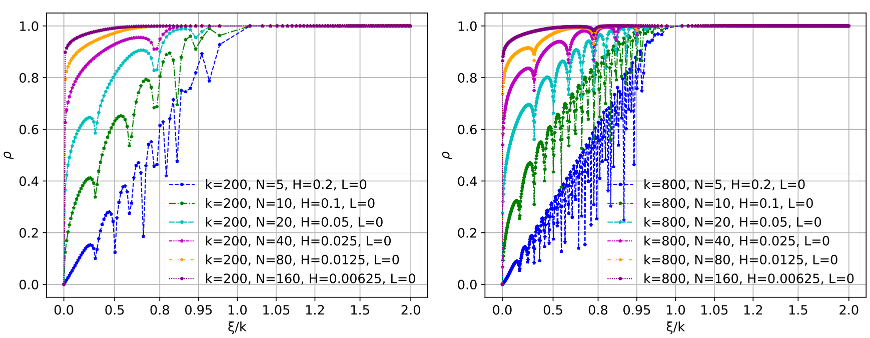

Fig. 10: ρ of Després' method with $\mathcal{B}^{\mp}$ the 2nd-order Taylor for waveguide on $[0, 1]^2$.

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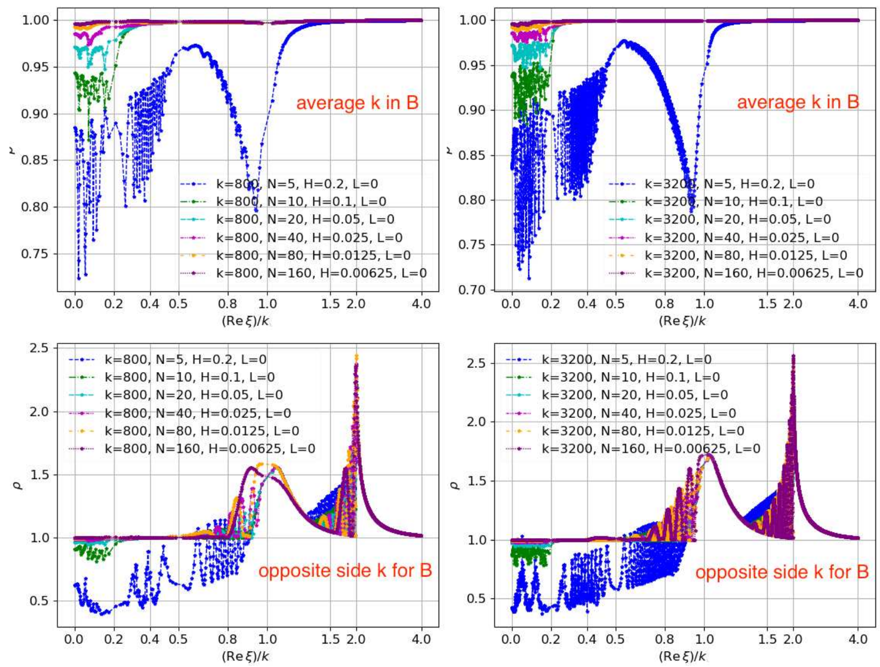

Fig. 11: ρ of Després' method on $[0, 1]^2$ with exterior wavenumber k & interior $k/[1, 5, 0.5, 2, 1]$.

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