Optimized Schwarz Methods for Isogeometric Analysis

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1 Introduction

Isogeometric Analysis (IGA) is a novel computational technique for solving partial differential equations (PDEs) first introduced by Hughes et al, see [6]. It integrates computer-aided design (CAD) and simulation. In IGA, a geometric model created within a CAD environment is used as the basis for analysis, and B-splines or non-uniform rational B-splines (NURBS) are employed as basis functions. IGA offers a new type of refinement strategy, in addition to the traditional mesh refinement (h-refinement) and p-refinement in Finite Element Analysis (FEA), namely k-refinement, which allows for changing the smoothness of the basis functions. The aim of IGA is to improve the accuracy and efficiency of simulation by using CAD models directly in the analysis process. In Section 2.1 we give a brief description of the B-spline functions. For an extensive overview on the approximation theory based on IGA, see [2].

Domain decomposition methods (DDM) are based on dividing the domain into subdomains which leads to solve small local problems. The classical Schwarz methods use Dirichlet boundary conditions at the artificial interfaces, see [8], while the Optimized Schwarz Methods (OSM) use Robin ($\partial_n u + \lambda u$) or higher order boundary conditions at the artificial interfaces. The challenge is to find the optimal value of the parameter λ, this latter can be solved by virtue of Fourier transform, see [4] for more details. Rather than relying on the existing literature on DDM for IGA as described in [3], we adopt an approach that enforces C^{-1} smoothness of the B-spline in the interface condition. For a more comprehensive understanding of it, please

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refer to [1]. For our analysis, we consider Algebraic Optimized Schwarz methods (AOSM) which mimic OSM algebraically.

Our approach involves combining IGA and AOSM to solve partial differential equations with complex geometries. The efficiency of the resulting algorithm is due to the robustness of AOSM/OSM and the flexibility of IGA.

2 IGA analysis and algebraic optimized Schwarz methods

For our analysis we need to introduce B-spline and algebraic optimized Schwarz methods.

2.1 B-spline based IGA

Let m and p be two positive integers, and Ξ be a set of non-decreasing real numbers such that $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_{m+p+1}$. The ξ_j's are called the *knots*, the set Ξ is the knot vector, and the interval $[\xi_j, \xi_{j+1})$ is the j-th knot span. Note that if ξ_j is repeated $k > 1$ times in the knot vector (i.e. $\xi_j = \xi_{j+1} = \ldots = \xi_{j+k-1}$), ξ_j is a multiple knot of multiplicity k with no corresponding knot span; otherwise, it is a simple knot if ξ_j appears only once (or $k = 1$). A knot vector is said to be *uniform* if its knots are uniformly spaced; otherwise, it is called a *nonuniform* knot vector. A knot vector is considered to be *open* if its first and last knots have multiplicity $p + 1$. The interval (ξ_1, ξ_{m+p+1}) is called the *patch*. The maximum multiplicity allowed is $p + 1$.

Once a knot vector is available, the B-spline basis functions can be defined recursively, beginning with the first order, $p = 0$ (piecewise constant)

$$N_j^0(\xi) := \chi_{[\xi_j, \xi_{j+1})} = \begin{cases} 1, & \text{if } \xi_j \leq \xi < \xi_{j+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For $p \geq 1$,

$$N_j^p(\xi) := \begin{cases} \dfrac{\xi - \xi_j}{\xi_{j+p} - \xi_j} N_j^{p-1}(\xi) + \dfrac{\xi_{j+p+1} - \xi}{\xi_{j+p+1} - \xi_{j+1}} N_{j+1}^{p-1}(\xi), & \text{if } \xi_j \leq \xi < \xi_{j+p+1}, \\ 0, \quad \text{otherwise.} \end{cases} \tag{2}$$

we adopt the convention $\frac{0}{0} = 0$ in (2).
According to (2), all B-spline functions are to be (i) non-negative, (ii) have a local support in $[\xi_j, \xi_{j+p+1}]$ (compact support) for all $j = 1, \ldots, m$, (iii) form a *partition of unity*, and (iv) be linear independent, as shown in [9]. The basis functions of order p, in general, have $p - k$ continuous derivatives $\mathscr{C}^{p-k}$ across knot ξ_j. When the multiplicity of a knot value is exactly p, the basis at that knot is interpolatory. If the multiplicity of a basis is $p + 1$, it can result the basis become discontinuous in the $\mathscr{C}^{-1}$ space. In Figure 1, we present an example of cubic basis functions generated by $p = 3$ from the uniform open knot vector $\Xi = \{0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6, 6\}$.

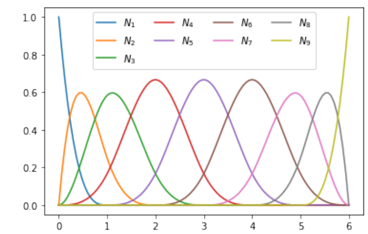

Fig. 1 Cubic basis functions formed from $\Xi = \{0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6, 6\}$.

2.2 Algebraic optimized Schwarz methods

Descritizing PDEs using IGA analysis leads to solve linear systems of the form

$$Au = f, \tag{3}$$

where A is a block banded matrix of size $n \times n$ given by

$$A = \begin{bmatrix} A_{11} & A_{12} & & \\ A_{21} & A_{22} & A_{23} & \\ & A_{32} & A_{33} & A_{34} \\ & & A_{43} & A_{44} \end{bmatrix}, \tag{4}$$

where A_{ij} are blocks of size $n_i \times n_j$, $i, j = 1, \ldots, 4$, and $n = \sum_i n_i$. For a two-subdomain decomposition with overlap we have $n_1 \gg n_2$ and $n_4 \gg n_3$. To illustrate this decomposition let us solve the Poison equation in $\Omega = \mathbb{R} \times (0, 1)$ with homogeneous Dirichlet at the boundary conditions. We discretize the continuous operator on a grid with an interval of size h in both the x and y directions and we assume that $h = 1/(N+1)$ so that there are N degrees of freedom in y-direction. For instance the stiffness matrix obtained when we discretize with the finite element method using piecewise linear functions, and using the subdomains $\Omega_1 = (-\infty, h) \times (0, 1)$ and $\Omega_2 = (0, +\infty) \times (0, 1)$, leading to the decomposition

$$A = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & O & O \\ \hline A_{21} & A_{22} & A_{23} & O \\ \hline O & A_{32} & A_{33} & A_{34} \\ \hline O & O & A_{43} & A_{44} \end{array}\right] = \left[\begin{array}{ccc|c|c|ccc} \ddots & \ddots & \ddots & & & & & \\ & -I & J & -I & & & & \\ \hline & & -I & J & -I & & & \\ \hline & & & -I & J & -I & & \\ \hline & & & & -I & J & -I & \\ & & & & & \ddots & \ddots & \ddots \end{array}\right], \tag{5}$$

where I is the $N \times N$ identity matrix and J is the $N \times N$ tridiagonal $J = \text{tridiag}(-1, 4, -1)$. We have in this case $n_2 = n_3 = N$. The Algebraic Optimized Schwarz methods are iterative methods [5, Section 2, page 4], and the optimized restricted additive and multiplicative Schwarz methods are defined by

$$T_{\mathrm{ORAS}} = I - \sum_{i=1}^{2} \tilde{R}_i^T \tilde{A}_i^{-1} R_i A, \quad \text{and} \quad T_{\mathrm{ORMS}} = \prod_{i=2}^{1} (I - \tilde{R}_i^T \tilde{A}_i^{-1} R_i A), \tag{6}$$

where the restriction operators with overlap are $R_1 = [I\ O]$ and $R_2 = [O\ I]$, of size $(n_1 + n_2 + n_3) \times n$ and $(n_2 + n_3 + n_4) \times n$ respectively, using the prolongations $\tilde{R}_i^T$ without the overlap, which are defined as

$$\tilde{R}_1 = \begin{bmatrix} I & O \\ O & O \end{bmatrix} \quad \text{and} \quad \tilde{R}_2 = \begin{bmatrix} O & O \\ O & I \end{bmatrix},$$

having the same order as the matrices R_i, and where the identity in $\tilde{R}_1$ is of order $n_1 + n_2$ and that in $\tilde{R}_2$ is of order $n_3 + n_4$. The matrices $\tilde{A}_i$ are defined by

$$\tilde{A}_1 = \begin{bmatrix} A_{11} & A_{12} & \\ A_{21} & A_{22} & A_{23} \\ & A_{32} & A_{33} + D_1 \end{bmatrix}, \qquad \tilde{A}_2 = \begin{bmatrix} A_{22} + D_2 & A_{23} & \\ A_{32} & A_{33} & A_{34} \\ & A_{43} & A_{44} \end{bmatrix}, \tag{7}$$

for which the transmission blocks D_1 and D_2 have to be determined for fast convergence. It has been shown in [5, Theorem 3.2] that the asymptotic convergence factor of AOSM depends on the product of the two norms

$$\| (I + D_1 B_{33})^{-1} [D_1 B_{12} - A_{34} B_{13}] \|, \ \| (I + D_2 B_{11})^{-1} [D_2 B_{32} - A_{21} B_{31}] \|. \tag{8}$$

The blocks B_{ij} depend on the inverses A_{11}^{-1} and A_{44}^{-1} which are expensive to calculate. Minimizing the linear part of equation (8) on matrices D_1 and D_2 within the spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ with distinct sparsity patterns leads to various forms of AOSM. The $O0s$ approach uses a scalar α_i in $D_i = \alpha_i I$, while the O0 method employs a general diagonal matrix D_i and the O2 scheme uses a general tridiagonal matrix D_i. The optimal method, i.e., $D_1 = -A_{34} A_{44}^{-1} A_{43}$ and $D_2 = -A_{21} A_{11}^{-1} A_{12}$, converges in two iterations [5].

3 IGA approximation of transmission conditions

3.1 AOSM approximations of D_1 and D_2

The challenge in approximating the transmission blocks D_1 and D_2 is to capture efficiently the sparsity of the related matrices. In Figure 2 we present different sparsity patterns for the model problem $-\Delta u = f$ in a square domain $\Omega = (0,1)^2$ for an IGA discretization with 32×32 elements with respect to B-spline degrees $p = 4, 5, 6$. Because of the structure of the matrices we need to use adapted algorithms which capture efficiently the sparsity of the transmissions blocks D_1 and D_2. For this purpose we introduce a new method, which we call O_{p+1}, that consists in approximating the blocks D_1 and D_2 using $2p + 1$ diagonals, where p is B-spline degree.

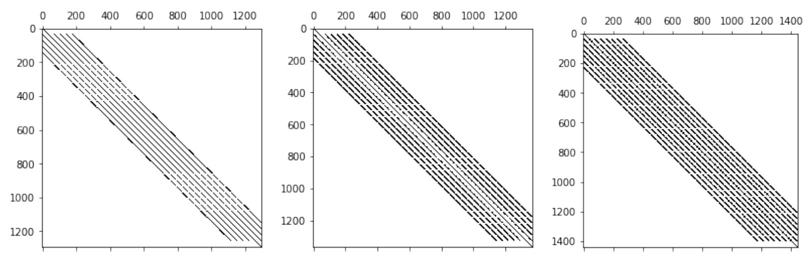

Fig. 2 The sparsity pattern of stiffness matrix in 2D with number of elements 32×32 with respect to spline polynomial degree $p = 4, 5, 6$, and we allows maximum regularity $k = 1$ at the internal knots.

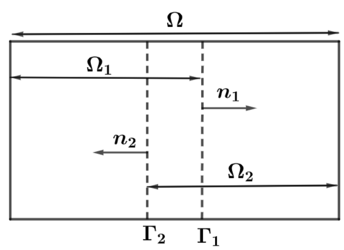

Fig. 3 Domain decomposition into two overlapping subdomains.

3.2 Optimized Schwarz methods for IGA

In this section we consider the Poisson equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{9}$$

in a square domain $\Omega = (0,1)^2$ with Dirichlet boundary conditions. We decompose the domain Ω into two overlapping subdomains $\Omega_1 = (0,\alpha) \times (0,1)$ and $\Omega_2 = (\beta, 1) \times (0,1)$, see Figure 3. The size of the overlap is defined by $\delta = \alpha - \beta$, where $\alpha \geq \beta$ allowing $\alpha = \beta$ for non-overlapping decomposition.

The parallel Schwarz method, introduced by P. Lions, 1990 [7], equipped with Robin boundary conditions for the model problem and the decomposition is

$$\begin{cases} \begin{cases} -\Delta u_1^{n+1} & = f, \text{ in } \Omega_1 = (0,\alpha) \times (0,1), \\ u_1^{n+1} & = 0, \text{ on } \partial\Omega_1 \\ \left(\partial_{n_1} + \lambda_1\right) u_1^{n+1} & = \left(\partial_{n_1} + \lambda_1\right) u_2^n, \text{ on } \Gamma_1 = \{\alpha\} \times (0,1), \end{cases} \\ \begin{cases} -\Delta u_2^{n+1} & = f, \; in \;\; \Omega_2 = (\beta, 1) \times (0,1), \\ u_2^{n+1} & = 0, \text{ on } \partial\Omega_2 \\ \left(\partial_{n_2} + \lambda_2\right) u_2^{n+1} & = \left(\partial_{n_2} + \lambda_2\right) u_1^n, \; on \; \Gamma_2 = \{\beta\} \times (0,1). \end{cases} \end{cases} \tag{10}$$

The OSM is based on finding the optimal parameter set (λ_1, λ_2) that yields a rapid convergence, M. Gander [4] provides an explicit formulas for λ_1 and λ_2 based on Fourier analysis for the model problem $(\eta - \Delta)u = f$. But in our case, no formulas are found yet. Thus, we relied on a numerical approximation supposing that $\lambda_1 = \lambda_2 = \lambda$, then conducting a grid search over a subset of λ to find the best value.

4 Numerical experiments

For our numerical experiments we consider the model problem (9) with two-overlapping decomposition as described before. We allow the parameter $\delta = \alpha - \beta$ to be zero for a non-overlapping decomposition. First we illustrate the performance of the new method O_{p+1} compared the optimal method, O0, O0s, and O2, for the methods labeled "Nonoverlapping" and "Overlapping" correspond to the nonoverlapping block Jacobi and RAS methods respectively (for further details, consult[5, Section 2.1]), see Figures 4, 5. Because of the banded sparsity of the matrices, the optimal method does not converge in two iterations as it is known, see [5, page 10, Proposition 4.4]. The algorithm O_{p+1} has similar behavior as the optimal algorithm. In table 1, we show the number of iterations taken by various methods when used as iterative solvers and as preconditioners for GMRES in order to achieve a residual of 10^{-8}. We can see that AOSMs work well combined with the IGA method, outperforming the classical Schwarz methods.

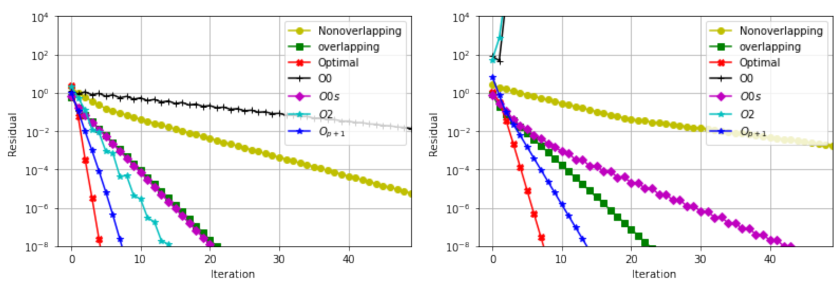

Fig. 4 Convergence history of Additive (7) AOSM with respect to $p = 4, 5$.

In Tables 2 and 3 we present the numerical experiments and the behavior of L^2-norms for the parallel algorithm (10) using isogeometric analysis. We show results for overlapping and non-overlapping decompositions, with the exact solution $u(x, y) = x\,(1 - x)\,y\,(1 - y)$, and $\lambda_1 = \lambda_2 = 0.075$.

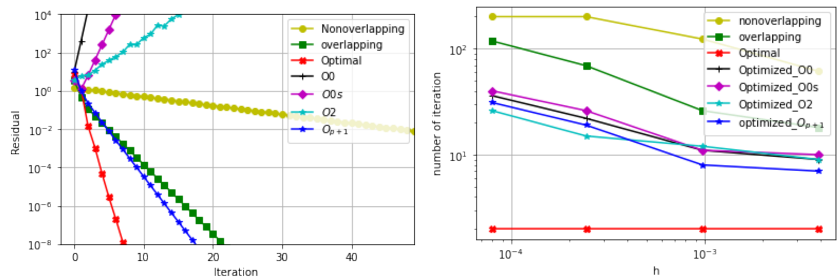

Fig. 5 Left: Convergence history of additive (7) with respect to $p = 6$. Right: The asymptotic behaviors of all methods with respect to h and $p = 2$.

Table 1 Number of iterations to attends a residual of 10^{-8} for: Additive (7) AOSM+IGA used as iterative method: top 32 elements, bottom 64 element in each direction (left), additive (7) AOSM+IGA used as preconditioner method: top 32 elements, bottom 64 element in each direction (right).

degree	Nonoverlap	Overlap	Optimal	O0	O0s	O2	O_{p+1}
1	61	29	2	50	29	NC	NC
2	75	26	2	11	10	12	8
3	87	28	7	28	24	13	9
4	92	24	6	167	24	16	9
5	111	26	9	NC	49	NC	16
6	132	25	9	NC	NC	NC	20
1	122	48	2	187	153	NC	NC
2	135	47	2	122	21	97	9
3	148	43	2	89	29	72	11
4	147	45	5	33	44	17	11
5	146	41	6	41	41	NC	12
6	187	63	4	NC	32	NC	22

degree	Nonoverlap	Overlap	Optimal	O0	O0s	O2	O_{p+1}
1	22	13	2	34	17	NC	NC
2	25	11	2	7	7	8	5
3	26	12	4	12	9	8	5
4	26	12	4	55	10	10	6
5	27	13	5	NC	33	NC	8
6	29	13	5	NC	NC	NC	9
1	23	16	2	68	49	NC	NC
2	27	18	2	73	15	51	3
3	20	19	2	40	15	45	3
4	22	12	3	34	22	8	4
5	25	17	3	21	16	NC	5
6	30	26	2	NC	12	NC	3

Table 2 L^2-norm without overlap after 10 iterations with respect to the number of element 16×16 (left), and 32×32 (right) for OSM method.

degree	$\|u - u_1^h\|_{L^2(\Omega_1)}$	$\|u - u_2^h\|_{L^2(\Omega_2)}$
2	2.80753e-07	3.13938e-06
3	3.82578e-07	1.66811e-06
4	4.89011e-07	1.12886e-06
5	5.38786e-07	8.99693e-07
6	5.71149e-07	7.84679e-07

degree	$\|u - u_1^h\|_{L^2(\Omega_1)}$	$\|u - u_2^h\|_{L^2(\Omega_2)}$
2	5.52967e-07	8.66084e-07
3	5.87442e-07	7.20705e-07
4	6.05761e-07	6.65119e-07
5	6.08555e-07	6.47844e-07
6	6.41174e-07	6.28697e-07

Table 3 L^2-norm with overlap $\delta = 0.2$ after 10 iterations with respect to the number of elements 16×16 (left), and 32×32 (right) for OSM method.

degree	$\|u - u_1^h\|_{L^2(\Omega_1)}$	$\|u - u_2^h\|_{L^2(\Omega_2)}$
2	2.32116e-09	2.32095e-09
3	3.95799e-08	3.88673e-08
4	3.08054e-08	3.08057e-08
5	1.53245e-12	4.18834e-12
6	7.09721e-10	3.29485e-08

degree	$\|u - u_1^h\|_{L^2(\Omega_1)}$	$\|u - u_2^h\|_{L^2(\Omega_2)}$
2	6.62172e-09	8.50223e-10
3	7.10352e-09	7.10352e-09
4	2.85556e-08	3.08303e-08
5	2.40483e-08	2.4419e-08
6	3.0926e-08	1.34111e-09

Concluding remarks

We presented an algebraic computational technique for solving a model problem that has been discetized using IGA. Our numerical experiments suggest that AOSM are well-suited for IGA. However, we found that the methods O0, O0s, and O2 are not effective in capturing the sparsity of IGA matrices, resulting in deteriorating performance. On the other hand, the O_{p+1} method efficiently captures the sparsity of the matrices. Our simulations of OSM for the model problem are encouraging for further analysis of OSM with IGA.

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