Parareal Schwarz Waveform Relaxation Method for the Time-Periodic Parabolic Problem

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1 Introduction and Model Problem

Time-periodic problems appear typically in special physical situations, for example in eddy current simulations [1], or when periodic forcing is used, like for periodically forced reactors, see [14, 15]. The numerical simulation of time-periodic problems is a special area of research, since the time periodicity modifies the problem structure and solution methods significantly. When the scale of the problems increases, it is desirable to use parallel methods to solve such problems.

For the time-dependent problems, Schwarz waveform relaxation algorithms are parallel algorithms based on a spatial domain decomposition [10]. More recently, time-parallel methods were also considered to increase the parallelism in time [5], i.e., the parareal method proposed by Lions, Maday, and Turinici in the context of virtual control to solve evolution problems in parallel; see [12]. Two parareal algorithms for time-periodic problems was proposed in [9]: one with a periodic coarse problem (PP-PC), and one with a non-periodic coarse problem (PP-IC). Further, based on these two algorithms, new applications and parallel methods for time-periodic problems were also considered; see [2, 11].

In [13], it was the first time that the combination of Schwarz waveform relaxation and parareal. Further, in [7], a new parallel algorithm named Parareal Schwarz waveform relaxation algorithm (PSWR), where there is no order between the Schwarz

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waveform relaxation algorithm and the parareal algorithm was introduced, and a superlinear convergence estimate of such algorithm has been provided in [8]. Recently, a new space-time algorithm which uses the optimized Schwarz waveform relaxation algorithm as the inner iteration of the parareal algorithm was also provided[4].

In this work, we consider a new PSWR algorithm for the following time-periodic parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{L}u + f && \text{in } \Omega \times (0,T), \\ u(x,0) &= u(x,T) && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{1}$$

where $\mathcal{L}$ is the Laplace operator, $f(x,0) = f(x,T)$, $g(x,0) = g(x,T)$, and $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$.

2 PSWR for Time-Periodic Parabolic Problem

We first introduce a parareal algorithm for time-periodic problems [7]. We decompose the time interval $[0,T]$ into N subintervals $[T_n, T_{n+1}]$, $n = 0,1,\ldots,N-1$, with $0 = T_0 < T_1 < \ldots < T_{N-1} < T_N = T$. We define so called coarse propagator $G(T_{n+1}, T_n, U_n, f, g)$ which provides a rough approximation in time of the solution $u_n(x, T_{n+1})$ of (2)

$$\frac{du_n}{dt} = \mathcal{L}u_n + f \text{ in } \Omega \times (T_n, T_{n+1}), u_n(x,T_n) = U_n(x) \text{ in } \Omega, u_n = g \text{ on } \partial\Omega \times (T_n, T_{n+1}). \tag{2}$$

with a given initial condition $u_n(x,T_n) = U_n(x)$, right hand side source term f and boundary conditions g. And we also define a fine propagator $F(T_{n+1}, T_n, U_n, f, g)$, which gives a more accurate approximation in time of the same solution of (2).

Then starting with an initial guess U_n^0 at the coarse time points $T_0, T_1, T_2, \ldots, T_{N-1}$, e.g., solving the model problem on the coarse time points, the periodic parareal algorithm with initial-value coarse problem (PP-IC) for the time-periodic problem (1) performs for $k = 0,1,2,\ldots$ the correction iteration

$$\begin{aligned} U_0^{k+1} &= U_N^k, \\ U_{n+1}^{k+1} &= F(T_{n+1}, T_n, U_n^k, f, g) + G(T_{n+1}, T_n, U_n^{k+1}, f, g) - G(T_{n+1}, T_n, U_n^k, f, g), \\ &\qquad n = 0,1,\ldots,N-1. \end{aligned} \tag{3}$$

Furthermore, we introduce the Schwarz waveform relaxation algorithm for the model problem (1) is based on a spatial decomposition only, in the most general case into overlapping subdomains $\Omega = \cup_{i=1}^I \Omega_i$. The Schwarz waveform relaxation algorithm solves iteratively for $k = 0,1,2,\ldots$ the space-time subdomain problems

$$\begin{aligned} \frac{\partial u_i^{k+1}}{\partial t} &= \mathcal{L}u_i^{k+1} + f && \text{in } \Omega_i \times (0,T), \\ u_i^{k+1}(x,0) &= u_0 && \text{in } \Omega_i, \\ \mathcal{B}_i u_i^{k+1} &= \mathcal{B}_i \bar{u}^k && \text{on } \partial\Omega_i \times (0,T). \end{aligned}$$

Here $\bar{u}^k$ denotes a composed approximate solution from the previous subdomain solutions u_i^k using for example a partition of unity, and an initial guess $\bar{u}^0$ is needed to start the iteration. The operators $\mathcal{B}_i$ are transmission operators: in the case of the identity, it will be Dirichlet transmission condition and we have the classical Schwarz waveform relaxation algorithm; for Robin or higher order transmission conditions, we obtain an optimized Schwarz waveform relaxation algorithm, if the parameters in the transmission conditions are chosen to optimize the convergence of the algorithm.

Finally, according to the reference [8], which designed the PSWR algorithm for the parabolic problems, we construct here PSWR for the time-periodic parabolic problem (1). We decompose the spatial domain Ω into I overlapping subdomains $\Omega = \cup_{i=1}^I \Omega_i$, and the time interval $(0,T)$ is divided into N time subintervals (T_n, T_{n+1}) with $0 = T_0 < T_1 < \cdots < T_N = T$. Therefore we can get a sequence of space-time subdomains $\Omega_{i,n} = \Omega_i \times (T_n, T_{n+1})$, $i = 1, 2, \ldots, I$, $n = 0, \ldots, N-1$.

Like in the parareal algorithm, we introduce a fine subdomain solver $F_{i,n}(U_{i,n}^k, \mathcal{B}_i \bar{u}_n^k)$ and a coarse subdomain solver $G_{i,n}(U_{i,n}^k, \mathcal{B}_i \bar{u}_n^k)$, where we do not explicitly state the dependence of these solvers on the time interval and the right hand side f and original Dirichlet boundary condition g to not increase the complexity of the notation further. There is also a further important notational difference with parareal: here the fine solver F returns the entire solution in space-time, not just at the final time, since this solution is also needed in the transmission conditions of the algorithm. Then for any initial guess of the initial values $U_{i,n}^0$ and the interface values $\mathcal{B}_i \bar{u}_n^0$, a new PSWR algorithm (named PSWR-IC) for the time-periodic parabolic problem (1) computes for iteration index $k = 0, 1, 2, \ldots$ and all spatial and time indices $i = 1, 2, \ldots, I$, $n = 0, 1, \ldots, N-1$, Step I. Use the more accurate evolution operator to calculate

$$u_{i,n}^{k+1} = F_{i,n}(U_{i,n}^k, \mathcal{B}_i \bar{u}_n^k);$$

Step II. Update new initial conditions using a parareal step both in space and time for $n = 0, 1, \ldots, N-1$

$$U_{i,n+1}^{k+1} = u_{i,n}^{k+1}(\cdot, T_{n+1}) + G_{i,n}(U_{i,n}^{k+1}, \mathcal{B}_i \bar{u}_n^{k+1}) - G_{i,n}(U_{i,n}^k, \mathcal{B}_i \bar{u}_n^k),$$

Step III. Update initial conditions at $t = 0$: $U_{i,0}^{k+1} = U_{i,N}^k$.

Here $\bar{u}_n^k$ is a composed approximate solution from the subdomain solutions $u_{i,n}^k$ using for example a partition of unity, e.g., $\bar{u}_n^k = u_{i,n}^k$ in $\Omega_{i,n} \cup_{j=1, j\neq i}^I \backslash (\Omega_{i,n} \cap \Omega_{j,n})$, and $\bar{u}_n^k$ is the average value of $u_{i,n}^k$ and $u_{j,n}^k$ in the overlap $\Omega_{i,n} \cap \Omega_{j,n}$, $j = 1, 2, \ldots, I$ and $j \neq i$. And an initial guess $\bar{u}_n^0$ and $U_{i,n}^0$ is needed to start the iteration (the latter can for example be computed by a time-periodic problem on the coarse using the coarse propagator once the former is chosen). Note that the first step in the proposed

PSWR-IC algorithm, which is the expensive step involving the fine propagator $F_{i,n}$, can be performed in parallel over all space-time subdomains $\Omega_{i,n}$, since both the initial and boundary data are available from the previous iteration. The cheap second step in the proposed PSWR-IC algorithm involving only the coarse propagator $G_{i,n}$ to compute a new initial condition for most space-time subdomains on $T_1, T_2, \ldots, T_{N-1}$, is still in parallel in space, but now sequential in time, like in the parareal algorithm. In step III, we use the idea of the PP-IC algorithm in [7] to update the initial condition at $t = 0$, which is a relaxation of $U_{i,0}^{k+1} = U_{i,N}^{k+1}$, avoiding solving a coupled system on the time coarse points T_i.

We have the following convergence result for the PSWR-IC algorithm as follows.

Remark 1 If the fine propagator F is the exact solver, and the coarse propagator G is Backward Euler, then PSWR-IC with Dirichlet transmission conditions and overlap L in two subdomain case for the 1-dimensional heat equation converges linearly on bounded time intervals $(0, T)$. The proof is technical [16], for an illustration see Section 3.

3 Numerical Experiments

To investigate numerically how the convergence of the PSWR-IC algorithm for time-periodic problems depends on the various parameters in the space-time decomposition, we use the following time-periodic 1-dimensional model problem

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) && (x,t) \in \Omega \times (0,1), \\ u(x,t) &= 0 && (x,t) \in \partial\Omega \times (0,1), \\ u(x,0) &= u(x,T) && x \in \Omega, \end{aligned} \tag{4}$$

where the domain $\Omega = (0, 3)$, and the exact solution of the model problem is $u = x(x-3)\sin(2\pi t)$. The model problem (4) is discretized by a second-order centered finite difference scheme with mesh size $h = 3/128$ in space and by the Backward Euler method with $\Delta t = 1/100$ in time. The time interval is divided into N time subintervals, while the domain Ω is decomposed into I equal spatial subdomains with overlap L. We define the relative error of the infinity norm of the errors along the interface and initial time in the space-time subdomains as the iterative error of our new algorithm.

We decompose the domain Ω into 2 spatial subdomains with overlap $L = 2h$. The total time interval length is $T = 1$. We show in Figure 1 on the left the convergence of the PSWR-IC algorithm when the number of time subintervals equals 1 (classical Schwarz waveform relaxation for time-periodic problems), 2, 4, 10, and 20. This shows that the convergence of the PSWR-IC algorithm does indeed not depend on the number of time subintervals, which is the same as the PSWR algorithm for the initial value problem. Here we also observe that the PSWR-IC algorithm converges

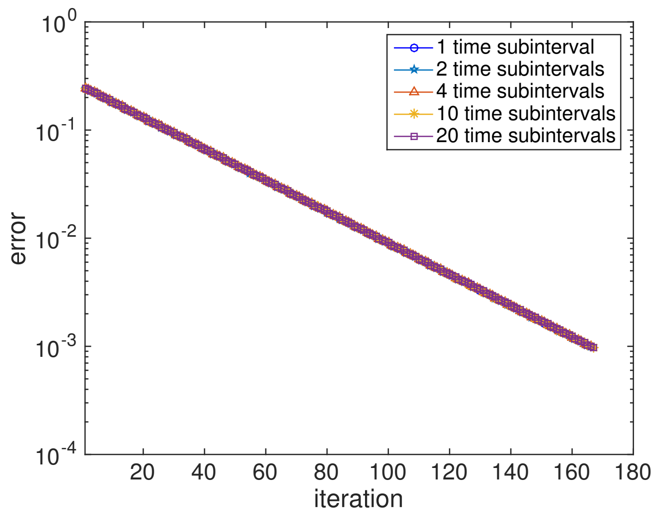

Fig. 1: Dependence of the PSWR-IC algorithm for the time-periodic problem (4) on the number of time subintervals.

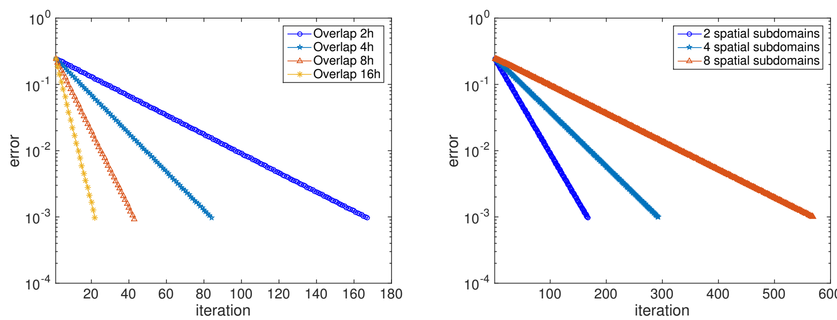

Fig. 2: Dependence of the PSWR-IC algorithm for the time-periodic problem (4) on the overlap (left), and on the number of spatial subdomains (right).

linearly, which is contrast to that of the PSWR algorithm for the initial value problem with the superlinear convergence.

We next study the dependence on the overlap. We use $L = 2h$, $4h$, $8h$ and $16h$, and divide the time interval $(0, T)$ with $T = 1$ into 10 time subintervals, still using the same two subdomain decomposition of Ω as before. We see on the left in Figure 2 that increasing the overlap substantially improves the convergence speed of the algorithm. This increases however also the cost of the method, since bigger subdomain problems need to be solved.

We then investigate numerically if a similar convergence result we derived for two subdomains also holds for the case of many subdomains. We decompose the domain

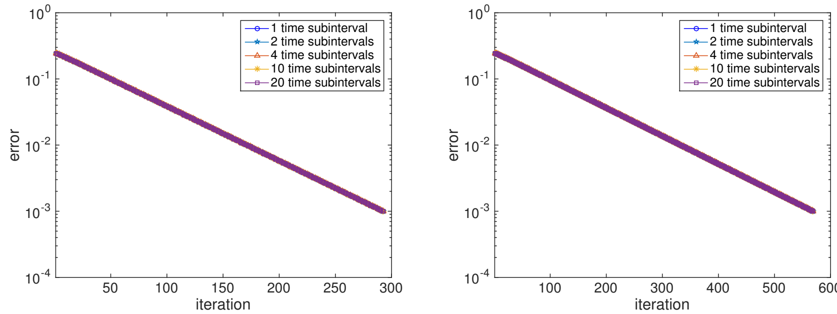

Fig. 3: Independence of the PSWR-IC algorithm on the number of time subintervals for four spatial subdomains (left), and eight spatial subdomains (right).

Ω into 2, 4, and 8 spatial subdomains, keeping again the overlap $L = 2h$. For each case, we divide the time interval $(0, T)$ with $T = 1$ into 10 time subintervals. We see in Figure 2 on the right that using more spatial subdomains makes the algorithm converge more slowly, like the PSWR algorithm for the initial value problem.

We further investigate whether the convergence of the algorithm still does not depend on the number of time subintervals for the case of many subdomains. We see in Figure 3 that the convergence behavior for four spatial subdomains (left), and eight spatial subdomains (right) is the same as the convergence behavior for two spatial subdomains.

Finally, we compare the convergence behavior of the PSWR-IC algorithm for the time-periodic problem (4) with Dirichlet and optimized transmission conditions. Using optimized transmission conditions leads to much faster, so called optimized Schwarz waveform relaxation methods, see for example [6, 3]. We divide the time interval $(0, T)$ with $T = 1$ into 10 time subintervals, and the domain Ω is decomposed into 2, 4 and 8 spatial subdomains. We use first order transmission conditions and choose for the parameters $p = 1$, $q = 1.75$ (for the terminology, see [3]), which is the same as optimized Schwarz waveform relaxation and optimized PSWR for initial value problem. In Figure 4, we show the corresponding convergence curves show that using optimized transmission conditions of these parameters even could not converge. Then we chose numerically optimized parameters $p = 10.5$, $q = 0$, which leads to substantially better performance of the PSWR-IC algorithm, even better than very generous overlap, and this at no additional cost. We also investigate the dependence on the number of time subintervals (on the right in Figure 5), where we choose the problem configuration as in the case of the Dirichlet transmission conditions in Figure 1. We observe that convergence is much faster with optimized transmission conditions (less than 10 iterations instead of over 100), and convergence is still linear, indicating that there is a different convergence mechanism dominating now, due to the optimized transmission conditions.

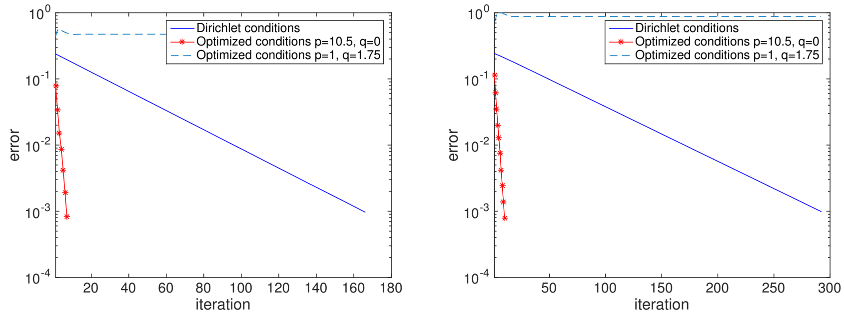

Fig. 4: Comparison of the PSWR-IC algorithm with Dirichlet and optimized transmission conditions for two spatial subdomains (left) and and four spatial subdomains (right).

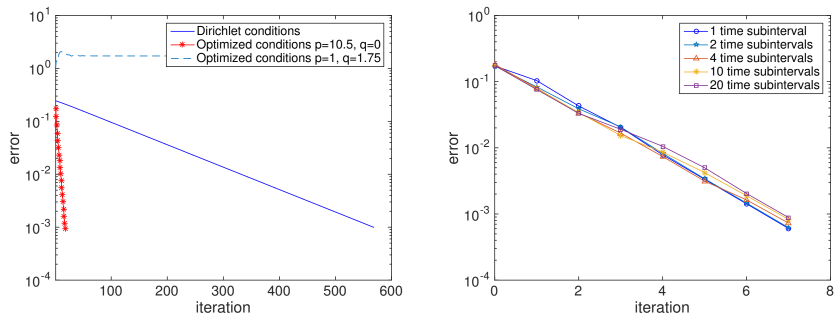

Fig. 5: Left: comparison of the PSWR-IC algorithm with Dirichlet and optimized transmission conditions for eight spatial subdomains. Right: dependence of the PSWR-IC algorithm on the number of time subintervals with optimized transmission conditions.

4 Conclusions

We designed a new parareal PSWR algorithm for time-periodic problems, i.e., the PSWR-IC algorithm. This algorithm is based on a domain decomposition of the entire space-time domain into smaller space-time subdomains, i.e., the decomposition is both in space and in time. The new algorithm iterates on these space-time subdomains using two different updating mechanisms: the Schwarz waveform relaxation approach for boundary condition updates, and the parareal mechanism for initial condition updates. All space-time subdomains are solved in parallel, both in space and in time. For the time-periodic problem, in particular, we use the periodic parareal algorithm with initial-value coarse problem to update initial condition at $t = 0$. The numerical results illustrate that the PSWR-IC algorithm converges linearly on bounded time intervals when using Dirichlet transmission conditions in space which is contrast to PSWR for initial value problem with the superlinear convergence, and optimized transmission conditions improve the convergence behavior significantly.

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References

1. Bachinger, F., Langer, U., Schöberl, J.: Efficient solvers for nonlinear time-periodic eddy current problems. Computing and Visualization in Science **9**(4), 197–207 (2006)
2. Bast, D., Kulchytska-Ruchka, I., Schöps, S., Rain, O.: Accelerated steady-state torque computation for induction machines using parallel-in-time algorithms. IEEE Transactions on Magnetics **56**(2), 1–9 (2020)
3. Bennequin, D., Gander, M.J., Halpern, L.: A homographic best approximation problem with application to optimized Schwarz waveform relaxation. Mathematics of Computation **78**(265), 185–223 (2009)
4. Bui, D.Q., Japhet, C., Maday, Y., Omnes, P.: Coupling parareal with optimized schwarz waveform relaxation for parabolic problems. hal.inria.fr/hal-03167571 (2021)
5. Gander, M.J.: 50 years of time parallel time integration. In: Multiple Shooting and Time Domain Decomposition Methods, pp. 69–113. Springer (2015)
6. Gander, M.J., Halpern, L.: Optimized Schwarz waveform relaxation methods for advection reaction diffusion problems. SIAM J. Numer. Anal. **45**(2), 666–697 (2007)
7. Gander, M.J., Jiang, Y.L., Li, R.J.: Parareal Schwarz waveform relaxation methods. In: Domain Decomposition Methods in Science and Engineering XX, pp. 451–458. Springer (2013)
8. Gander, M.J., Jiang, Y.L., Song, B.: A superlinear convergence estimate for the parareal Schwarz waveform relaxation algorithm. SIAM Journal on Scientific Computing **41**(2), A1148–A1169 (2019)
9. Gander, M.J., Jiang, Y.L., Song, B., Zhang, H.: Analysis of two parareal algorithms for time-periodic problems. SIAM Journal on Scientific Computing **35**(5), A2393–A2415 (2013)
10. Gander, M.J., Stuart, A.M.: Space-time continuous analysis of waveform relaxation for the heat equation. SIAM Journal on Scientific Computing **19**(6), 2014–2031 (1998)
11. Kulchytska-Ruchka, I., Schöps, S.: Efficient parallel-in-time solution of time-periodic problems using a multiharmonic coarse grid correction. SIAM Journal on Scientific Computing **43**(1), C61–C88 (2021)
12. Lions, J.L., Maday, Y., Turinici, G.: A "parareal" in time discretization of PDE's. Comptes Rendus de l'Académie des Sciences-Series I-Mathematics **332**(7), 661–668 (2001)
13. Maday, Y., Turinici, G.: The parareal in time iterative solver: a further direction to parallel implementation. Lecture Notes in Computational Science and Engineering **40**, 441–448 (2005)
14. van Noorden, T.L., Lunel, S.V., Bliek, A.: The efficient computation of periodic states of cyclically operated chemical processes. IMA Journal of Applied Mathematics **68**(2), 149–166 (2003)
15. van de Rotten, B.A., Lunel, S.M.V., Bliek, A.: Efficient simulation of periodically forced reactors with radial gradients. Chemical engineering science **61**(21), 6981–6994 (2006)
16. Song, B., Jiang, Y.L.: Parareal Schwarz waveform relaxation method for the heat equation. in preparation (2021)