Optimized Neumann-Neumann Method for the Stokes-Darcy Problem

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1 Introduction and problem setting

The Stokes-Darcy problem [9, 15] is a good example of multi-physics problem where splitting methods typical of domain decomposition naturally apply. The problem is defined in a computational domain formed by a fluid region Ω_f and a porous-medium region Ω_p that are non-overlapping and separated by an interface Γ. In Ω_f, an incompressible fluid with constant viscosity and density is modelled by the dimensionless Stokes equations:

$$-\boldsymbol{\nabla} \cdot (2\mu_f \nabla^s \mathbf{u}_f - p_f \boldsymbol{I}) = \mathbf{f}_f \,, \qquad \nabla \cdot \mathbf{u}_f = 0 \quad \text{in } \Omega_f \,, \tag{1}$$

where $\mu_f = Re^{-1}$, Re being the Reynolds number, $\mathbf{u}_f$ and p_f are the fluid velocity and pressure, $\boldsymbol{I}$ and $\nabla^s \mathbf{u}_f = \frac{1}{2}(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T)$ are the identity and the strain rate tensor, and $\mathbf{f}_f$ is an external force. In the porous medium domain Ω_p, we consider the dimensionless Darcy's model:

$$-\nabla \cdot (\boldsymbol{\eta}_p \nabla p_p) = f_p \qquad \text{in } \Omega_p \,, \tag{2}$$

where p_p is the fluid pressure in the porous medium, $\boldsymbol{\eta}_p$ is the permeability tensor, and f_p is an external force. The two local problems are coupled through the classical Beaver-Joseph-Saffman conditions at the interface [1, 14, 17]:

$$\mathbf{u}_f \cdot \mathbf{n} = -(\boldsymbol{\eta}_p \nabla p_p) \cdot \mathbf{n} \quad \text{on } \Gamma, \tag{3}$$

$$-\mathbf{n} \cdot (2\mu_f \nabla^s \mathbf{u}_f - p_f \boldsymbol{I}) \cdot \mathbf{n} = p_p \quad \text{on } \Gamma, \tag{4}$$

$$-((2\mu_f \nabla^s \mathbf{u}_f - p_f \boldsymbol{I}) \cdot \mathbf{n})_\tau = \xi_f (\mathbf{u}_f)_\tau \quad \text{on } \Gamma, \tag{5}$$

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where $\xi_f = \alpha_{BJ} \, (\mu_f / (\boldsymbol{\tau} \cdot \boldsymbol{\eta}_p \cdot \boldsymbol{\tau}))^{1/2}$, α_{BJ} is the Beavers-Joseph constant, $\mathbf{n}$ denotes the unit normal vector pointing outward of Ω_f, while $(\mathbf{v})_\tau$ indicates the tangential component of any vector $\mathbf{v}$ at Γ. Finally, we impose $\mathbf{u}_f = \mathbf{0}$ on Γ_f^D, $(2\mu_f \nabla^s \mathbf{u}_f - p_f \boldsymbol{I}) \cdot \mathbf{n} = \mathbf{0}$ on Γ_f^N, $p_p = 0$ on Γ_p^D, $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p^N, where $\Gamma_f^D \cup \Gamma_f^N = \partial\Omega_f \setminus \Gamma$ and $\Gamma_p^D \cup \Gamma_p^N = \partial\Omega_p \setminus \Gamma$.

Classical Dirichlet-Neumann type methods [16] for the Stokes-Darcy problem were studied in [7, 9, 10] where it was pointed out that their convergence can be slow for small values of the fluid viscosity and of the porous medium permeability. Robin-Robin methods were then proposed as an alternative [3, 4, 5, 6, 7, 11], and they were analysed in the framework of optimized Schwarz methods in [8, 12, 13].

In this work, we focus on a Neumann-Neumann approach that allows to solve a scalar interface problem like in the case of Dirichlet-Neumann methods. This reduces the number of interface unknowns compared to the system associated with Robin-Robin iterations, and it allows to use preconditioned conjugate gradient (PCG) iterations instead of the more expensive GMRES iterations used in the Robin-Robin context (see, e.g., [8]). However, to define effective Neumann-Neumann methods, the contribution of each subproblem must be suitably weighted. For single-physics problems, this is typically done using algebraic strategies that can take into account coefficient jumps across interfaces (see, e.g., [18]). However, no clear strategies are available for multi-physics problems. In this work, we extend techniques for the analysis of optimized Schwarz methods with the aim of characterizing optimal weighting parameters to define a robust Neumann-Neumann preconditioner.

2 Optimized Neumann-Neumann method

Let α_f and α_p be two positive parameters: $\alpha_f, \alpha_p \in \mathbb{R}$, $\alpha_f, \alpha_p > 0$. The Neumann-Neumann method for the Stokes-Darcy problem considering the normal velocity on Γ as interface variable reads as follows. Given λ^0 on Γ, for $m \geq 1$ until convergence,

1. Find $\mathbf{u}_f^{(m)}$ and $p_f^{(m)}$ such that

$$\begin{aligned} -\boldsymbol{\nabla} \cdot (2\mu_f \nabla^s \mathbf{u}_f^{(m)} - p_f^{(m)} \boldsymbol{I}) = \mathbf{f}_f \,, \quad \nabla \cdot \mathbf{u}_f^{(m)} = 0 \quad & \text{in } \Omega_f \,, \\ -(\mathbf{n} \cdot (2\mu_f \nabla^s \mathbf{u}_f^{(m)} - p_f^{(m)} \boldsymbol{I}))_\tau = \xi_f \, (\mathbf{u}_f^{(m)})_\tau \quad & \text{on } \Gamma \,, \\ \mathbf{u}_f^{(m)} \cdot \mathbf{n} = \lambda^{(m-1)} \quad & \text{on } \Gamma \,. \end{aligned} \tag{6}$$

2. Find $p_p^{(m)}$ such that

$$\begin{aligned} -\nabla \cdot (\boldsymbol{\eta}_p \nabla p_p^{(m)}) = f_p \quad & \text{in } \Omega_p \,, \\ -(\boldsymbol{\eta}_p \nabla p_p^{(m)}) \cdot \mathbf{n} = \lambda^{(m)} \quad & \text{on } \Gamma \,. \end{aligned} \tag{7}$$

3. Compute

$$\sigma^{(m)} = -\mathbf{n} \cdot (2\mu_f \nabla^s \mathbf{u}_f^{(m)} - p_f^{(m)} \boldsymbol{I}) \cdot \mathbf{n} - p_p^{(m)} \quad \text{on } \Gamma \,. \tag{8}$$

4. Find $\mathbf{v}_f^{(m)}$ and $q_f^{(m)}$ such that

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(2\mu_f\nabla^s\mathbf{v}_f^{(m)}-q_f^{(m)}\boldsymbol{I}) = \mathbf{0}\,, \quad \nabla\cdot\mathbf{v}_f^{(m)} = 0 \qquad & \text{in } \Omega_f\,,\\ -(\mathbf{n}\cdot(2\mu_f\nabla^s\mathbf{v}_f^{(m)}-q_f^{(m)}\boldsymbol{I}))_\tau = \xi_f\,(\mathbf{v}_f^{(m)})_\tau \quad & \text{on } \Gamma\,,\\ -\mathbf{n}\cdot(2\mu_f\nabla^s\mathbf{v}_f^{(m)}-q_f^{(m)}\boldsymbol{I})\cdot\mathbf{n} = \sigma^{(m)} \qquad & \text{on } \Gamma\,. \end{aligned} \tag{9}$$

5. Find $q_p^{(m)}$ such that

$$\begin{aligned} -\nabla\cdot(\boldsymbol{\eta}_p\nabla q_p^{(m)}) = 0 \qquad & \text{in } \Omega_p\,,\\ q_p^{(m)} = \sigma^{(m)} \ & \text{on } \Gamma\,. \end{aligned} \tag{10}$$

6. Set

$$\lambda^{(m+1)} = \lambda^{(m)} - (\alpha_f\,(\mathbf{v}_f^{(m)}\cdot\mathbf{n}) + \alpha_p(\boldsymbol{\eta}_p\nabla q_p^{(m)})\cdot\mathbf{n}) \quad \text{on } \Gamma\,. \tag{11}$$

Problems (6), (7), (9) and (10) are supplemented with homogeneous boundary conditions on $\partial\Omega_f\setminus\Gamma$ and $\partial\Omega_p\setminus\Gamma$ as indicated in Sect. 1.

2.1 Convergence analysis and optimization of the parameters

We analyse the Neumann-Neumann method (6)-(11) with the aim of characterizing optimal parameters α_f and α_p. To this purpose, we extend the methodology used to study optimized Schwarz methods for the Stokes-Darcy problem in [8, 12, 13]. Since all the problems are linear, we can study the convergence on the error equation to the zero solution when the forcing terms are $\mathbf{f}_f = \mathbf{0}$ and $f_p = 0$.

We consider the simplified setting where $\Omega_f = \{(x,y)\in\mathbb{R}^2\colon x<0\}$, $\Omega_p = \{(x,y)\in\mathbb{R}^2\colon x>0\}$, $\Gamma = \{(x,y)\in\mathbb{R}^2\colon x=0\}$, and $\mathbf{n}=(1,0)$ and $\boldsymbol{\tau}=(0,1)$. We assume $\boldsymbol{\eta}_p = \mathrm{diag}(\eta_1,\eta_2)$ with constant $\eta_1\neq\eta_2$, and let $\mathbf{u}_f(x,y) = (u_1(x,y),u_2(x,y))^T$, $\mathbf{v}_f(x,y) = (v_1(x,y),v_2(x,y))^T$. In this setting, the Neumann-Neumann algorithm (6)–(11) becomes: given λ^0 on Γ, for $m\geq 1$ until convergence,

1. Solve the Stokes problem

$$\begin{aligned} -\mu_f\begin{pmatrix}(\partial_{xx}+\partial_{yy})u_1^{(m)}\\(\partial_{xx}+\partial_{yy})u_2^{(m)}\end{pmatrix} + \begin{pmatrix}\partial_x p_f^{(m)}\\ \partial_y p_f^{(m)}\end{pmatrix} = 0,\ \partial_x u_1^{(m)}+\partial_y u_2^{(m)} = 0,\ \text{in } (-\infty,0)\times\mathbb{R}\,,\\ -\mu_f\,(\partial_x u_2^{(m)}+\partial_y u_1^{(m)}) = \xi_f\ u_2^{(m)},\quad u_1^{(m)} = \lambda^{(m)},\ \text{on } \{0\}\times\mathbb{R}\,. \end{aligned} \tag{12}$$

2. Solve Darcy's problem

$$\begin{aligned} -(\eta_1\,\partial_{xx}+\eta_2\,\partial_{yy})\,p_p^{(m)} = 0 \qquad & \text{in } (0,+\infty)\times\mathbb{R}\,,\\ -\eta_1\partial_x p_p^{(m)} = \lambda^{(m)} \ & \text{on } \{0\}\times\mathbb{R}\,. \end{aligned} \tag{13}$$

3. Compute

$$\sigma^{(m)} = -2\mu_f\,\partial_x u_1^{(m)} + p_f^{(m)} - p_p^{(m)} \quad \text{on } \{0\}\times\mathbb{R}\,. \tag{14}$$

4. Solve the Stokes problem

$$
\begin{aligned}
-\mu_f \begin{pmatrix} (\partial_{xx}+\partial_{yy})v_1^{(m)} \\ (\partial_{xx}+\partial_{yy})v_2^{(m)} \end{pmatrix} + \begin{pmatrix} \partial_x q_f^{(m)} \\ \partial_y q_f^{(m)} \end{pmatrix} = 0,\ \partial_x v_1^{(m)} + \partial_y v_2^{(m)} = 0\,,\ \text{ in } (-\infty,0)\times\mathbb{R}\,, \\
-\mu_f\,(\partial_x v_2^{(m)} + \partial_y v_1^{(m)}) = \xi_f\ v_2^{(m)}\,,\ \text{ on } \{0\}\times\mathbb{R}\,, \\
-2\mu_f\,\partial_x v_1^{(m)} + q_f^{(m)} = \sigma^{(m)}\,,\ \text{ on } \{0\}\times\mathbb{R}\,.
\end{aligned}
\tag{15}
$$

5. Solve Darcy's problem

$$
\begin{aligned}
-(\eta_1\,\partial_{xx} + \eta_2\,\partial_{yy})\,q_p^{(m)} &= 0 \quad \text{ in } (0,+\infty)\times\mathbb{R}\,, \\
q_p^{(m)} &= \sigma^{(m)} \ \text{ on } \{0\}\times\mathbb{R}\,.
\end{aligned}
\tag{16}
$$

6. Set

$$
\lambda^{(m+1)} = \lambda^{(m)} - (\alpha_f\, v_1^{(m)} + \alpha_p\,\eta_1 \partial_x q_p^{(m)}) \quad \text{on } \{0\}\times\mathbb{R}\,. \tag{17}
$$

For the convergence analysis, we consider the Fourier transform in the direction tangential to the interface (corresponding to the y variable):

$$
\mathcal{F}:\, w(x,y) \mapsto \widehat{w}(x,k) = \int_{\mathbb{R}} e^{-iky} w(x,y)\,dy\,, \qquad \forall\, w(x,y) \in L^2(\mathbb{R}^2)\,,
$$

where k is the frequency variable. We quantify the error in the frequency space between two successive approximations $\widehat{\lambda}^{m+1}$ and $\widehat{\lambda}^{m}$ at Γ and characterize the reduction factor at iteration m for each frequency k. Finally, we identify optimal values of α_f and α_p by minimizing the reduction factor at each iteration over all the relevant Fourier modes.

Proposition 1 *Let* $\eta_p = \sqrt{\eta_1\eta_2}$. *The reduction factor of algorithm* (12)–(16) *does not depend on the iteration m, and it is given by* $|\rho(\alpha_f, \alpha_p, k)|$ *with*

$$
\rho(\alpha_f,\alpha_p,k) = 1 - \alpha_p(1 + 2\,\mu_f\ \eta_p\ k^2) - \alpha_f\,(1 + (2\,\mu_f\ \eta_p\ k^2\,)^{-1})\,. \tag{18}
$$

Proof Following the same steps of the proof of Proposition 3.1 of [8], we find

$$
\widehat{u}_1^{(m)}(x,k) = \left(U_1^{(m)}(k) + \frac{P^{(m)}(k)}{2\mu_f}\,x\right) e^{|k|x}\,, \qquad \widehat{p}_p^{(m)}(x,k) = \Phi^{(m)}(k)\, e^{-\sqrt{\frac{\eta_2}{\eta_1}}|k|x}\,,
$$

and $\widehat{p}_f^{(m)}(x,k) = P^{(m)}(k)\, e^{|k|x}$. The interface conditions $(12)_4$ and $(13)_2$ give $U_1^{(m)}(k) = \widehat{\lambda}^{(m)}$ and $\Phi^{(m)}(k) = \frac{\widehat{\lambda}^{(m)}}{\eta_p |k|}$. Then, using the Fourier transform of (14), we can obtain

$$
\widehat{\sigma}^{(m)} = -(2\mu_f\,|k| + (\eta_p|k|)^{-1})\,\widehat{\lambda}^{(m)}\,.
$$

Proceeding in analogous way, the solutions of problems (15) and (16) become

$$
\widehat{v}_1^{(m)}(x,k) = \left(\overline{P}^{(m)}(k)\,x - \frac{\widehat{\sigma}^{(m)}}{|k|}\right)\frac{e^{|k|x}}{2\mu_f}\,, \qquad \widehat{q}_p^{(m)}(x,k) = \widehat{\sigma}^{m} e^{-\sqrt{\frac{\eta_2}{\eta_1}}|k|x}\,,
$$

and $\widehat{q}_f^{(m)}(x,k) = \overline{P}^m(k)e^{|k|x}$. Substituting into the Fourier transform of (17), we find $\widehat{\lambda}^{(m+1)} = \rho(\alpha_f, \alpha_p, k)\widehat{\lambda}^{(m)}$ with $\rho(\alpha_f, \alpha_p, k)$ defined in (18). □

Using a classical approach in optimized Schwarz methods, we now aim at optimizing the parameters α_f and α_p by minimizing the reduction factor for all the relevant frequencies k with $0 < \underline{k} \leq |k| \leq \overline{k}$, where $\underline{k}$ and $\overline{k}$ are the minimum and maximum relevant frequencies, respectively, with $\underline{k} = \pi/L$ (L being the length of the interface) and $\overline{k} = \pi/h$ (h being the size of the mesh). Since the function $\rho(\alpha_f, \alpha_p, k)$ is even with respect to k, we only consider $k > 0$ without loss of generality, and we proceed to solve the min-max problem

$$\min_{\alpha_f, \alpha_p > 0} \max_{k \in [\underline{k}, \overline{k}]} |\rho(\alpha_f, \alpha_p, k)| . \tag{19}$$

The following result holds.

Proposition 2 *The solution of the min-max problem* (19) *is given by*

$$\begin{aligned} \alpha_f^{NN} &= (2\,\mu_f\,\eta_p\,\underline{k}\,\overline{k})^2\,(\,1 + (2\,\mu_f\,\eta_p\,\underline{k}\,\overline{k})^2 + \mu_f\,\eta_p\,(\underline{k}+\overline{k})^2\,)^{-1}\,, \\ \alpha_p^{NN} &= (\,1 + (2\,\mu_f\,\eta_p\,\underline{k}\,\overline{k})^2 + \mu_f\,\eta_p\,(\underline{k}+\overline{k})^2\,)^{-1}\,. \end{aligned} \tag{20}$$

Moreover, $|\rho(\alpha_f^{NN}, \alpha_p^{NN}, k)| < 1$ *for all* $k \in [\underline{k}, \overline{k}]$*, and, asymptotically, when* $h \to 0$*,*

$$\begin{aligned} \alpha_f^{NN} &= 4\pi^2\mu_f\eta_p\,C_{NN}\,(1 - 2\,L\,C_{NN}\,h) + O(h^2) \\ \alpha_p^{NN} &= L^2(\pi^2\mu_f\eta_p)^{-1}C_{NN}\,h^2 + O(h^3) \\ \rho(\alpha_f^{NN}, \alpha_p^{NN}, \overline{k}) &= -L^2\,C_{NN} + (8\pi^2\mu_f\eta_p L + 4L^3)\,C_{NN}^2\,h + O(h^2)\,, \end{aligned}$$

with $C_{NN} = (4\pi^2\mu_f\,\eta_p + L^2)^{-1}$.

Proof For all $\alpha_f, \alpha_p > 0$, $\lim_{k\to 0}\rho(\alpha_f, \alpha_p, k) = \lim_{k\to\infty}\rho(\alpha_f, \alpha_p, k) = -\infty$, and the function $\rho(\alpha_f, \alpha_p, k)$ has a local maximum at $k^* = (\alpha_f/(\alpha_p\,(2\,\mu_f\,\eta_p)^2))^{1/4}$ where

$$\rho(\alpha_f, \alpha_p, k^*) = 1 - (\sqrt{\alpha_f} + \sqrt{\alpha_p})^2\,. \tag{21}$$

We distinguish two cases.

Case 1: $\sqrt{\alpha_f} + \sqrt{\alpha_p} \geq 1$. In this case, $\rho(\alpha_f, \alpha_p, k) \leq 0$ for all $\underline{k} \leq k \leq \overline{k}$, and $\rho(\alpha_f, \alpha_p, k) = 0$ if $\sqrt{\alpha_f} + \sqrt{\alpha_p} = 1$. Taking $\sqrt{\alpha_f} + \sqrt{\alpha_p} = 1$ would result in a null convergence rate for $k = k^*$, and we could then choose α_f and α_p by imposing $|\rho(\alpha_f, \alpha_p, \underline{k})| = |\rho(\alpha_f, \alpha_p, \overline{k})|$ (which would also ensure that $\underline{k} < k^* < \overline{k}$). This approach leads to $\alpha_p = (1 + 2\,\mu_f\,\eta_p\,\underline{k}\,\overline{k})^{-2}$ and $\alpha_f = (2\,\mu_f\,\eta_p\,\underline{k}\,\overline{k})^2(1 + 2\,\mu_f\,\eta_p\,\underline{k}\,\overline{k})^{-2}$, but, unfortunately, it does not guarantee that $|\rho(\alpha_f, \alpha_p, k)| < 1$ for all $k \in [\underline{k}, \overline{k}]$, which would be true when $1 + 2\,\mu_f\,\eta_p\,\underline{k}\,\overline{k} > \sqrt{2\,\mu_f\,\eta_p}\,(\overline{k} - \underline{k})$.

Case 2: $0 < \sqrt{\alpha_f} + \sqrt{\alpha_p} < 1$. In this case, $\rho(\alpha_f, \alpha_p, k^*) > 0$, and the function $\rho(\alpha_f, \alpha_p, k)$ has two positive zeros

$$k_{1,2} = (1 - \alpha_f - \alpha_p \pm ((1 - \alpha_f - \alpha_p)^2 - 4\,\alpha_f\,\alpha_p)^{1/2})^{1/2} \,/\, (4\,\mu_f\,\eta_p\,\alpha_p)^{1/2} \;,$$

whose position depends on the values of α_f and α_p. Therefore, we proceed by equioscillation and we look for α_f and α_p such that $-\rho(\alpha_f, \alpha_p, \underline{k}) = \rho(\alpha_f, \alpha_p, k^*)$ and $-\rho(\alpha_f, \alpha_p, \overline{k}) = \rho(\alpha_f, \alpha_p, k^*)$. This gives the values (20). Simple algebraic manipulations permit to verify that, for such values of the parameters, $k^* = (\underline{k}\,\overline{k})^{1/2}$ so that $\underline{k} < k_1 < k^* < k_2 < \overline{k}$. Moreover, $|\rho(\alpha_f, \alpha_p, k)| \le \rho(\alpha_f, \alpha_p, k^*)$ for all $\underline{k} \le k \le \overline{k}$ and, owing to (21), we can conclude that $|\rho(\alpha_f, \alpha_p, k)| < 1$ for all frequencies of interest. □

3 Numerical results

We consider a finite element approximation based on the inf-sup stable $\mathbb{Q}_2 - \mathbb{Q}_1$ Taylor-Hood elements [2] for Stokes, and $\mathbb{Q}_2$ elements Darcy. Denoting by the indices I_f, I_p and Γ the degrees of freedom in Ω_f, Ω_p and on Γ, respectively, the algebraic form of the discrete Stokes-Darcy problem (1)–(5) becomes

$$\begin{pmatrix} A^f_{I_f I_f} & A^f_{I_f \Gamma} & G^f_{I_f} & 0 & 0 \\ A^f_{\Gamma I_f} & A^f_{\Gamma\Gamma} & G^f_{\Gamma} & 0 & C_{fp} \\ (G^f_{I_f})^T & (G^f_{\Gamma})^T & 0 & 0 & 0 \\ 0 & 0 & 0 & A^p_{I_p I_p} & A^p_{I_p \Gamma} \\ 0 & -C^T_{fp} & 0 & A^p_{\Gamma I_p} & A^p_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{f,I_f} \\ \mathbf{u}_{f,\Gamma} \\ \mathbf{p}_f \\ \mathbf{p}_{p,I_p} \\ \mathbf{p}_{p,\Gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{f,I_f} \\ \mathbf{f}_{f,\Gamma} \\ \mathbf{0} \\ \mathbf{f}_{p,I_p} \\ \mathbf{f}_{p,\Gamma} \end{pmatrix}, \tag{22}$$

where $\mathbf{u}_{f,\Gamma}$ denotes the vector of degrees of freedom of the normal velocity on Γ. The Schur complement system with respect to $\mathbf{u}_{f,\Gamma}$ is

$$(\Sigma_f + \Sigma_p)\,\mathbf{u}_{f,\Gamma} = \mathbf{b}_\Gamma \tag{23}$$

where Σ_f and Σ_p are the symmetric and positive definite matrices (see [7]):

$$\Sigma_f = A^f_{\Gamma\Gamma} - \begin{pmatrix} A^f_{\Gamma I_f} & G^f_{\Gamma} \end{pmatrix} \begin{pmatrix} A^f_{I_f I_f} & G^f_{I_f} \\ (G^f_{I_f})^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} A^f_{I_f \Gamma} \\ (G^f_{\Gamma})^T \end{pmatrix},$$

$$\Sigma_p = \begin{pmatrix} 0 & C_{fp} \end{pmatrix} \begin{pmatrix} A^p_{I_p I_p} & A^p_{I_p \Gamma} \\ A^p_{\Gamma I_p} & A^p_{\Gamma\Gamma} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C^T_{fp} \end{pmatrix}.$$

Following a classical approach in domain decomposition (see, e.g., [7, 16]), the Neumann-Neumann method (6)–(10) can be equivalently reformulated as a Richardson method for the Schur complement system (23) with preconditioner

$$P = \alpha_f\,\Sigma_f^{-1} + \alpha_p\,\Sigma_p^{-1}\,. \tag{24}$$

The PCG method with preconditioner P can then be used to solve (23).

We consider the computational domains $\Omega_f = (0, 0.5) \times (1, 1.5)$ and $\Omega_p = (0, 0.5) \times (0.5, 1)$ so that $\Gamma = (0, 0.5) \times \{1\}$, and we choose the forces $\mathbf{f}_f$ and f_p and the boundary conditions in such a way that the Stokes-Darcy problem has analytic solution $\mathbf{u}_f = (\sqrt{\eta_p}, \alpha_{BJ} x)^T$, $p_f = 2\,\mu_f\,(x + y - 1) + (3\,\eta_p)^{-1}$, and $p_p = \eta_p^{-1}(-\alpha_{BJ} x(y-1) + y^3/3 - y^2 + y) + 2\,\mu_f\,x$. The computational meshes are structured and characterized by $h = 0.1 \times 2^{1-j}$, $j = 1, \ldots, 4$, with 11, 21, 41, and 81 interface unknowns, respectively. We consider four configurations of physically significant dimensionless problem parameters (see also [12]): *(a)* $\mu_f = 10, \eta_p = 4 \times 10^{-10}$; *(b)* $\mu_f = 1, \eta_p = 4 \times 10^{-7}$; *(c)* $\mu_f = 10, \eta_p = 4 \times 10^{-9}$; *(d)* $\mu_f = 0.2, \eta_p = 2 \times 10^{-7}$.

Table 1 reports the computed values of the optimal parameters α_f^{NN} and α_p^{NN} (20) and the number of CG iterations with preconditioner (24) and without preconditioner (in brackets). For comparison, we indicate also the values of the optimal parameters α_f^{RR} and α_p^{RR} and the number of GMRES iterations obtained with the optimized Schwarz (Robin-Robin) method studied in [8]. (Notice that $\alpha_f^{NN} \sim c_f^0 + c_f^1 h$ and $\alpha_p^{NN} \sim c_p^0 + c_p^1 h$ when $h \to 0$ for suitable constants c_f^0, c_f^1, c_p^0 and c_p^1 that depend on μ_f, η_p and L.)

The number of PCG iterations using optimized parameters α_f^{NN}, α_p^{NN} is almost independent of both the mesh size and of the values of μ_f and η_p.

Moreover, the optimized Neumann-Neumann method performs better than the Robin-Robin method with lower computational cost per iteration. We also observe that, considering the Robin interface conditions $(3.3)_4$ and $(3.4)_4$ in [8] and the values of α_f^{RR} and α_p^{RR} (especially, the large values of α_f^{RR}), the Robin-Robin method actually behaves like a Dirichlet-Robin method with interface condition on

Table 1 Optimal parameters α_f^{NN} and α_p^{NN} and number of PCG iterations, and optimal parameters α_f^{RR} and α_p^{RR} for the Robin-Robin method with corresponding GMRES iterations ($tol = 10^{-9}$).

Case	Mesh	α_f^{NN}	α_p^{NN}	PCG iter	α_f^{RR}	α_p^{RR}	GMRES iter
(a)	h_1	9.97×10^{-12}	$1.00 \times 10^{+0}$	2 (12)	$7.23 \times 10^{+7}$	$6.91 \times 10^{+2}$	4
	h_2	3.99×10^{-11}	$1.00 \times 10^{+0}$	2 (17)	$3.79 \times 10^{+7}$	$1.32 \times 10^{+3}$	4
	h_3	1.60×10^{-10}	$1.00 \times 10^{+0}$	3 (22)	$1.94 \times 10^{+7}$	$2.58 \times 10^{+3}$	4
	h_4	6.38×10^{-10}	9.99×10^{-1}	3 (31)	$9.83 \times 10^{+6}$	$5.09 \times 10^{+3}$	4
(b)	h_1	9.96×10^{-8}	9.98×10^{-1}	3 (12)	$7.24 \times 10^{+4}$	$6.91 \times 10^{+1}$	6
	h_2	3.96×10^{-7}	9.93×10^{-1}	4 (17)	$3.80 \times 10^{+4}$	$1.32 \times 10^{+2}$	6
	h_3	1.55×10^{-6}	9.74×10^{-1}	4 (24)	$1.96 \times 10^{+4}$	$2.55 \times 10^{+2}$	8
	h_4	5.78×10^{-6}	9.06×10^{-1}	5 (30)	$1.03 \times 10^{+4}$	$4.86 \times 10^{+2}$	8
(c)	h_1	9.97×10^{-10}	$1.00 \times 10^{+0}$	3 (12)	$7.23 \times 10^{+6}$	$6.91 \times 10^{+2}$	4
	h_2	3.99×10^{-9}	9.99×10^{-1}	3 (17)	$3.79 \times 10^{+6}$	$1.32 \times 10^{+3}$	4
	h_3	1.59×10^{-8}	9.97×10^{-1}	3 (24)	$1.94 \times 10^{+6}$	$2.57 \times 10^{+3}$	6
	h_4	6.32×10^{-8}	9.90×10^{-1}	4 (30)	$9.87 \times 10^{+5}$	$5.06 \times 10^{+3}$	6
(d)	h_1	2.49×10^{-10}	$1.00 \times 10^{+0}$	2 (12)	$7.23 \times 10^{+5}$	$3.46 \times 10^{+1}$	4
	h_2	9.97×10^{-10}	$1.00 \times 10^{+0}$	3 (17)	$3.79 \times 10^{+5}$	$6.60 \times 10^{+1}$	4
	h_3	3.98×10^{-9}	9.99×10^{-1}	3 (22)	$1.94 \times 10^{+5}$	$1.29 \times 10^{+2}$	6
	h_4	1.59×10^{-8}	9.95×10^{-1}	4 (29)	$9.85 \times 10^{+4}$	$2.54 \times 10^{+2}$	6

the normal velocity $\mathbf{u}_f \cdot \mathbf{n}$ for the Stokes problem. This confirms that condition $(6)_3$ in the Neumann-Neumann algorithm is a valid choice for the Stokes problem.

Finally, the optimal values α_f^{NN}, α_p^{NN} suggest that the preconditioner (24) behaves like $P \approx \Sigma_p^{-1}$. Thus, while Σ_f^{-1} is an effective preconditioner for large values of μ_f and η_p (see [7, 10]), Σ_p^{-1} is a much better choice for small values, which is the case in most applications. This can lead to a Dirichlet-Neumann-type method different from the one in [7, 10] that will be discussed in a future work.

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References

1. Beavers, G. S. and Joseph, D. D. Boundary conditions at a naturally permeable wall. *J. Fluid Mech.* **30**, 197–207 (1967).
2. Boffi, D., Brezzi, F., and Fortin, M. *Mixed Finite Element Methods and Applications*. Springer, Berlin and Heidelberg (2013).
3. Caiazzo, A., John, V., and Wilbrandt, U. On classical iterative subdomain methods for the Stokes-Darcy problem. *Comput. Geosci.* **18**, 711–728 (2014).
4. Cao, Y., Gunzburger, M., Hu, X., Hua, F., Wang, X., and Zhao, W. Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions. *SIAM J. Numer. Anal.* **47**(6), 4239–4256 (2010).
5. Cao, Y., Gunzburger, M., Hua, F., and Wang, X. Coupled Stokes-Darcy model with Beavers-Joseph interface boundary conditions. *Comm. Math. Sci.* **8**(1), 1–25 (2010).
6. Chen, W., Gunzburger, M., Hua, F., and Wang, X. A parallel Robin-Robin domain decomposition method for the Stokes-Darcy system. *SIAM J. Numer. Anal.* **49**(3), 1064–1084 (2011).
7. Discacciati, M. *Domain Decomposition Methods for the Coupling of Surface and Groundwater Flows*. Ph.D. thesis, École Polytechnique Fédérale de Lausanne, Switzerland (2004).
8. Discacciati, M. and Gerardo-Giorda, L. Optimized Schwarz methods for the Stokes-Darcy coupling. *IMA J. Numer. Anal.* **38**(4), 1959–1983 (2018).
9. Discacciati, M., Miglio, E., and Quarteroni, A. Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.* **43**, 57–74 (2002).
10. Discacciati, M. and Quarteroni, A. Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations. *Comput. Visual. Sci.* **6**, 93–103 (2004).
11. Discacciati, M., Quarteroni, A., and Valli, A. Robin-Robin domain decomposition methods for the Stokes-Darcy coupling. *SIAM J. Numer. Anal.* **45**(3), 1246–1268 (2007).
12. Discacciati, M. and Vanzan, T. Optimized Schwarz methods for the time-dependent Stokes-Darcy coupling. Tech. rep. (2022). Submitted.
13. Gander, M. J. and Vanzan, T. On the derivation of optimized transmission conditions for the Stokes-Darcy coupling. In: et al., R. H. (ed.), *Domain Decomposition Methods in Science and Engineering XXV. DD 2018*. Springer (2020).
14. Jäger, W. and Mikelić, A. On the boundary conditions at the contact interface between a porous medium and a free fluid. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **23**, 403–465 (1996).
15. Layton, W. L., Schieweck, F., and Yotov, I. Coupling fluid flow with porous media flow. *SIAM J. Num. Anal.* **40**, 2195–2218 (2003).
16. Quarteroni, A. and Valli, A. *Domain Decomposition Methods for Partial Differential Equations*. The Clarendon Press, Oxford University Press, New York (1999).
17. Saffman, P. G. On the boundary condition at the interface of a porous medium. *Stud. Appl. Math.* **1**, 93–101 (1971).
18. Toselli, A. and Widlund, O. *Domain Decomposition Methods – Algorithms and Theory*, *Springer Series in Computational Mathematics*, vol. 34. Springer, Berlin (2005).