An Alternating Approach for Optimizing Transmission Conditions in Algebraic Schwarz Methods

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1 Introduction

Approximating transmission conditions is very important for Optimized Schwarz Methods (OSM) [2]. For the Algebraic Optimized Schwarz Method (AOSM) [4], approximations need to be done purely algebraically, leading to a challenging minimization problem. A first approach we proposed is to use SPAI [6] to approximate certain intermediate inverses [3]. The resulting method does however not capture the classical behavior of optimized Schwarz methods. In [5] another approach is explored using low-rank approximations, see also [4] for approximate factorization techniques, and [1, 7] for algebraically formulated transmission conditions. We propose here a new approach, based on an alternating method. In section 2 we describe two variants of the alternating method used to approximate the transmission blocks needed in AOSM: a theoretical one using exact inverse information, and a more practical one using SPAI approximations. In section 3 we present numerical evidence to support our findings.

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2 The alternating algorithm to approximate transmission blocks

To describe the alternating algorithm, we consider linear systems of the form

$$Au = f,$$

where the $n \times n$ matrix A usually comes from finite element or finite difference discretizations of a partial differential equation. We further assume that A has a block banded shape of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & & \\ A_{21} & A_{22} & A_{23} & \\ & A_{32} & A_{33} & A_{34} \\ & & A_{43} & A_{44} \end{bmatrix}, \tag{1}$$

with A_{ij} blocks of size $n_i \times n_j$, $i, j = 1, \dots, 4$, and $n = \sum_i n_i$. The structure of the matrix A corresponds to a two-subdomain decomposition where we assume that $n_1 \gg n_2$ and $n_4 \gg n_3$, i.e. $n_2 + n_3$ is related to the overlap size. For generalizations to more subdomains, see [4, Section 6]. The iteration operators corresponding to the additive and the multiplicative AOSM are given by

$$T_{\text{ORAS}} = I - \sum_{i=1}^{2} \tilde{R}_i^T \tilde{A}_i^{-1} R_i A, \text{ and } T_{\text{ORMS}} = \prod_{i=1}^{2} (I - \tilde{R}_i^T \tilde{A}_i^{-1} R_i A), \tag{2}$$

where the classical restriction operators are $R_1 := [I \; O]$ and $R_2 := [O \; I]$, which have order $(n_1 + n_2)n$ and $(n_3 + n_4)n$. The transpose of these operators, R_i^T, are prolongation operators, and $\tilde{R}_i^T$ are RAS-variants thereof, see [4] for more details. The matrices $\tilde{A}_i$ are defined by

$$\tilde{A}_1 = \begin{bmatrix} A_{11} & A_{12} & \\ A_{21} & A_{22} & A_{23} \\ & A_{32} & A_{33} + D_1 \end{bmatrix}, \qquad \tilde{A}_2 = \begin{bmatrix} A_{22} + D_2 & A_{23} & \\ A_{32} & A_{33} & A_{34} \\ & A_{43} & A_{44} \end{bmatrix}, \tag{3}$$

for which the transmission blocks D_1 and D_2 have to be determined for fast convergence. It has been shown in [4, Theorem 3.2] that the asymptotic convergence factor of AOSM depends on the product of the two norms

$$\| (I + D_1 B_{33})^{-1} [D_1 B_{12} - A_{34} B_{13}] \|_2, \; \| (I + D_2 B_{11})^{-1} [D_2 B_{32} - A_{21} B_{31}] \|_2. \tag{4}$$

The goal is to find D_1 and D_2 to minimize the norms in (4), where the B matrices are given by

$$\begin{bmatrix} B_{31} \\ B_{32} \\ B_{33} \end{bmatrix} := \begin{bmatrix} A_{11} & A_{12} & \\ A_{21} & A_{22} & A_{23} \\ & A_{32} & A_{33} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad \begin{bmatrix} B_{11} \\ B_{12} \\ B_{13} \end{bmatrix} := \begin{bmatrix} A_{22} & A_{23} & \\ A_{32} & A_{33} & A_{34} \\ & A_{43} & A_{44} \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}. \tag{5}$$

This implies that

$$B_{13} = -A_{44}^{-1} A_{43} B_{12} \quad \text{and} \quad B_{31} = -A_{11}^{-1} A_{12} B_{32}. \tag{6}$$

Substituting B_{13} and B_{31} into (4), we obtain for the convergence factor estimates

$$\begin{aligned} &\| (I + D_1 B_{33})^{-1} (D_1 + A_{34} A_{44}^{-1} A_{43}) B_{12} \|_2, \\ &\qquad\qquad \| (I + D_2 B_{11})^{-1} (D_2 + A_{21} A_{11}^{-1} A_{12}) B_{32} \|_2. \end{aligned} \tag{7}$$

The optimal choice for the transmission matrices making the norms vanish is therefore

$$D_{1,\text{opt}} = -A_{34} A_{44}^{-1} A_{43} \qquad \text{and} \qquad D_{2,\text{opt}} = -A_{21} A_{11}^{-1} A_{12}, \tag{8}$$

which requires however components of the expensive inverses of the large matrices A_{11} and A_{44} and is thus not very practical.

2.1 Alternating algorithm with exact blocks B_{ij}

We start by describing the new alternating algorithm to compute simple diagonal approximations to the optimal $D_{1,\text{opt}}$ in (8) (the algorithm for approximations to $D_{2,\text{opt}}$ is analogous):

Initialization: Set $D_{1,0} := -A_{34} \tilde{A}_{44}^{-1} A_{43}$, where $\tilde{A}_{44}^{-1}$ is a diagonal SPAI approximation of A_{44}^{-1}. Due to the sparsity of A_{34} and A_{43} and the SPAI approximation, $D_{1,0}$ is diagonal and almost constant on the diagonal, except for the two endpoints. For this reason we consider constant diagonal matrices $D_{1,m}$ for $m \geq 1$.

Iteration: For iteration index $m = 1, 2, \ldots$, compute

$$\begin{aligned} p_m &:= \operatorname{argmin}_{p \in \mathbb{R}} \| \left(I + D_{1,m-1} B_{33}\right)^{-1} \left(p\, I + A_{34} A_{44}^{-1} A_{43}\right) B_{12} \|_2; \\ D_{1,m} &:= p_m I; \end{aligned} \tag{9}$$

In (9), we use the exact inverse of the block A_{44}, and we do so also for the blocks B_{12} and B_{33}. The calculation of these blocks is very expensive which makes this first approach expensive. In the next subsection we will present a more practical approach using SPAI approximations for these blocks. Thus the cost in evaluating (9) is reduced significantly.

The minimization problems in (9) are scalar problems for $p \in \mathbb{R}$, but we can obtain tridiagonal and pentadiagonal alternating approximation algorithms by replacing pI in the algorithm above by matrices with tridiagonal and pentadiagonal matrices with constant diagonals leading to 3 and 5 degrees of freedom, respectively. We will use the name Alternating SPAI(1) for diagonal approximations, Alternating SPAI(3) for tridiagonal ones, and Alternating SPAI(5) for pentadiagonal ones.

We next investigate how the alternating algorithm converges to the minimum obtained by globally minimizing the norm in (7). We consider the model prob-

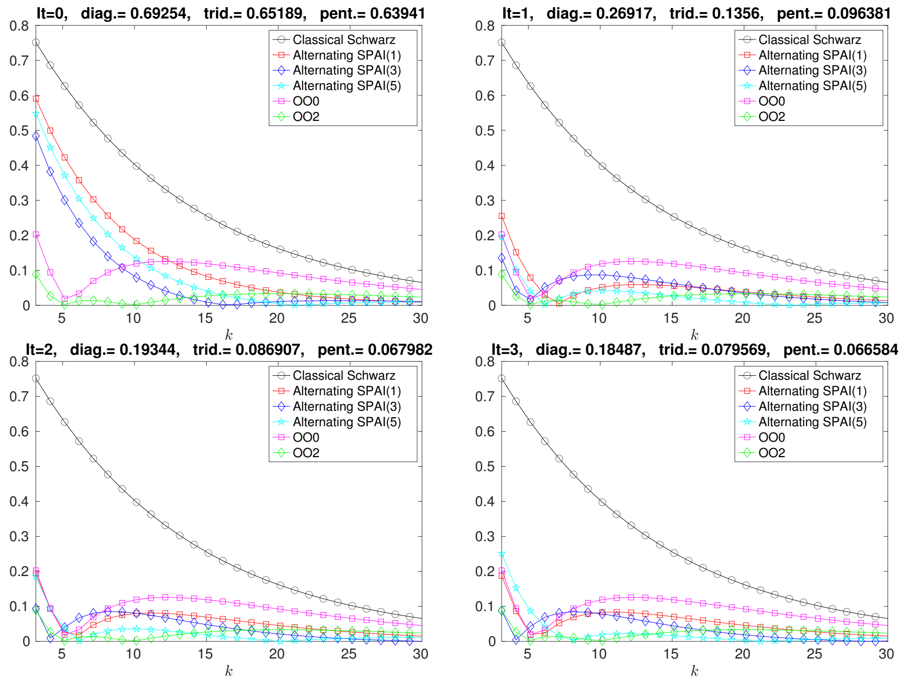

Fig. 1 From top left to bottom right: convergence factor estimates for the initial approximation with SPAI, and then the first three iterations of the new alternating approach.

lem $-\Delta u = f$ in a square domain $\Omega = (0, 1)^2$, discretized using standard centered finite differences with mesh size $h = \frac{1}{N+1}$ for $N = 2^5$. We decompose the domain into two equal overlapping subdomains in the x direction with overlap $3h$. In order to visualize the convergence and compare the convergence factor estimates obtained by the alternating method with the convergence factors of the OO0 and OO2 OSM algorithms from [2], we plot them in Fourier space as function of the Fourier variable k in the y direction, see [3] for more details. We show in Figure 1 the results for the initial approximation with SPAI, and then the first 3 iterations of our new alternating algorithm. We see that for the SPAI initial guess, the behavior of the diagonal, tridiagonal and pentadiagonal methods is not like for OSM, their convergence for low frequencies, k small, is more like for the classical Schwarz method. This is consistent with the analysis presented in [3]. With the first correction of our new alternating procedure however, we can see a great improvement for low frequency behavior, the methods obtained from the alternating procedure now behave like OSM. The second and third iterations give further improvements.

In Figure 2, we show on the left the maximum of the two norms in (4) for the first 8 iterations of the alternating algorithm. The algorithm converges very rapidly to the global minimization of the norm (4) shown in Figure 2 on the right.

Iteration	Diagonal	Tridiagonal	Pentadiagonal
0	0.69254	0.651890	0.639410
1	0.26917	0.135600	0.096381
2	0.19344	0.086907	0.067982
3	0.18487	0.079569	0.066584
4	0.18395	0.078645	0.066586
5	0.18385	0.078135	0.066578
6	0.18384	0.078124	0.066579
7	0.18384	0.078123	0.066579
8	0.18384	0.078123	0.066579

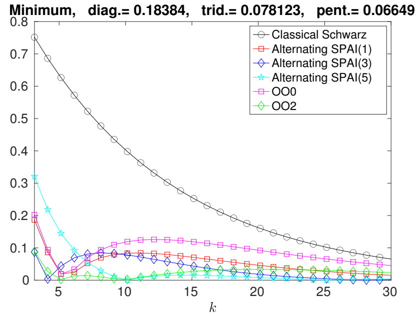

Fig. 2 Left: Maximum of the two norms in (4) for the first 8 iterations of the alternating algorithm. Right: Convergence factors for the global minimization of the norm.

2.2 Alternating algorithm using SPAI approximations for B_{ij}

The alternating algorithm described above requires the calculation of subblocks of A_{11}^{-1} and A_{44}^{-1} and the resulting blocks B_{ij} which is expensive. We now consider SPAI approximations $\tilde{B}_{ij}$ for the blocks B_{ij} and we modify the minimization problem in (9) of the alternating algorithm to

$$p_m = \operatorname*{argmin}_{p\in\mathbb{R}} \| \left(I + D_{1,m-1}\tilde{B}_{33}\right)^{-1} \left(p\,\tilde{B}_{12} - A_{34}\tilde{B}_{13}\right) \|_2. \tag{10}$$

This step thus does no longer require to calculate the inverses A_{11}^{-1} and A_{44}^{-1}, and the modified alternating algorithm requires to compute approximations of the blocks B_{ij} only once.

In Figure 3 we present the behavior of the convergence factor corresponding to each method with respect to the fill-in[1] used in the SPAI approximations for the blocks B_{ij} after 8 iterations. On the top left, we used a diagonal SPAI approximation, and we see that this is not enough for the alternating procedure to improve the low frequency behavior toward OSM. On the top right we used a tridiagonal SPAI approximation and we see that this also does not suffice. In order to obtain good low frequency behavior like OSM, we need to use sufficient fill-in in the SPAI approximations for B_{ij}, as we see in the bottom left and right panels of Figure 3. Note that this is a one time approximation and because of the nature of the SPAI algorithm we can approximate the columns one by one independently, and thus in parallel. In the numerical experiments section we present a comparison between sequential and parallel estimations of B_{ij}.

For the minimization of the linear problems involved in the alternating algorithm we used the Nelder-Mead algorithm implemented in `fminsearch` in Matlab. In the numerical experiments we show that minimizing the norm globally takes more time compared to the time if we minimize 8 linear problems associated to 8 iterations to obtain convergence of the alternating algorithm.

[1] Here, i fill-in means i fill-in entries per column are allowed.

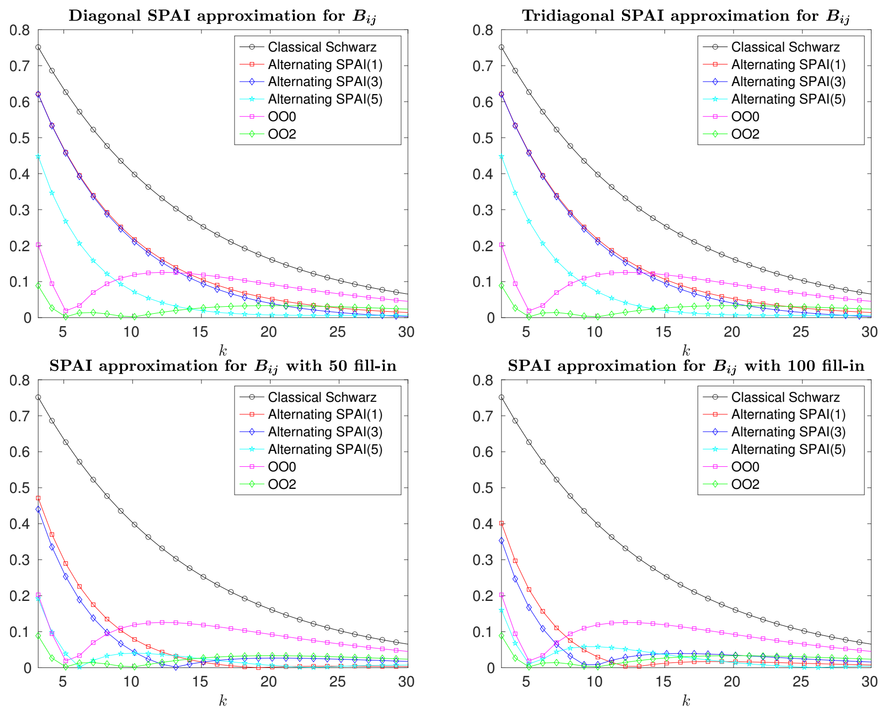

Fig. 3 The behavior of the convergence factor with respect to the fill-in used in the SPAI approximation of the blocks B_{ij} after 3 iterations.

Note that this minimization process can be performed offline, it is independent of the solution process when the Schwarz method is running, and “alternating” here refers to the optimization process, not to the Schwarz method, which can run in parallel or alternating fashion.

3 Numerical experiments

For our numerical experiments we consider the advection-reaction-diffusion equation,

$$\eta u - \nabla \cdot (a \nabla u) + b \cdot \nabla u = f,$$

where $a = a(x, y) > 0$, $b = [b_1(x, y), b_2(x, y)]^T$, $\eta = \eta(x, y) \geq 0$, with

$$b_1 = y - \frac{1}{2}, \quad b_2 = -x + \frac{1}{2}, \quad \eta = x^2 \cos(x + y)^2, \quad a = 1 + (x + y)^2 e^{x-y}.$$

We decompose the unit square domain $\Omega = (0, 1) \times (0, 1)$ into two subdomains $\Omega_1 = (0, \beta) \times (0, 1)$ and $\Omega_2 = (\alpha, 1) \times (0, 1)$, where $0 < \alpha \leq \beta < 1$. Using

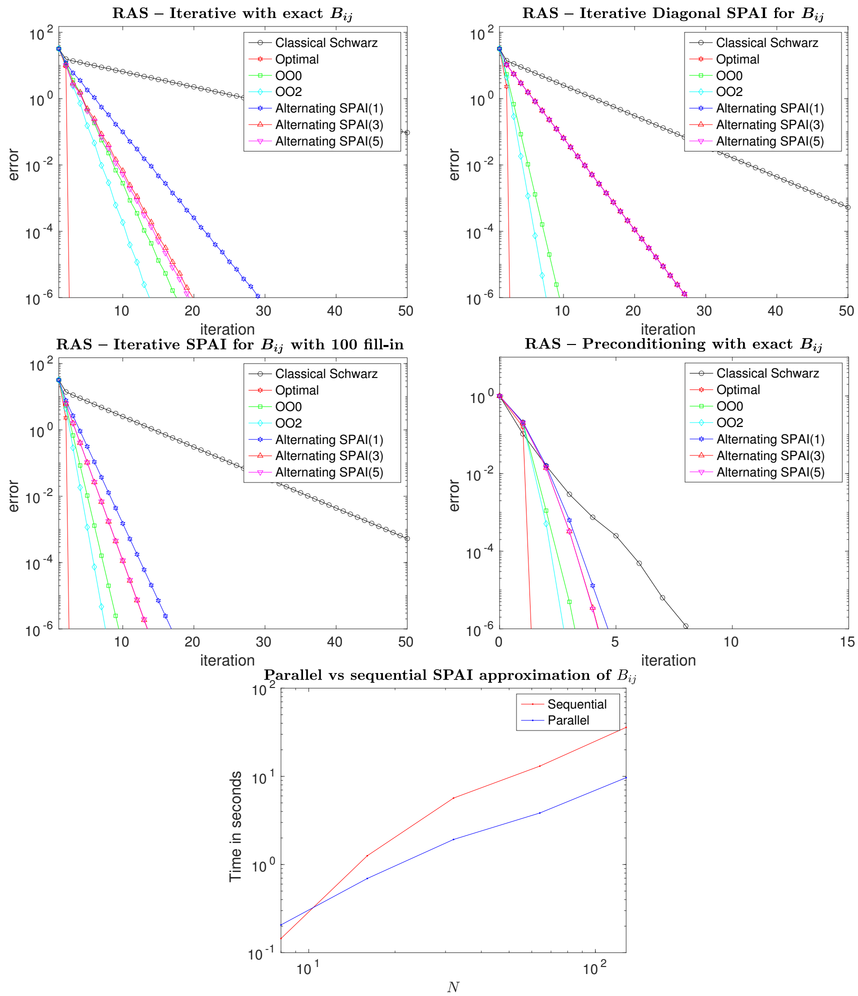

Fig. 4 Convergence of the various methods for the advection-reaction-diffusion model problem. Top left: exact B_{ij}. Top right: diagonal SPAI approximations for B_{ij}. Middle left: SPAI approximations for B_{ij} with 100 fill-in. Middle right: All methods used as preconditioners. Bottom: computational time to compute the corresponding B_{ij} sequentially and in parallel.

a finite difference method, the corresponding matrix A is of size 1024×1024, with a decomposition into two subdomains where the blocks A_{11}, A_{12}, A_{21}, and A_{22} are of size 480×480, 480×32, 32×480, and 32×32 respectively.

In Figure 4 we present the error as a function of the iteration index for the various methods based on the alternating technique. We compare these methods again with OO0, OO2, and also the optimal Schwarz method obtained with the choice (8). We see on the top left in Figure 4 that the alternating SPAI methods are optimized Schwarz methods if we use the exact values of B_{ij}. For alternating

SPAI(1) in the top right in Figure 4, convergence is not as good, but we need only $0.005034 \times 8 = 0.0403$ seconds to calculate the parameter p where 8 is the number of iterations for the alternating algorithm to converge to the minimum. In contrast, we need 4.152525 seconds to calculate the same value of the parameter p if we globally minimize the norm in (4). Using more fill-in in the SPAI approximation, rapid convergence can be recovered, see the bottom-left of Figure 4. This is more expensive, but one can calculate the SPAI approximations for the blocks B_{ij} in parallel. For instance the time needed to calculate the blocks B_{ij} for a 100 fill-in without using `parfor`, in Matlab, is 2.540780 seconds, while with `parfor` we need only 0.005207 seconds.

4 Concluding remarks

We proposed an alternating SPAI technique to minimize the convergence factor estimate for the algebraic optimized Schwarz methods from [4]. By alternating between terms involved in the convergence factor estimate, we reduce the minimization process to solve linear problems instead of non-linear ones. The required time to calculate the parameters of AOSM is thus reduced drastically, but we have also shown that one still needs quite accurate SPAI estimates of the terms in the convergence factor estimate for AOSM in order to obtain good optimized parameters.

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