Optimized Schwarz Method in Time for Transport Control

Duc-Quang Bui, Bérangère Delourme, Laurence Halpern, and Felix Kwok

1 Introduction

Parallel-in-time methods for solving optimal control problems under time-dependent PDE constraints have gained much interest in the past decade (see, e.g., ParaOpt [5]). Among all the possible approaches, it is natural to consider Schwarz time domain decomposition techniques when one deals with transport equations, since the original control problem is equivalent to an elliptic problem in which the initial and target conditions play the role of boundary conditions (see e.g. [1]).

In this paper, we consider the following one-dimensional transport control problem. Let $T > 0$, and let y_{ini} and y_{tar} be two periodic functions in $\in L^2_{\text{loc}}(\mathbb{R})$ with period one. We want to find a control $v \in L^2_{\text{loc}}(\mathbb{R} \times (0, T))$, periodic in space of period one, such that the function y defined by

$$\begin{cases} \partial_t y + \partial_x y = v & \text{in } \mathbb{R} \times (0, T), \\ y(., 0) = y_{\text{ini}}, \end{cases} \tag{1}$$

verifies the exact constraint

$$y(., T) = y_{\text{tar}}. \tag{2}$$

Over all the possible controls v, we shall seek the one with minimal L^2-norm, namely, we minimize the functional

$$J(v) = \frac{1}{2} \int_0^T \|v\|^2_{L^2(0,1)}. \tag{3}$$

Duc Quang Bui, Bérangère Delourme, Laurence Halpern
Université Sorbonne Paris Nord, Villetaneuse, France, e-mail: bui@math.univ-paris13.fr, delourme@math.univ-paris13.fr, halpern@math.univ-paris13.fr

Felix Kwok
Université Laval, Québec, Canada, e-mail: felix.kwok@mat.ulaval.ca

The optimization problem (1)-(2)-(3) admits a unique solution v_* that can be deduced from the following optimality system: find (y, λ), 1-periodic in space, such that

$$\begin{cases} \partial_t y + \partial_x y = \lambda & \text{in } \mathbb{R} \times (0, T), \\ \partial_t \lambda + \partial_x \lambda = 0 & \text{in } \mathbb{R} \times (0, T), \\ y(.,0) = y_{\text{ini}}, \\ y(.,T) = y_{\text{tar}}, \end{cases} \qquad v_* = \lambda. \tag{4}$$

2 Domain decomposition in time for the continuous problem

We apply Schwarz-in-time domain decomposition methods to (4). To do so, we decompose the time interval $(0, T)$ into two subdomains $(0, T_1)$ and (T_1, T) with $T_1 = \Delta T = \frac{T}{2}$. To start with, we solve the system (4) using the optimized Schwarz method with Robin transmission conditions on the interface $t = \Delta T$, with a single parameter $\mathfrak{p}$. More specifically, at iteration k, the functions y_1^k and λ_1^k (resp. y_2^k and λ_2^k) are solutions to (4) on $(0, T_1)$ (resp (T_1, T)) together with the following boundary condition:

$$\mathfrak{p} y_1^k + \lambda_1^k = \mathfrak{p} y_2^{k-1} + \lambda_2^{k-1}, \quad -\mathfrak{p} y_2^k + \lambda_2^k = -\mathfrak{p} y_1^{k-1} + \lambda_1^{k-1}. \tag{5}$$

Theorem 1 *Let* $\mathfrak{p} = \frac{1}{\Delta T}$. *Then the Schwarz iterative algorithm based on* (5) *and applied to the system* (4) *converges after* 1 *iteration.*

The theorem is proven by calculating explicitly the solutions of the sub-domain problems. We point out that in [6], a convergence proof using energy estimates has been given for all $\mathfrak{p} > 0$. On the other hand, to our knowledge, there has not been a detailed analysis of the convergence factor on the corresponding discrete systems (see [4, 7] for a convergence proof for semi-discrete schemes in the parabolic case). Understanding the behaviour of the discrete systems is the subject of the next sections.

3 Time-domain decomposition for a discrete problem

3.1 Discrete control problem

To discretize our problem, we consider a spatial discretization based on the upwind scheme with N uniform nodes and a mesh size of $\Delta x = 1/N$. We denote by $\mathcal{A}_{\Delta x} \in \mathcal{M}_N(\mathbb{R})$ the corresponding matrix: its diagonal terms are Δx^{-1}, its lower sub-diagonal ones are equal to $-\Delta x^{-1}$, and $[\mathcal{A}_{\Delta x}]_{1,N} = -\Delta x^{-1}$ (to take into account the periodicity), and zero coefficients elsewhere. The time discretization is made using the semi-implicit Euler scheme (explicit in y and implicit in v), using $M + 1$

uniform nodes on $[0, T]$ and a mesh size of $\Delta t = \frac{T}{M}$. We denote by $\mathbf{y}_{\text{ini}}$, $\mathbf{y}_{\text{tar}}$ (vectors of $\mathbb{R}^N$), the discretization of y_{ini} and y_{tar}. We mimic the continuous minimization problem (1)-(2)-(3) by considering the following discrete one:

$$\min_{\mathbf{v}=(\mathbf{v}_i^n)\in\mathbb{R}^{N\times M}} J(\mathbf{v}) = \frac{1}{2}\Delta t\,\Delta x\,\|\mathbf{v}\|^2, \tag{6}$$

where the control $\mathbf{v} = (\mathbf{v}^1, \ldots, \mathbf{v}^M)$ is such that $\mathbf{y} = (\mathbf{y}^0, \ldots, \mathbf{y}^M) \in (\mathbb{R}^N)^{M+1}$ satisfies

$$\begin{cases} \dfrac{\mathbf{y}^m - \mathbf{y}^{m-1}}{\Delta t} + \mathcal{A}_{\Delta x}\mathbf{y}^{m-1} = \mathbf{v}^m \quad m = 1, \ldots, M, \\ \mathbf{y}^0 = \mathbf{y}_{\text{ini}}, \end{cases} \tag{7}$$

as well as the target constraint

$$\mathbf{y}^M = \mathbf{y}_{\text{tar}}. \tag{8}$$

In the problem (6), $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^{N\times M}$. As in the continuous case, Problem (6)-(7)-(8) admits a unique solution $\mathbf{v}_*^m = \boldsymbol{\lambda}^m$, where $(\mathbf{y}^m, \boldsymbol{\lambda}^m)$ is the solution of the following optimality system (see [3]):

$$\begin{cases} \mathbf{y}^m - (\mathcal{I} - \Delta t\,\mathcal{A}_{\Delta x})\mathbf{y}^{m-1} = \Delta t\boldsymbol{\lambda}^m \; m = 1, \ldots, M, \\ \boldsymbol{\lambda}^{m-1} - (\mathcal{I} - \Delta t\,\mathcal{A}_{\Delta x}^t)\boldsymbol{\lambda}^m = 0 \qquad m = 1, \ldots, M, \\ \mathbf{y}^0 = \mathbf{y}_{\text{ini}}, \\ \mathbf{y}^M = \mathbf{y}_{\text{tar}}. \end{cases} \tag{9}$$

In the sequel, in order to guarantee the convergence of the scheme, we shall consider the standard relation between Δt and Δx given by

$$\frac{\Delta t}{\Delta x} = r, \tag{10}$$

where r is a given real parameter in $(0, 1)$.

3.2 Schwarz domain decomposition

We apply the Schwarz method strategy (5) to the system (9). For the sake of simplicity, let us consider $M = 2L$, so that the interface $T/2$ corresponds exactly to the node L. The algorithm then reads: starting from an initial guess $(\boldsymbol{\xi}_1^0, \boldsymbol{\xi}_2^0) \in \mathbb{R}^{2N}$, at each iteration $k \geq 1$, we construct $(\mathbf{y}_1^{k,m}, \boldsymbol{\lambda}_1^{k,m})$ (respectively $(\mathbf{y}_2^{k,m}, \boldsymbol{\lambda}_2^{k,m})$) solution to (9) for $m = 1, \ldots, L$ (resp. $m = L+1, \ldots, M$) together with the transmission conditions

$$\mathfrak{p}\mathbf{y}_1^{k,L} + \boldsymbol{\lambda}_1^{k,L} = \boldsymbol{\xi}_1^{k-1}, \quad -\mathfrak{p}\mathbf{y}_2^{k,L} + \boldsymbol{\lambda}_2^{k,L} = \boldsymbol{\xi}_2^{k-1}. \tag{11}$$

Then, we update $\boldsymbol{\xi}_1^k$ by taking

$$\boldsymbol{\xi}_1^k = \mathfrak{p}\mathbf{y}_2^{k,L} + \boldsymbol{\lambda}_2^{k,L}, \quad \boldsymbol{\xi}_2^k = -\mathfrak{p}\mathbf{y}_1^{k,L} + \boldsymbol{\lambda}_1^{k,L}. \tag{12}$$

Remark 1 The local subdomain problems are indeed optimality systems associated with local control problems (see [6]).

The convergence analysis of the algorithm (9)-(11)-(12) relies on the Discrete Fourier transform in space $\mathbb{R}^N \to \mathbb{R}^N$, $(u_1, \ldots, u_{N-1}) \mapsto (\hat{u}_0, \ldots \hat{u}_{N-1})$ defined by $\hat{u}_\ell = \sum_{n=0}^{N-1} u_n \exp(-2\pi \mathrm{i} \ell n \Delta x)$. Indeed, (9)-(11)-(12) can be transformed as follows: at iteration k, in subdomain Ω_i, for any ℓ between 0 and $N-1$ (spatial frequency), $\hat{y}_{i,\ell}^{k,m}$ (with m denoting the time step) solves

$$\begin{cases} \hat{y}_{i,\ell}^{k,m} - (1 - \sigma(\ell)\,\Delta t)\,\hat{y}_{i,\ell}^{k,m-1} = \Delta t \hat{\lambda}_{i,\ell}^{k,m}, \\ (1 - \overline{\sigma(\ell)}\Delta t)\hat{\lambda}_{i,\ell}^{k,m} - \hat{\lambda}_{i,\ell}^{k,m-1} = 0, \end{cases} \quad \text{where} \ \ \sigma(\ell) = \frac{1 - \exp(-2\pi \mathrm{i} \ell \Delta x)}{\Delta x}, \tag{13}$$

together with boundary conditions

$$\begin{cases} \hat{y}_{1,\ell}^{k,0} = \hat{y}_{\mathrm{ini},\ell}, \\ \mathfrak{p}\hat{y}_{1,\ell}^{k,L} + \hat{\lambda}_{1,\ell}^{k,L} = \hat{\xi}_{1,\ell}^{k-1}, \end{cases} \qquad \begin{cases} -\mathfrak{p}\hat{y}_{2,\ell}^{k,L} + \hat{\lambda}_{2,\ell}^{k,L} = \hat{\xi}_{2,\ell}^{k-1}, \\ \hat{y}_{2,\ell}^{k,M} = \hat{y}_{\mathrm{tar},\ell}. \end{cases} \tag{14}$$

Then,

$$\hat{\xi}_{1,\ell}^{k} = \mathfrak{p}\hat{y}_{2,\ell}^{k,L} + \hat{\lambda}_{2,\ell}^{k,L}, \quad \hat{\xi}_2^{k} = -\mathfrak{p}\hat{y}_{1,\ell}^{k,L} + \hat{\lambda}_{1,\ell}^{k,L}. \tag{15}$$

As the problem is linear, the convergence analysis of the algorithm reduces to investigating the case $\hat{y}_{\mathrm{ini},\ell} = \hat{y}_{\mathrm{tar},\ell} = 0$, starting from given data $\hat{\xi}_{1,\ell}^0$ and $\hat{\xi}_{2,\ell}^0$. Eliminating $\hat{y}_{i,\ell}^{k,0}$ and $\hat{\lambda}_{i,\ell}^{k,0}$ by solving explicitly the recurrence equations (13)–(15), we see that $\hat{\xi}_{i,\ell}^{k+2}$ follows the geometric progression

$$\hat{\xi}_{i,\ell}^{k+2} = \rho_{\Delta t}(\mathfrak{p}, \ell)\,\hat{\xi}_{i,\ell}^{k} \quad \text{with} \quad \rho_{\Delta t}(\mathfrak{p}, \ell) = \left(\frac{1 - \mathfrak{p}\gamma_{\Delta t}(\ell)}{1 + \mathfrak{p}\gamma_{\Delta t}(\ell)}\right)\left(\frac{|\beta_{\Delta t}(\ell)|^2 - \mathfrak{p}\gamma_{\Delta t}(\ell)}{|\beta_{\Delta t}(\ell)|^2 + \mathfrak{p}\gamma_{\Delta t}(\ell)}\right),$$

where $\beta_{\Delta t}(\ell) = (1 - \sigma(\ell)\Delta t)^L$, and $\gamma_{\Delta t}(\ell) = \Delta t \sum_{m=0}^{L-1} |1 - \sigma(\ell)\Delta t|^{2m}$. As in [2], our objective is to minimize $|\rho_{\Delta t}|$ uniformly in ℓ, namely, to solve the problem

$$\min_{\mathfrak{p}>0} \left(\max_{\ell=0,\ldots,N-1} |\rho_{\Delta t}(\mathfrak{p}, \ell)| \right). \tag{16}$$

To analyse (16), and in view of Theorem 1, we first make the change of variables $p = \mathfrak{p}\Delta T$. Then, under the assumption (10), we see that

$$|1 - \sigma(\ell)\Delta t|^2 = 1 - 4 \cdot \frac{\Delta t}{\Delta x}\left(1 - \frac{\Delta t}{\Delta x}\right)\sin^2(\pi \ell \Delta x) = 1 - 4r(1-r)\sin^2\left(\pi \ell \frac{\Delta t}{r}\right).$$

It motivates us to introduce the new variable

$$z = 4r(1-r)\sin^2\left(\pi\ell\frac{\Delta t}{r}\right),$$

which varies between 0 and $z_{\max} = 4r(1-r)$ (take $\ell = N/2$) as ℓ varies from 0 to $N-1$. For the sake of simplicity, we choose to optimize $\rho_{\Delta t}$ over the whole interval $[0, z_{\max}]$ and to study

$$\min_{p>0}\left(\max_{0\le z\le z_{\max}}|\rho_{\Delta t}(p,z)|\right) \quad \rho_{\Delta t}(p,z) = \frac{\varphi_{\Delta t}(z)-p}{\varphi_{\Delta t}(z)+p}\cdot\frac{\psi_{\Delta t}(z)-p}{\psi_{\Delta t}(z)+p} \tag{17}$$

with

$$\varphi_{\Delta t}(z) = \frac{\Delta T}{\gamma_{\Delta t}(z)}, \quad \psi_{\Delta t}(z) = \frac{|\beta_{\Delta t}(z)|^2\Delta T}{\gamma_{\Delta t}(z)},$$

and $|\beta_{\Delta t}(z)|^2 = (1-z)^L$, $\gamma_{\Delta t}(z) = \Delta t\sum_{m=0}^{L-1}(1-z)^m$.

4 Existence, uniqueness and asymptotic study of the optimized parameter

The following theorem proves the well-posedness of the problem (17) and describes the asymptotic behaviour of the optimal convergence factor as Δt goes to 0.

Theorem 2 *For any $\Delta t > 0$, Problem* (17) *has a unique solution $p^*_{\Delta t}$, which is the unique solution larger than* 1 *of the following alternation equation*

$$\max_{0\le z\le z_{\max}}\rho_{\Delta t}(p,z) = -\min_{0\le z\le z_{\max}}\rho_{\Delta t}(p,z). \tag{18}$$

Moreover, as Δt goes to 0*,*

$$p^*_{\Delta t} = \sqrt{2\Delta T\, z_{\max}}\;\Delta t^{-1/2} + o\left(\Delta t^{-1/2}\right), \tag{19}$$

$$\max_{0\le z\le z_{\max}}|\rho_{\Delta t}(p^*_{\Delta t},z)| = 1 - \frac{2\sqrt{2}}{\sqrt{\Delta T\, z_{\max}}}\;\Delta t^{1/2} + o\left(\Delta t^{1/2}\right). \tag{20}$$

Remark 2 In (19)-(20), $o(\Delta t^s)$ (with $s = \pm 1/2$) means that the remainder is negligible relative to Δt^s. We also point out that, unless $r = 1$ (in which case the scheme is exact), we have $\lim_{\Delta t\to 0} p^*_{\Delta t} \neq 1$, meaning we do not recover the optimal parameter associated with the continuous DD algorithm.

The remainder of this section is dedicated to the sketch of the proof of Theorem 2.

Step 1: *We prove that the alternation Equation* (18) *has a unique solution $p^*_{\Delta t}$ larger than 1.* Let us introduce

$$\rho_{\Delta t,\max}(p) = \max_{0\le z\le z_{\max}} \rho_{\Delta t}(p,z), \quad \rho_{\Delta t,\min}(p) = \min_{0\le z\le z_{\max}} \rho_{\Delta t}(p,z),$$

and the function $s(p) = \rho_{\Delta t,\max}(p) + \rho_{\Delta t,\min}(p)$. We prove that s has a unique zero larger than one (and, consequently (18) has a unique root). Indeed,

- For $p > 1$, the function s is a continuous and strictly increasing function of p. In fact, for $p > 1$, a direct computation shows that $\partial_p \rho_{\Delta t}(p,z) > 0$. Therefore, $\rho_{\Delta t,\max}$, $\rho_{\Delta t,\min}$, and their sum s are strictly increasing functions of p.
- $s(1) < 0$ ($\rho_{\Delta t,\max}(1) \le 0$ and $\rho_{\Delta t,\min}(1) < 0$).
- $s(\varphi_{\Delta t}(z_{\max})) > 0$ ($\rho_{\Delta t,\max}(\varphi_{\Delta t}(z_{\max})) > 0$ and $\rho_{\Delta t,\min}(\varphi_{\Delta t}(z_{\max})) = 0$).

Thus, (18) has a unique solution $p^*_{\Delta t} > 1$.

Step 2: *We show that $p^*_{\Delta t}$ is the unique solution to Problem* (17). First, based on the properties of $\varphi_{\Delta t}$ and $\psi_{\Delta t}$, we can prove (by contradiction) that any solution p of (17) must be in the interval $(1, \varphi_{\Delta t}(z_{\max}))$. But,

- For $p \in (1, p^*_{\Delta t})$, a careful investigation leads to

$$\max_{0\le z\le z_{\max}} |\rho_{\Delta t}(p,z)| = -\rho_{\Delta t,\min}(p) > -\rho_{\Delta t,\min}(p^*_{\Delta t}) = \max_{0\le z\le z_{\max}} |\rho_{\Delta t}(p^*_{\Delta t},z)|.$$

- Similarly, for $p \in (p^*_{\Delta t}, \varphi_{\Delta t}(z_{\max}))$, we obtain

$$\max_{0\le z\le z_{\max}} |\rho_{\Delta t}(p,z)| = \rho_{\Delta t,\max}(p) > \rho_{\Delta t,\max}(p^*_{\Delta t}) = \max_{0\le z\le z_{\max}} |\rho_{\Delta t}(p^*_{\Delta t},z)|.$$

Therefore, $p^*_{\Delta t}$ is the unique global minimum of (17).

Step 3: *Asymptotics of the optimal parameter $p^*_{\Delta t}$ and its corresponding convergence factor with respect to Δt.* We first remark that Equation (18) is defined implicitly in p, so it is a priori difficult to tackle directly. However, we can approximate $\rho_{\Delta t,\max}(p)$ by $\rho_{\Delta t}(p,0)$: indeed, an attentive analysis shows that there exists $\Delta t_0 > 0$ and a constant C such that for $\Delta t < \Delta t_0$,

$$|\rho_{\Delta t,\max}(p) - \rho_{\Delta t}(p,0)| \le C p^{-1}\Delta t. \tag{21}$$

Consequently, for small Δt, it is sufficient to consider the 'approximate' equation

$$\rho_{\Delta t}(p,0) = -\rho_{\Delta t}(p,z_{\max}), \tag{22}$$

which turns out to be explicitly solvable. Its solution $p^*_{\mathrm{eq},\Delta t}$ is given by

$$p^*_{\mathrm{eq},\Delta t} = \left(S_{m,\Delta t} - \frac{P_{m,\Delta t}}{2} - \frac{1}{2} + \left(\left(S_{m,\Delta t} - \frac{P_{m,\Delta t}}{2} - \frac{1}{2} \right)^2 - P_{m,\Delta t} \right)^{1/2} \right)^{1/2},$$

where $S_{m,\Delta t} = \psi_{\Delta t}(z_{\max}) + \varphi_{\Delta t}(z_{\max})$ and $P_{m,\Delta t} = \psi_{\Delta t}(z_{\max})\varphi_{\Delta t}(z_{\max})$. From the asymptotic behaviour of $\psi_{\Delta t}(z_{\max})$ and $\varphi_{\Delta t}(z_{\max})$, we deduce that when $\Delta t \to 0$,

$$p^*_{\mathrm{eq},\Delta t} = \sqrt{2\Delta T\ z_{\max}}.\Delta t^{-1/2} + o\left(\Delta t^{-1/2}\right),$$

which implies

$$-\rho_{\Delta t}(p^*_{\mathrm{eq},\Delta t}, z_{\max}) = \rho_{\Delta t}(p^*_{\mathrm{eq},\Delta t}, 0) = 1 - \frac{2\sqrt{2}}{\sqrt{\Delta T\ z_{\max}}} \cdot \Delta t^{1/2} + o\left(\Delta t^{1/2}\right).$$

Finally, the asymptotic formulas (19)–(20) result from (21).

5 Numerical illustration

We illustrate the results of Theorem 2 in the case of $T = 1$. In the left panel of Figure 1, we plot $1 - |\rho_{\Delta t}|_{\max}(p^*_{\Delta t})$ with respect to Δt (in logarithmic scale) for three different values of r. In each case, the optimized parameter $p^*_{\Delta t}$ is computed using `fminsearch` in Matlab. As expected, whatever the choice of $r \in (0, 1)$, we obtain straight lines with slope equal to that of the curve $y = \sqrt{\Delta t}$.

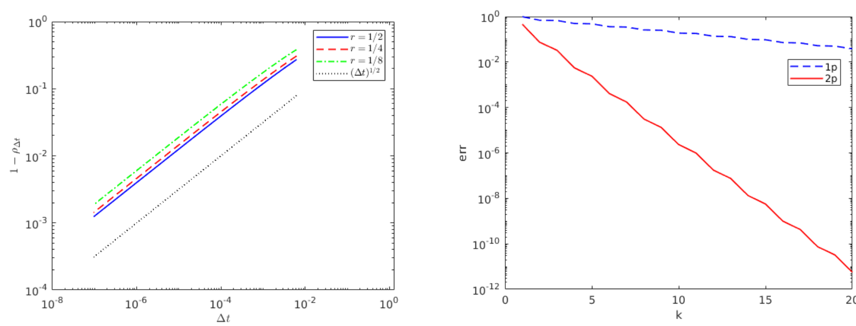

Fig. 1 Left: Asymptotic behaviour of $1 - |\rho_{\Delta t}|_{\max}(p^*_{\Delta t})$. Right: performance of $p^*_{\Delta t}$ for $\Delta t = 1/160$, $r = 1/2$.

Next, we test the performance of our domain decomposition-in-time algorithm. For the simulation, we take $\Delta t = 1/160$, $r = 1/2$, $y_{\mathrm{ini}} = y_{\mathrm{tar}} = 0$, and we start from a random initial guess $\boldsymbol{\xi}_i^0$ (i.e. we compute the zero solution). In the right panel of Figure 1, we display in blue the evolution of the error with respect to the number of iterations; in the present case, it just consists of computing the maximum of the L^2 norm of $\boldsymbol{\xi}_1^k$ and $\boldsymbol{\xi}_2^k$. The performance is as predicted by the theory. On the other hand, the convergence rate can be drastically improved by using a two-sided algorithm, where we allow for two different values p and q instead of $\mathfrak{p}$

in the formulas (5). The `fminsearch` function provides us with two optimized parameters $(p^*_{\Delta t}, q^*_{\Delta t}) = (1.1831, 8.5024 \times 10^{-2})$, leading to a convergence factor of 7.0728×10^{-2}. The performance of the two-sided algorithm for this value is displayed in red, and appears to be much better than the optimized one-sided one. The proof of that result will be given in a forthcoming publication.

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