A Parallel Space-Time Finite Element Method for the Simulation of an Electric Motor

Peter Gangl, Mario Gobrial, and Olaf Steinbach

1 Introduction

Shape and topology optimization of electrical machines [3] as well as the optimal control [4] subject to parabolic evolution equations require an efficient solution of the direct simulation problem which is forward in time, and in most cases also of the adjoint problem which is backward in time. Space-time discretization methods [8] are therefore a method of choice to solve the overall system at once, and also to allow for adaptive refinements and a parallel solution simultaneously in space and time. In the case of a fixed spatial domain the numerical analysis of a space-time finite element method was given, e.g., in [7], see also the review article [8] and the references given therein. In this note we present an extension of this approach in order to simulate an electric motor where one part, the rotor, is rotating in time, while the stator is fixed. In addition to the stator and the rotor we have to include an air domain which is non-conducting. Hence we have to deal with an elliptic-parabolic interface problem for the eddy current approximation of the Maxwell system in two space dimensions. In this paper we present its space-time finite element discretization and provide first numerical results using parallel solution strategies in order to handle the overall system in the space-time domain. More details on the numerical analysis of the proposed method and further numerical results will be given in our forthcoming paper [2].

This paper is structured as follows: In Section 2 we describe the mathematical model and its space-time variational formulation. Unique solvability is based on the Babuška–Nečas theorem, i.e., on injectivity and surjectivity of the operator which is

Peter Gangl
Johann Radon Institute for Computational and Applied Mathematics, Altenberger Straße 69, 4040 Linz, Austria, e-mail: peter.gangl@oeaw.ac.at

Mario Gobrial, Olaf Steinbach
Institute of Applied Mathematics, TU Graz, Steyrergasse 30, 8010 Graz, Austria
e-mail: gobrial@math.tugraz.at, o.steinbach@tugraz.at

associated to the bilinear form of the variational formulation. The space-time finite element discretization is given in Section 3, which also provides an a priori error estimate, i.e., Cea's lemma, for the numerical solution. Numerical results are given in Section 4, and finally we provide some conclusions and comment on ongoing work.

2 Mathematical model and space-time variational formulation

To model the electromagnetic fields in a rotating electric machine, we consider the eddy current approximation of the Maxwell equations,

$$\mathrm{curl}_y H(y,t) = j(y,t), \quad \mathrm{curl}_y E(y,t) = -\partial_t B(y,t), \quad \mathrm{div}_y B(y,t) = 0,$$

subject to the constitutive equations

$$B(y,t) = \mu(y)H(y,t)+M(y,t),\ j(y,t) = j_i(y,t)+\sigma(y)\Big[E(y,t)+v(y,t)\times B(y,t)\Big],$$

with the material dependent magnetic permeability μ, the electric conductivity σ, and an impressed electric current j_i. Moreover, M is the magnetization which vanishes outside permanent magnets. For a reference point $x \in \mathbb{R}^3$ we consider the trajectory $y(t) = \varphi(t,x)$, where the deformation φ is assumed to be bijective and sufficiently regular for all $t \in (0,T)$, satisfying $\varphi(0,x) = x$. Here, $T > 0$ is a given time horizon. Finally, we introduce the velocity $v(y,t) = \frac{d}{dt}y(t)$. In addition we consider appropriate boundary and initial conditions to be specified.

When using the vector potential ansatz $B(y,t) = \mathrm{curl}_y A(y,t)$, and following the standard approach to consider a spatially two-dimensional reference domain $\Omega \subset \mathbb{R}^2$ describing the cross section of the electric machine, this gives an evolution equation to find $u(y,t)$ as third component of $A = (0,0,u)^\top$ such that

$$\sigma(y)\,\frac{d}{dt}u(y,t) - \mathrm{div}_y\,[\nu(y)\nabla_y u(y,t)] = j_i(y,t) - \mathrm{div}_y\,[\nu(y)M^\perp(y,t)] \qquad (1)$$

is satisfied in the space-time domain

$$Q := \Big\{(y,t) \in \mathbb{R}^3 : y = \varphi(t,x) \in \Omega(t),\ x \in \Omega \subset \mathbb{R}^2,\ t \in (0,T)\Big\}.$$

Note that in (1) we use the reluctivity $\nu = 1/\mu$, and the total time derivative

$$\frac{d}{dt}u(y,t) := \partial_t u(y,t) + v(y,t)\cdot\nabla_y u(y,t).$$

Moreover, $M^\perp = (-M_2, M_1)^\top$ is the perpendicular of the first two components of the magnetization M. In addition to (1) we consider homogeneous Dirichlet boundary conditions $u = 0$ on $\Sigma := \partial\Omega \times (0,T)$, and the homogeneous initial condition $u(x,0) = 0$ whenever $\sigma(x) > 0$ is satisfied for $x \in \Omega$.

The electric motor consists of a rotor in $\Omega_r(t)$, the stator in Ω_s, and the air domain Ω_{air} which is non-conducting, i.e., $\sigma = 0$ in Ω_{air}. This shows that we can formulate (1) as an elliptic-parabolic interface problem. While the stator domain Ω_s is fixed in time, i.e., $v \equiv 0$, the rotating subdomain $\Omega_r(t)$ can be described, when using polar coordinates, as

$$x = r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad y(t) = \varphi(t,x) = r \begin{pmatrix} \cos(\varphi + \alpha t) \\ \sin(\varphi + \alpha t) \end{pmatrix} \in \Omega_r(t),\ t \in (0,T),$$

with $\alpha > 0$ describing the velocity

$$v(y,t) = \frac{d}{dt} y(t) = \alpha r \begin{pmatrix} -\sin(\varphi + \alpha t) \\ \cos(\varphi + \alpha t) \end{pmatrix} = \alpha \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}.$$

The variational formulation of the parabolic-elliptic interface problem (1) is to find $u \in X$ such that

$$\begin{aligned} b(u,z) &:= \int_0^T \int_{\Omega(t)} \left[\sigma \frac{d}{dt} u\, z + \nu\, \nabla_y u \cdot \nabla_y z \right] dy\, dt \qquad (2) \\ &= \int_0^T \int_{\Omega(t)} \left[j_i\, z + \nu\, M^\perp \cdot \nabla_y z \right] dy\, dt \end{aligned}$$

is satisfied for all $z \in Y$, where $Y := L^2(0,T; H_0^1(\Omega(t)))$ and

$$X := \left\{ u \in Y\colon \sigma \frac{d}{dt} u \in Y^*,\ u(x,0) = 0 \text{ for } x \in \Omega : \sigma(x) > 0 \right\}.$$

Related norms are given by

$$\|z\|_Y^2 := \int_0^T \int_{\Omega(t)} \nu\, |\nabla_y z|^2\, dy\, dt, \quad \|u\|_X^2 := \|u\|_Y^2 + \|w_u\|_Y^2,$$

where $w_u \in Y$ is the unique solution of the variational formulation

$$\int_0^T \int_{\Omega(t)} \nu\, \nabla_y w_u \cdot \nabla_y z\, dy\, dt = \int_0^T \int_{\Omega(t)} \sigma\, \frac{d}{dt} u\, z\, dy\, dt \quad \text{for all } z \in Y. \qquad (3)$$

The bilinear form $b(\cdot,\cdot)$ as defined in (2) is bounded and satisfies an inf-sup stability condition, see [2, 7], i.e., for all $u \in X$ and $z \in Y$ there holds

$$|b(u,z)| \le \sqrt{2}\, \|u\|_X \|z\|_Y, \qquad \frac{1}{\sqrt{2}} \|u\|_X \le \sup_{0 \ne z \in Y} \frac{b(u,z)}{\|z\|_Y}.$$

Moreover, the bilinear form $b(\cdot,\cdot)$ is surjective, i.e., for any $z \in Y$ there exists a $u_z \in X$ such that $b(u_z,z) > 0$ is satisfied, see [2]. Hence, all assumptions of the Babuška–Nečas theorem [1, 5] are satisfied, i.e. we conclude unique solvability of the space-time variational formulation (2).

3 Space-time finite element discretization

For the space-time finite element discretization of the variational formulation (2) we introduce conforming finite dimensional spaces $X_h \subset X$ and $Y_h \subset Y$ where we assume as in the continuous case $X_h \subset Y_h$. For our specific purpose we even consider $X_h = Y_h := S_h^1(Q_h) \cap X = \text{span}\{\varphi_k\}_{k=1}^M$ as the space of piecewise linear and continuous basis functions φ_k which are defined with respect to some admissible locally quasi-uniform decomposition $\mathcal{T}_h = \{\tau_\ell\}_{\ell=1}^N$ of the space-time domain Q into shape-regular simplicial finite elements τ_ℓ of mesh size h_ℓ, see, e.g., [6].

The Galerkin space-time finite element variational formulation of (2) reads to find $u_h \in X_h$, such that

$$b(u_h, z_h) = \int_0^T \int_{\Omega(t)} \Big[j_i \, z_h \, + \, \nu \, M^\perp \cdot \nabla_y z_h \Big] \, dy \, dt \quad \text{for all } z_h \in Y_h. \qquad (4)$$

Unique solvability of (4) is based on the discrete inf-sup stability condition

$$\frac{1}{\sqrt{2}} \|u_h\|_{X_h} \le \sup_{0 \ne z_h \in Y_h} \frac{b(u_h, z_h)}{\|z_h\|_Y} \quad \text{for all } u_h \in X_h,$$

which follows as in the continuous case [2, 7], but makes use of the discrete norm

$$\|u_h\|_{X_h}^2 := \|u_h\|_Y^2 + \|w_{u_h h}\|_Y^2 \le \|u_h\|_Y^2 + \|w_{u_h}\|_Y^2 = \|u_h\|_X^2,$$

where $w_{u_h h} \in Y_h$ is the unique solution of the variational formulation

$$\langle \nu \nabla_y w_{u_h h}, \nabla_y z_h \rangle_{L^2(Q)} = \langle \sigma \frac{d}{dt} u_h, z_h \rangle_Q \quad \text{for all } z_h \in Y_h. \qquad (5)$$

As in [7] we can then derive Cea's lemma,

$$\|u - u_h\|_{X_h} \le 3 \inf_{z_h \in X_h} \|u - z_h\|_X,$$

from which we conclude optimal order of convergence when assuming sufficient regularity for the solution. In particular we obtain linear convergence in the space-time mesh size h when assuming $u \in H^2(Q)$, see [2, 7].

4 Numerical results

As numerical example we consider an electric motor, where both the rotor and the stator are made of iron, with 16 magnets and 48 coils, see Fig. 1.
The motor is pulled up in time, where the rotation of the rotating parts, i.e., the rotor, the magnets and the air around the magnets, is already considered within the mesh for a 90 degree rotation. The time component is treated as the third spatial dimension

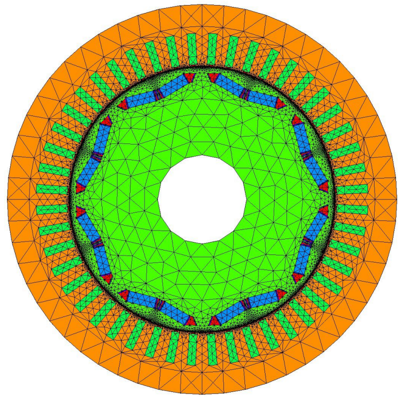

Fig. 1 The unstructured mesh of the bottom of the motor, showing the different materials.

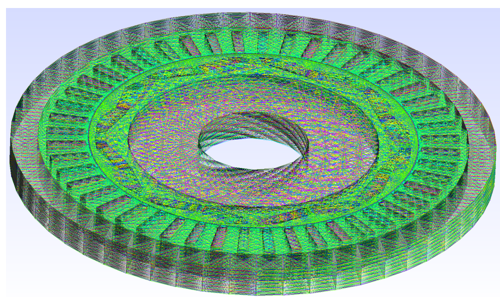

Fig. 2 The full space-time cylinder of the motor for a 90 degree rotation with 333, 288 nodes and 1, 978, 689 elements. The time is treated as the third spatial component divided into 30 time slices. The rotating parts are already considered within the mesh.

with a time span $(0, T)$, $T = 0.015$ seconds. Moreover, 30 time slices are inserted in order to have a good temporal resolution, where the mesh is differently unstructured at every time $t \in (0, T)$, see Figure 2.
The electric motor consists of isotropic materials, hence we choose the linear material parameters as given in Table 1.

Table 1 Material parameters

material	σ	ν
air	0	$10^7/(4\pi)$
coils	$5.8 \cdot 10^7$	$10^7/(4\pi)$
magnets	10^6	$10^7/(4\pi)$
iron	10^7	$10^7/(20400\pi)$

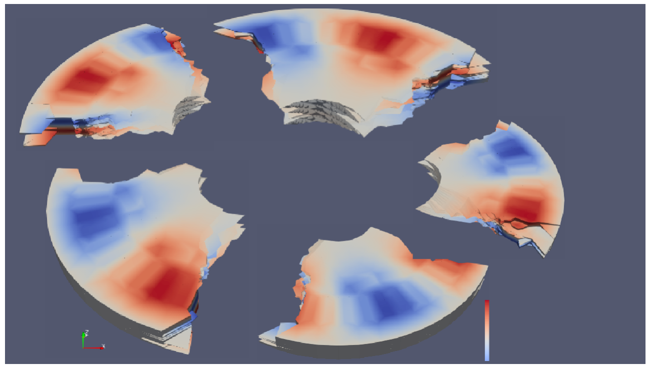

Fig. 3 Space-time mesh decomposition into 5 subdomains.

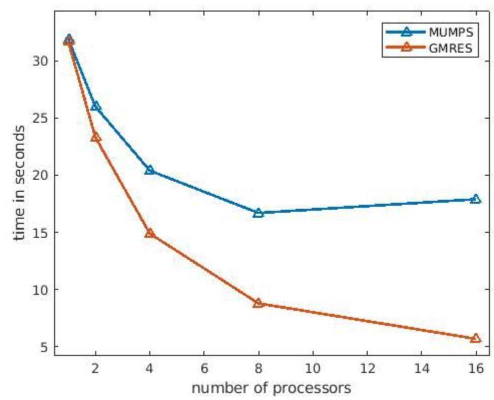

Fig. 4 The computation times for solving the linear system using the MUMPS solver and GMRES solver.

We solve the resulting linear system in parallel, using a mesh decomposition method provided by the finite element library Netgen/NGSolve[1], see Fig. 3. For our purpose, MPI parallelization is used, however the computations are done on one computer with 384 GiB RAM and two Intel Xenon Gold 5218 CPU's.

We use MUMPS[2] supported by PETSc[3], to solve the linear system. Figure 4 shows the time for solving the linear system in relation to the number of processors used for the parallel computation. For comparison, the GMRES solver provided by PETCs is used to solve the same linear systems with an error tolerance of 10^{-6} and a maximum of 1000 iterations.

The solution of the Galerkin space-time finite element formulation (4) for the time span $(0, T)$ with $T = 0.015$ is not sufficient to visualize, since in this short time the solution is close to zero due to the zero initial condition. Instead, one may consider

[1] https://www.ngsolve.org

[2] http://mumps-solver.org

[3] https://petsc.org

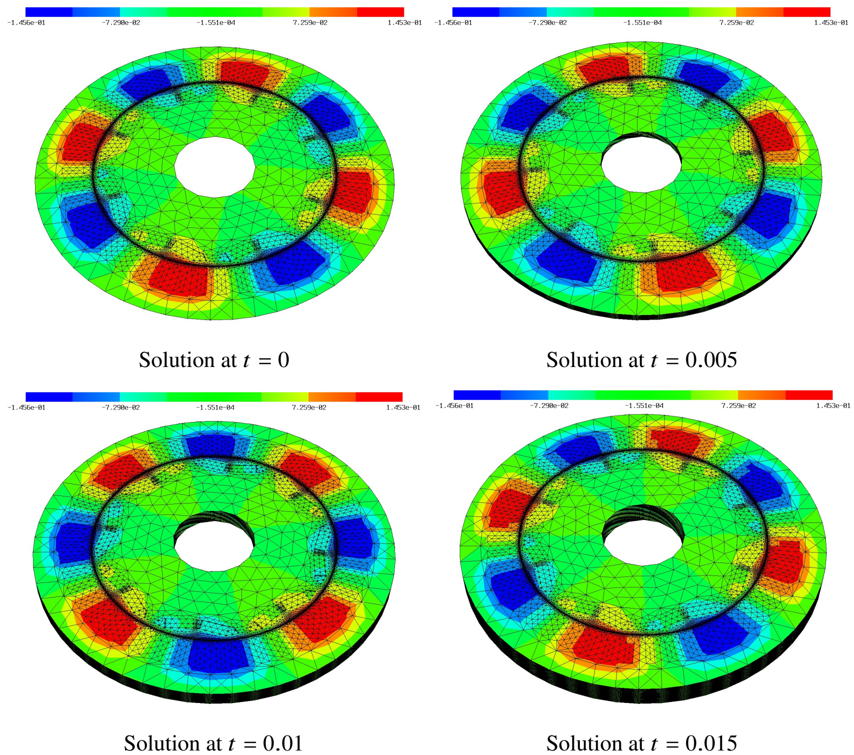

Fig. 5 The solution of the static problem visualized on different time slices.

periodic conditions $u(x, T) = u(x, 0)$ for $x \in \Omega$, see [2]. Here, we also present the results for the quasi-static problem using $\sigma = 0$ for all material regions at any time, see Fig. 5 for the solution at different time slices. Note that this corresponds to the problem of magnetostatics which is widely used in practical applications together with time-stepping methods.

5 Conclusions

In this note we have described a space-time finite element discretization of an elliptic-parabolic interface problem to model an electric motor. The computed electromagnetic fields can be used to compute other characteristic quantities such as the torque and iron losses in order to optimize the shape and the topology of electric machines. Instead of initial conditions and a linear description of the involved materials one can easily include periodic conditions in time and a nonlinear material model. Although we have provided first results for a parallel solution of the resulting linear system of algebraic equations also using mesh decomposition algorithms, further work is required in the design of more efficient solution strategies using appropriate preconditioning and domain decomposition methods.

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