Non-local Impedance Operator for Non-overlapping DDM for the Helmholtz Equation

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In the context of time harmonic wave equations, the pioneering work of B. Després [4] has shown that it is mandatory to use impedance type transmission conditions in the coupling of sub-domains in order to obtain convergence of non-overlapping domain decomposition methods (DDM). In later works [2, 3], it was observed that using non-local impedance operators leads to geometric convergence, a property which is unattainable with local operators. This result was recently extended to arbitrary geometric partitions, including configurations with cross-points, with provably uniform stability with respect to the discretization parameter [1].

We present a novel strategy to construct suitable non-local impedance operators that satisfy the theoretical requirements of [1] or [2, 3]. It is based on the solution of elliptic auxiliary problems posed in the vicinity of the transmission interfaces. The definition of the operators is generic, with simple adaptations to the acoustic or electromagnetic settings, even in the case of heterogeneous media. Besides, no complicated tuning of parameters is required to get efficiency. The implementation in practice is straightforward and applicable to sub-domains of arbitrary geometry, including ones with rough boundaries generated by automatic graph partitioners.

1 General approach for a two-domain decomposition

We consider the Helmholtz equation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, with a first order absorbing boundary condition imposed on the boundary Γ: Find $u \in H^1(\Omega)$ such that

$$(-\operatorname{div} \mathfrak{a}\nabla - \kappa^2 \mathfrak{n})u = f, \quad \text{in } \Omega, \qquad (\mathfrak{a}\partial_{\mathbf{n}} - \imath\kappa)u = g, \quad \text{on } \Gamma, \tag{1}$$

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where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, κ denotes the wavenumber, $\mathfrak{a}$ and $\mathfrak{n}$ are two strictly positive and bounded functions (so that the medium is purely propagative) and $\mathbf{n}$ is the outward normal to Γ. The well-posedness of this problem is guaranteed by application of the Fredholm alternative and a unique continuation principle.

A geometrically convergent DD method. We consider a non-overlapping partition in two domains, excluding the presence of (boundary) cross-points, by introducing a *closed* Lipschitz interface Σ that splits the domain Ω into an interior domain Ω_1 and exterior domain Ω_2, see Figure 1 (left). The Domain Decomposition (DD) method consists in solving iteratively the Helmholtz equation in parallel in each sub-domain by imposing two transmission conditions. Introducing a boundary operator T on Σ we consider here impedance-like transmission conditions:

$$\begin{cases} (+\mathfrak{a}\partial_{\mathbf{n}_1} - \imath\kappa T)u_1 = (-\mathfrak{a}\partial_{\mathbf{n}_2} - \imath\kappa T)u_2, \\ (-\mathfrak{a}\partial_{\mathbf{n}_1} - \imath\kappa T)u_1 = (+\mathfrak{a}\partial_{\mathbf{n}_2} - \imath\kappa T)u_2, \end{cases} \qquad \text{on } \Sigma,$$

where we denoted by $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) the outward unit normal vector to Ω_1 (resp. Ω_2).

The DD method is best analysed in the form of an interface problem. Let us introduce $(w_1, w_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$ a lifting of the source defined as follows

$$\begin{cases} (-\operatorname{div}\mathfrak{a}\nabla - \kappa^2\mathfrak{n})w_1 = f|_{\Omega_1}, & \text{in } \Omega_1, \\ (+\mathfrak{a}\partial_{\mathbf{n}_1} - \imath\kappa T)w_1 = 0, & \text{on } \Sigma, \end{cases} \quad \begin{cases} (-\operatorname{div}\mathfrak{a}\nabla - \kappa^2\mathfrak{n})w_2 = f|_{\Omega_2}, & \text{in } \Omega_2, \\ (+\mathfrak{a}\partial_{\mathbf{n}} - \imath\kappa)w_2 = g, & \text{on } \Gamma, \\ (+\mathfrak{a}\partial_{\mathbf{n}_2} - \imath\kappa T)w_2 = 0, & \text{on } \Sigma, \end{cases} \tag{2}$$

and we define for any $x_j \in H^{-1/2}(\Sigma)$, $L_j x_j := v_j \in H^1(\Omega_j)$, $j \in \{1, 2\}$, such that

$$\begin{cases} (-\operatorname{div}\mathfrak{a}\nabla - \kappa^2\mathfrak{n})v_1 = 0, & \text{in } \Omega_1, \\ (+\mathfrak{a}\partial_{\mathbf{n}_1} - \imath\kappa T)v_1 = x_1, & \text{on } \Sigma, \end{cases} \qquad \begin{cases} (-\operatorname{div}\mathfrak{a}\nabla - \kappa^2\mathfrak{n})v_2 = 0, & \text{in } \Omega_2, \\ (+\mathfrak{a}\partial_{\mathbf{n}} - \imath\kappa)v_2 = 0, & \text{on } \Gamma, \\ (+\mathfrak{a}\partial_{\mathbf{n}_2} - \imath\kappa T)v_2 = x_2, & \text{on } \Sigma. \end{cases} \tag{3}$$

Assuming that the operator T is *self-adjoint positive definite*, one can prove that the local sub-problems appearing in (2) and (3) are well posed [3, Lem. 2.5]. Finally let us introduce for any $x \in H^{-1/2}(\Sigma)$ the so-called local *scattering* operators, $j \in \{1, 2\}$

$$S_j x := (-\mathfrak{a}\partial_{\mathbf{n}_j} - \imath\kappa T)L_j x,$$

and set
$$\mathbf{S} := \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \qquad \mathbf{\Pi} := \begin{bmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}, \qquad \mathbf{b} := \begin{bmatrix} (-\mathfrak{a}\partial_{\mathbf{n}_2} - \imath\kappa T)w_2 \\ (-\mathfrak{a}\partial_{\mathbf{n}_1} - \imath\kappa T)w_1 \end{bmatrix}.$$

$\mathbf{S}$ is the *global* scattering operator and $\mathbf{\Pi}$ is referred to as the *exchange* operator since its action consists in swapping information between the two sub-domains. It can be shown [2, Th. 2] that if u satisfies the model problem (1) then the two (incoming) Robin traces $\mathbf{x} := ((+\mathfrak{a}\partial_{\mathbf{n}_1} - \imath\kappa T)u|_{\Omega_1}, (+\mathfrak{a}\partial_{\mathbf{n}_2} - \imath\kappa T)u|_{\Omega_2})$, satisfy the interface problem

$$(\mathrm{Id} - \mathbf{\Pi}\mathbf{S})\mathbf{x} = \mathbf{b}. \tag{4}$$

Reciprocally, if $\mathbf{x} = (x_1, x_2)$ satisfies the interface problem (4), then the concatenation of $(L_1 x_1 + w_1, L_2 x_2 + w_2)$ is solution to the original problem (1).

One of the simplest iterative method to solve (4) is the (relaxed) Jacobi algorithm. Let $\mathbf{x}^0$ and a relaxation parameter $0 < r < 1$ be given, a sequence $(\mathbf{x}^n)_{n\in\mathbb{N}}$ is constructed using the (relaxed) Jacobi algorithm as follows

$$\mathbf{x}^{n+1} = [(1-r)\mathrm{Id} + r\mathbf{\Pi}\mathbf{S}]\, \mathbf{x}^n + r\mathbf{b}, \qquad n \in \mathbb{N}. \tag{5}$$

Theorem 1 [3, Th. 2.1] *If T is a positive self-adjoint* isomorphism *between the trace spaces $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$, then the above algorithm converges geometrically*

$$\exists\, 0 \le \tau < 1,\ C > 0,\ \|u_1 - (L_1 x_1^n + w_1)\|_{H^1} + \|u_2 - (L_2 x_2^n + w_2)\|_{H^1} \le C\tau^n.$$

Note that the isomorphism property is essential to ensure the geometric nature of the convergence, and, together with the positivity and self-adjointness properties, necessarily requires T to be *non-local*. Alternatively, a more efficient algorithm to use in practice is the GMRES algorithm. The convergence rate of the GMRES algorithm is necessarily better (i.e. the algorithm is always faster) than the convergence of the Jacobi algorithm, but much more delicate to analyse.

A suitable impedance operator. We propose to construct impedance operators that satisfy the above theoretical requirements of the convergence analysis from elliptic (or dissipative) version of conventional Dirichlet-to-Neumann (DtN) maps. To do so, we introduce two strips $\mathcal{B}_1 \subset \Omega_1$ and $\mathcal{B}_2 \subset \Omega_2$ so that $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) has two disconnected (and not intersecting) boundaries Σ and Σ_1 (resp. Σ_2), see Figure 1 (left). We do not exclude the case $\Sigma_1 = \emptyset$ for which we have $\mathcal{B}_1 = \Omega_1$. We denote by $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) the outward unit normal vector to $\mathcal{B}_1$ (resp. $\mathcal{B}_2$).
We define two operators, for $j \in \{1, 2\}$ and any $x \in H^{1/2}(\Sigma)$,

$$T_j x := \kappa^{-1}\mathfrak{a}\partial_{\mathbf{n}_j} u_j, \qquad u_j \in H^1(\mathcal{B}_j),\ \begin{cases} (-\operatorname{div}\mathfrak{a}\nabla + \kappa^2\mathfrak{n})u_j = 0, & \text{in } \mathcal{B}_j,\\ \mathfrak{a}\partial_{\mathbf{n}_j} u_j + \kappa u_j = 0, & \text{on } \Gamma_j,\\ u_j = x, & \text{on } \Sigma. \end{cases} \tag{6}$$

It is a straightforward consequence of the surjectivity of the Dirichlet trace operator and the Lax-Milgram Lemma to prove the following result, which then guarantees that we fall within the situation of Theorem 1.

Proposition 1 *The impedance operator defined as $T = \frac{1}{2}(T_1 + T_2)$, is a self-adjoint positive isomorphism from $H^{1/2}(\Sigma)$ to $H^{-1/2}(\Sigma)$.*

2 Quantitative analysis for the wave-guide

The aim of this section is to derive convergence estimates to study in particular the influence of the width of the strip in the definition of the auxiliary problems, which has a direct influence on the computational cost of the proposed method. We consider the theoretical (because unbounded) configuration of an infinite wave guide of width L, so that $\Omega := \{(x, y) \in \mathbb{R}^2 | 0 < x < L\}$, see Figure 1 (right). The media is considered homogeneous ($\mathfrak{a} \equiv \mathfrak{n} \equiv 1$); we impose homogeneous Dirichlet boundary conditions on the sides $u(0, \cdot) = u(L, \cdot) = 0$ and require u to be outgoing [5].

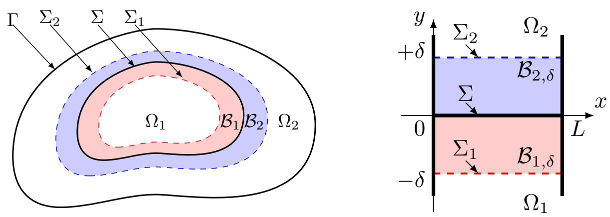

Fig. 1: Geometric configurations.

Remark 1 The above problem is well-posed except at cut-off frequencies $\kappa L \in \pi\mathbb{Z}$, configurations which are thus excluded in what follows.

The domain Ω is divided in its upper region $\Omega_2 := \{(x, y) \in \Omega | y > 0\}$ and lower region $\Omega_1 := \{(x, y) \in \Omega | y < 0\}$ and the interface is $\Sigma := (0, L) \times \{0\}$. Suppose that we have at hand a suitable impedance operator T (described below), in spite of the different geometry and the unboundedness, the same DD algorithm of Section 1 is formally applicable with minor adaptations. For completeness and because it will be important in the following, we simply provide the full definition of the local scattering operators, for $j \in \{1, 2\}$ and any $\mathrm{x} \in H^{-1/2}(\Sigma)$

$$S_j \mathrm{x} := (-\kappa^{-1}\partial_{\mathbf{n}_j} - \imath T) u_j|_{y=0}, \quad u_j \in H^1(\Omega_j), \quad \begin{cases} (-\Delta - \kappa^2) u_j = 0, & \text{in } \Omega_j, \\ u_j(0, \cdot) = u_j(L, \cdot) = 0, & \text{on } \partial\Omega_j \setminus \Sigma, \\ (\kappa^{-1}\partial_{\mathbf{n}_j} - \imath T) u_j = \mathrm{x}, & \text{on } \Sigma, \end{cases}$$

and u_j is supposed outgoing.

A family of suitable impedance operators. We introduce now several possible impedance operators on the model of (6). The domain of the auxiliary problem that defines the impedance operator is bounded in the y-direction, for a positive parameter $\delta > 0$, let $\mathcal{B}_{j,\delta} := \left\{(x, y) \in \Omega_j | 0 \leq |y| \leq \delta\right\}$, $j \in \{1, 2\}$. We consider the operators, indexed by the width δ and the type of boundary condition $* \in \{D, N, R\}$ (for Dirichlet, Neumann and Robin), for $j \in \{1, 2\}$ and any $\mathrm{x} \in H^{1/2}(\Sigma)$

$$T^*_{j,\delta} \mathrm{x} := \kappa^{-1} \partial_{\mathbf{n}_j} v^*_j |_{y=0},$$

where v^*_j solves the (elliptic) problem,

$$\begin{cases} (-\Delta+\kappa^2)v_j^* = 0, & \text{in } \mathcal{B}_{j,\delta}, \\ v_j^*(0,y) = v_j^*(L,y) = 0, & |y| \le \delta, \\ v_j^*(\cdot,0) = \mathrm{x}, & \text{on } \Sigma, \end{cases} \quad \text{and} \quad \begin{cases} v_j^D = 0, \\ \partial_{\mathbf{n}_j} v_j^N = 0, \\ (\partial_{\mathbf{n}_j}+\kappa)v_j^R = 0, \end{cases} \quad \text{on } \Sigma_j.$$

The impedance operators are then, $T_\delta^* := \frac{1}{2}(T_{1,\delta}^* + T_{2,\delta}^*)$. The aim of this section is to investigate the effect on the convergence of the type of boundary condition $* \in \{D, N, R\}$; as well as the shrinking of the width δ of the strips $\mathcal{B}_{1,\delta}$ and $\mathcal{B}_{2,\delta}$.

Modal analysis, convergence factor. Because of the separable geometry, we are able to conduct a quantitative study. The main tool for this is the Hilbert basis $\{\sin(k_m x)\}_{m\in\mathbb{N}}$ of $L^2(]0,L[)$ where we introduced the mode numbers $k_m := m\frac{\pi}{L}$, $m \in \mathbb{N}$. All the operators involved are diagonalized on this basis.

Symbol of the impedance operators. By symmetry, we need only to study the upper half-region. Standard computations show that the coefficients $(v_{m,2}^*)_{m\in\mathbb{N}}$ of v_2^* satisfy,

$$v_{m,2}^*(y) = \hat{x}_m \frac{e^{-\mu_m y} + \alpha_{\delta,m}^* e^{\mu_m y}}{1+\alpha_{\delta,m}^*}, \quad 0 \le y \le \delta \quad \text{where} \begin{cases} \alpha_{\delta,m}^D = -e^{-2\mu_m\delta}, \\ \alpha_{\delta,m}^N = e^{-2\mu_m\delta}, \\ \alpha_{\delta,m}^R = \frac{\kappa^{-1}\mu_m - 1}{\kappa^{-1}\mu_m+1} e^{-2\mu_m\delta}, \end{cases}$$

where we set $\mu_m := \sqrt{k_m^2+\kappa^2}$, and introduced in addition the coefficients $(\hat{x}_m)_{m\in\mathbb{N}}$ of the decomposition of x on the same modal basis. The symbol of the transmission operator T_δ^* is then

$$\hat{t}_{\delta,m}^* = \kappa^{-1}\mu_m \frac{1-\alpha_{\delta,m}^*}{1+\alpha_{\delta,m}^*} > 0, \qquad m \in \mathbb{N}.$$

Symbol of the scattering operators. Relying again on symmetry, we consider only $j = 2$. From the definitions of the scattering operators S_j, we formally have

$$S_j = -\left(\Lambda_j + \imath T_\delta^*\right)\left(\Lambda_j - \imath T_\delta^*\right)^{-1},$$

where we introduced the (propagative) DtN operators, for any $\mathrm{x} \in H^{1/2}(\Sigma)$

$$\Lambda_j \mathrm{x} := \kappa^{-1}\partial_{\mathbf{n}_j} u_j|_{y=0}, \quad u_j \in H^1(\Omega_j), \begin{cases} (-\Delta-\kappa^2)u_j = 0, & \text{in } \Omega_j, \\ u_j(0,\cdot) = u_j(L,\cdot) = 0, & \text{on } \partial\Omega_j \setminus \Sigma, \\ u_j(\cdot,0) = \mathrm{x}, & \text{on } \Sigma, \end{cases}$$

and u_j is supposed outgoing. The coefficients $(u_{2,m})_{m\in\mathbb{N}}$ of u_2 satisfy

$$u_{2,m}(y) = \hat{x}_m e^{-\xi_m y}, \qquad 0 \le y, \qquad m \in \mathbb{N},$$

and we set $$\xi_m := \begin{cases} -\imath\sqrt{\kappa^2-k_m^2}, & \text{if } k_m \le \kappa, \\ \sqrt{k_m^2-\kappa^2}, & \text{if } \kappa \le k_m, \end{cases} \qquad m \in \mathbb{N}.$$

The symbols of the operators Λ_j and the scattering operators S_j are then, for $m \in \mathbb{N}$,

$$\hat{\lambda}_{j,m} = \kappa^{-1}\xi_m, \quad \hat{s}^*_{\delta,j,m} = \frac{-\hat{\lambda}_{j,m} - \imath\hat{t}^*_{\delta,m}}{\hat{\lambda}_{j,m} - \imath\hat{t}^*_{\delta,m}} = -\frac{\hat{z}^*_{\delta,j,m} - \imath}{\hat{z}^*_{\delta,j,m} + \imath}, \quad \text{with} \quad \hat{z}^*_{\delta,j,m} = -\frac{\hat{\lambda}_{j,m}}{\hat{t}^*_{\delta,m}}.$$

Modal and global convergence factors. Finally, the modal and global convergence factors of the algorithm (5) can be estimated respectively by (we skip the technical details which can be found in [3, Th. 4.2])

$$\hat{\tau}^*_{\delta,m} := \max_{\pm}\left|(1-r) \pm r\sqrt{\hat{s}^*_{\delta,1,m}\hat{s}^*_{\delta,2,m}}\right|, \qquad \text{and} \qquad \hat{\tau}^*_{\delta} := \sup_{m\in\mathbb{N}} \hat{\tau}^*_{\delta,m}.$$

Study of the convergence factor $\hat{\tau}^*_{\delta}$. We stress that, ultimately, much of the analysis boils down to the properties of the Cayley transform $z \mapsto \frac{z-\imath}{z+\imath}$ in the complex plane, allowing to get a rather deep understanding of the convergence [6, Lem. 6.5]. For instance, the positivity of T^*_{δ} implies that in the propagative regime ($k_m < \kappa$) the ratio $\hat{z}^*_{\delta,j,m} \in \imath\mathbb{R}^+ \setminus \{0\}$, whereas in the evanescent regime ($\kappa < k_m$) the ratio $\hat{z}^*_{\delta,j,m} \in \mathbb{R} \setminus \{0\}$. The properties of the Cayley transform imply, in turn, that the scattering operators S_j are contractions ($|\hat{s}^*_{\delta,j,m}| < 1$) [6, Cor. 6.6] so that all modal convergence factors satisfy $\hat{\tau}^*_{\delta,m} < 1$. To study the global convergence factor we will use the following technical result whose proof rests on simple Taylor expansions.

Lemma 1 *Let $z(\epsilon) \in \mathbb{C}$, $\epsilon > 0$. The asymptotic behavior of the modal convergence factor of the form*

$$\tau_z = \max_{\pm}\left|(1-r) \pm r\frac{z(\epsilon)-\imath}{z(\epsilon)+\imath}\right|,$$

as ϵ goes to 0 can be deduced from the one of $z(\epsilon)$: we have

$$\begin{array}{l} z(\epsilon) \in \imath\mathbb{R}^+, \\ \quad (\zeta \in \mathbb{R}^+) \end{array} \begin{cases} z(\epsilon) \sim \imath\zeta, \\ z(\epsilon) \sim \imath\zeta\epsilon, \\ z(\epsilon) \sim \imath\zeta\epsilon^{-1}, \end{cases} \Rightarrow \begin{cases} \tau_z = 1 - 2r(1+\zeta)^{-1}\min(1,\zeta) + O(\epsilon), \\ \tau_z = 1 - 2r\zeta\epsilon + O(\epsilon^2), \\ \tau_z = 1 - 2r\zeta^{-1}\epsilon + O(\epsilon^2), \end{cases}$$

$$\begin{array}{l} z(\epsilon) \in \mathbb{R}, \\ \quad (\zeta \in \mathbb{R}) \end{array} \begin{cases} z(\epsilon) \sim \zeta, \\ z(\epsilon) \sim \zeta\epsilon, \\ z(\epsilon) \sim \zeta\epsilon^{-1}, \end{cases} \Rightarrow \begin{cases} \tau_z = 1 - 2r(1-r)(1+\zeta^2)^{-1}\min(1,\zeta^2) + O(\epsilon), \\ \tau_z = 1 - 2r(1-r)\zeta^2\epsilon^2 + O(\epsilon^3), \\ \tau_z = 1 - 2r(1-r)\zeta^{-2}\epsilon^2 + O(\epsilon^3). \end{cases}$$

Interest in using non-local operators (δ fixed). It is immediate to check that

$$\hat{z}^*_{\delta,j,m} \sim -1, \qquad \text{as } m \to \infty, \qquad \text{for } * \in \{D, N, R\}.$$

Lemma 1 (with $z(\epsilon) \equiv \hat{z}^*_{\delta,j,m}$, $\epsilon \equiv 1/m$), implies that $\lim_{m\to+\infty} \hat{\tau}^*_{\delta,m} = 1 - r(1-r) < 1$. Notice that the limit is independent of both δ and the type of boundary condition. This is not surprising as the highest modes “do not see”, in some sense, the boundary condition. Since we have already established that $\hat{\tau}^*_{\delta,m} < 1$ for all m, it follows that,

$$\hat{\tau}^*_{\delta} < 1, \qquad \text{for } * \in \{D, N, R\}.$$

We see here a manifestation of the effect of choosing an operator with the "right" order that adequately deals with the highest frequency modes. For instance, if we were to use a multiple of the identity as proposed originally by Després [4], then in this case we would obtain $z(m^{-1}) \sim -\xi_m$ so that the asymptotic convergence factor would behave like $1 - O(m^{-2})$ and the global convergence rate would be 1.

Influence of the strip width δ. From the previous expressions, we obtain that all transmission operators become local in the limit $\delta \to 0$ and, for a fixed m,

$$\hat{z}^{D}_{\delta,j,m} \sim -\xi_m \delta, \qquad \hat{z}^{N}_{\delta,j,m} \sim -\xi_m \kappa^{-2}\delta^{-1}, \qquad \hat{z}^{R}_{\delta,j,m} \sim -\xi_m \kappa^{-1}, \qquad \text{as } \delta \to 0.$$

Lemma 1 (with $z(\epsilon) \equiv \hat{z}^{*}_{\delta,j,m}$, $\epsilon \equiv \delta$) implies that, in the cases $* \in \{D, N\}$, the modal convergence factor $\hat{\tau}^{*}_{\delta,m}$ converges to 1 as $O(\delta)$ in the propagative regime ($k_m < \kappa$) and as $O(\delta^2)$ in the evanescent regime ($\kappa < k_m$). In contrast, in the case $* = R$, the modal convergence factor $\hat{\tau}^{R}_{\delta,m}$ is bounded away from 1 in all regimes.

We wish to study now the global convergence factor $\hat{\tau}^{R}_{\delta}$. We report in Figure 2 (left) the mode number of the slowest converging mode with respect to δ/λ, for $\lambda := 2\pi/\kappa$ and $\kappa = 3\pi$. This reveals that, for $* \in \{D, N\}$, the maximum modal factor is attained for a fixed mode number m as $\delta \to 0$. Therefore, our theoretical and numerical analysis have demonstrated that

$$\hat{\tau}^{*}_{\delta} = 1 - O(\delta^2), \qquad \text{as } \delta \to 0, \qquad * \in \{D, N\}.$$

In contrast, in the case $* = R$, the maximum modal factor is attained for the mode number $m \propto \delta^{-1/2}$ as $\delta \to 0$. This motivates to study the case $\delta_m = k_m^{-2}$ in the limit $m \to +\infty$. We have

$$\hat{z}^{R}_{\delta_m,j,m} \sim \kappa(1+\kappa)^{-1}\ \delta_m^{-1/2}, \qquad \text{as } m \to +\infty \text{ with } \delta_m = k_m^{-2}.$$

Therefore, using Lemma 1 (with $z(\epsilon) \equiv \hat{z}^{R}_{\delta_m,j,m}$, $\epsilon \equiv \delta_m^{1/2}$), the above theoretical and numerical analysis shows that

$$\hat{\tau}^{R}_{\delta} = 1 - O(\delta), \qquad \text{as } \delta \to 0.$$

To conclude, we report in Figure 2 (right) the global convergence factor $\hat{\tau}^{*}_{\delta}$ with respect to δ/λ, for $\lambda := 2\pi/\kappa$ and $\kappa = 3\pi$. For δ large enough we observe that the convergence factor is constant and the same for all three cases. This can be explained by the dissipative nature of the auxiliary problems and the fact that the boundary condition $*$ is imposed far away from the source of the problem. For sufficiently small δ, the asymptotic regime is attained and corroborates our previous findings.

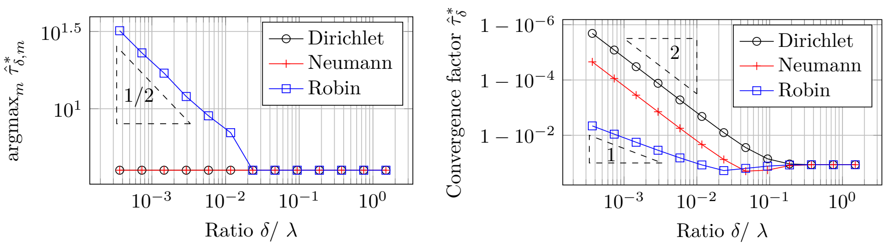

Fig. 2: Slowest-converging mode (left) and convergence factor $\hat{\tau}^*_\delta$ (right) for the wave-guide.

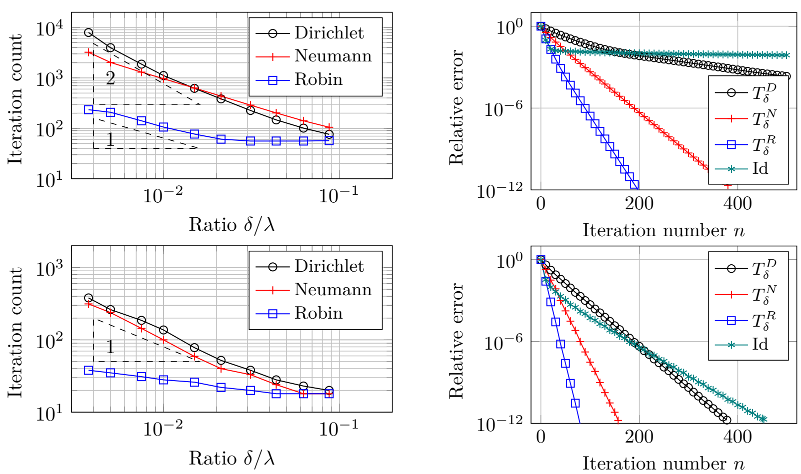

Fig. 3: Iteration count (left) and convergence history (right) in the circular configuration for the Jacobi algorithm (top row) and the Gmres algorithm (bottom row).

3 Finite element computations in a circular geometry

We provide the results of actual computations using $\mathbb{P}_1$-Lagrange finite elements with the relaxed Jacobi algorithm ($r = 1/2$) and the restarted Gmres algorithm (restart 20 iterations). The problem is (1) in a homogeneous ($\mathfrak{a} \equiv \mathfrak{n} \equiv 1$) disk of radius $R = 2$ with an interface at $R = 1$. We compute the relative error using the κ-weighted H^1-norm $\|u\|^2 := \|u\|^2_{L^2} + \kappa^{-2}\|\nabla u\|^2_{L^2}$. We report in the left column of Figure 3 the iteration count to reach a set tolerance of 10^{-8} with respect to δ/λ, with $\lambda := 2\pi/\kappa$ and $\kappa = 1$ and mesh size $h = \lambda/400$. We observe a quasi-quadratic growth for sufficiently smaller δ for the Dirichlet and Neumann conditions. In contrast, for the Robin condition, the growth is only linear and we still benefit of the non-local effect up to $\delta \approx \lambda/50$. We also report in the right column of Figure 3 the convergence history in the case $\kappa = 10$, mesh size $h = \lambda/40$ and $\delta = \lambda/20$ (i.e. strip width of two mesh cells). We added the results using the Després operator $T = \mathrm{Id}$

for comparison. The efficiency of the approach, using the Robin-type condition, is clearly demonstrated.

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