On Algebraic Bounds for POSM and MRAS

Martin J. Gander and Michal Outrata

1 Introduction and preliminaries

We consider the Poisson equation as our model problem, i.e.,

$$\Delta u = f \quad \text{in } \Omega := (-a, a) \times (0, b) \quad \text{and} \quad u = g \quad \text{on } \partial\Omega, \tag{1}$$

where f and g are given. We decompose Ω into two subdomains $\Omega_1 := (-a, L/2) \times (0, b)$ and $\Omega_2 := (-L/2, a) \times (0, b)$ with interfaces Γ_1 and Γ_2, overlap $O := (-L/2, L/2) \times (0, b)$ (if $L > 0$) and complements $\Theta_2 := \Omega\backslash\Omega_1$ and $\Theta_1 := \Omega\backslash\Omega_2$. Creating an equidistant mesh on Ω with mesh size h, we denote by $N_r + 1$ the number of grid rows and $N_c + 1$ the number of grid columns, see Figure 1. We also define the one-grid-column-prolonged subdomains $\Omega_1^h := (-a, L/2+h)\times(0, b)$ and $\Omega_2^h := (-L/2 - h, a) \times (0, b)$ and also their interfaces $\Gamma_1^h := (L/2 + h) \times (0, b)$ and $\Gamma_2^h := (-L/2 - h) \times (0, b)$. We discretize (1) with a finite difference scheme, obtaining the block tridiagonal system matrix

$$\begin{bmatrix} A_{\Theta_1} & A_{\Theta_1,\Gamma_2} & & & \\ A_{\Gamma_2,\Theta_1} & A_{\Gamma_2} & A_{\Gamma_2,O} & & \\ & A_{O,\Gamma_1} & A_O & A_{O,\Gamma_1} & \\ & & A_{\Gamma_1,O} & A_{\Gamma_1} & A_{\Gamma_1,\Theta_2} \\ & & & A_{\Theta_2,\Gamma_1} & A_{\Theta_2} \end{bmatrix}. \tag{2}$$

We follow the notation of [3, Section 6.1] and introduce the *parallel optimized Schwarz method* (POSM) with the transmission operators $\mathcal{P}_{\Gamma_1} = \mathcal{P}_{\Gamma_2} = p\mathcal{I}$ and $\mathcal{Q}_{\Gamma_1} = \mathcal{Q}_{\Gamma_2} = \mathcal{I}$ acting on the Dirichlet and Neumann data along the interfaces. Hence POSM is given by the iteration operator $\mathcal{T} : (u_1^{(n-1)}, u_2^{(n-1)}) \mapsto (u_1^{(n)}, u_2^{(n)})$, where $u_1^{(n)}, u_2^{(n)}$ are given as the solutions of the subdomain problems

Martin J. Gander, Michal Outrata
University of Geneva, e-mail: martin.gander@unige.ch, michal.outrata@unige.ch

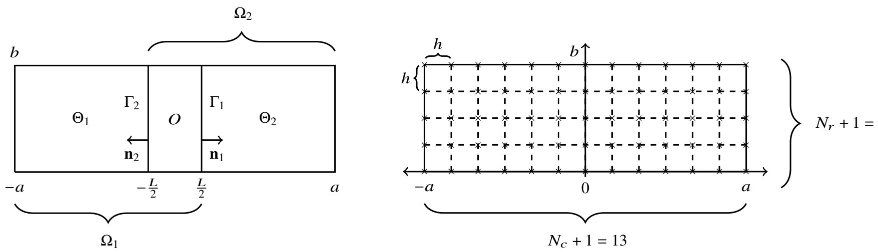

Fig. 1 The physical domain (left), and the discrete mesh (right).

$$\begin{aligned} \Delta u_i^{(n)} &= f \quad \text{in } \Omega_i, \quad u_i^{(n)} = g \quad \text{on } \partial\Omega_i \backslash \Gamma_i, \\ \mathbf{n}_i \cdot \nabla u_i^{(n)} + p u_i^{(n)} &= \mathbf{n}_i \cdot \nabla u_j^{(n-1)} + p u_j^{(n-1)} \quad \text{on } \Gamma_i, \end{aligned} \qquad \text{for } i, j = 1, 2, \ |i - j| = 1.$$

The convergence factor of POSM (see [1, Proposition 2]) as a function of $a, b, L/2$ and the Fourier mode $k \in \mathbb{N}$ is given by

$$\frac{\frac{k\pi}{b}\coth\left(\frac{k\pi}{b}(a - L/2)\right) - p}{\frac{k\pi}{b}\coth\left(\frac{k\pi}{b}(a + L/2)\right) + p} \cdot \frac{\sinh\left(\frac{k\pi}{b}(a - L/2)\right)}{\sinh\left(\frac{k\pi}{b}(a + L/2)\right)}. \tag{3}$$

Writing (2) in its *augmented form* and modifying the interface block rows we get

$$A_{\text{aug}} := \begin{bmatrix} \tilde{A}_{\Omega_1} & \tilde{A}_{\Omega_1,\Omega_2} \\ \tilde{A}_{\Omega_2,\Omega_1} & \tilde{A}_{\Omega_2} \end{bmatrix} := \begin{bmatrix} A_{\Theta_1} & A_{\Theta_1,\Gamma_2} & & & & & & \\ A_{\Gamma_2,\Theta_1} & A_{\Gamma_2} & A_{\Gamma_2,O} & & & & & \\ & A_{O,\Gamma_2} & A_O & A_{O,\Gamma_1} & & & & \\ & & A_{\Gamma_1,O} & \tilde{A}_{\Gamma_1} & & & \tilde{A}_{\Gamma_1,\Gamma_1} & A_{\Gamma_1,\Theta_2} \\ A_{\Gamma_2,\Theta_1} & \tilde{A}_{\Gamma_2,\Gamma_2} & & & \tilde{A}_{\Gamma_2} & A_{\Gamma_2,O} & & \\ & & & & A_{O,\Gamma_2} & A_O & A_{O,\Gamma_1} & \\ & & & & & A_{\Gamma_1,O} & A_{\Gamma_1} & A_{\Gamma_1,\Theta_2} \\ & & & & & & A_{\Theta_2,\Gamma_1} & A_{\Theta_2} \end{bmatrix}, \tag{4}$$

where we introduced the discrete transmission conditions in the last block row of $[A_{\Omega_1}\ A_{\Omega_1,\Omega_2}]$ and the first block row of $[A_{\Omega_2,\Omega_1}\ A_{\Omega_2}]$, which are now given by

$$\tilde{A}_{\Gamma_1} := A_{\Gamma_1} + D,\ \tilde{A}_{\Gamma_1,\Gamma_1} := -D \quad \text{and} \quad \tilde{A}_{\Gamma_2} := A_{\Gamma_2} + D,\ \tilde{A}_{\Gamma_2,\Gamma_2} := -D.$$

We are interested in the subdomain version of the *modified restricted additive Schwarz* (MRAS[1], see [2]), defined by its iteration matrix T,

$$T = I - \sum_{i=1}^{2} R_{\Omega_i}^T \tilde{A}_{\Omega_i}^{-1} R_{\Omega_i} \tilde{A}_{\text{aug}} \quad \text{with } R_{\Omega_1} = [I_{\Omega_1}\ 0_{\Omega_2}],\ R_{\Omega_2} = [0_{\Omega_1}\ I_{\Omega_2}]. \tag{5}$$

[1] MRAS was introduced in the so-called *globally deferred correction form*, where we iterate on the global solution unknowns, in contrast to iterating on the subdomain solution unknowns here. This is but a technicality and hence we keep the name; the equivalence is shown in [3, Section 6.1, 6.2].

Notice that the interface block structure of MRAS does *not* match the one in [3, Algorithm 2] but the transmission matrix D is chosen to get fast convergence, analogously to the parameter p in POSM. Setting

$$E_{\Gamma_2}^{\Omega_1} := \left[0_{\Theta_1} I_{\Gamma_2} 0_O 0_{\Gamma_1}\right]^T,\ E_{\Gamma_1}^{\Omega_1} := \left[0_{\Theta_1} 0_{\Gamma_2} 0_O I_{\Gamma_1}\right]^T,\ E_{\Theta_1}^{\Omega_1} := \left[A_{\Gamma_2,\Theta_1} 0_{\Gamma_2} 0_O 0_{\Gamma_1}\right]^T,$$
$$E_{\Gamma_2}^{\Omega_2} := \left[I_{\Gamma_2} 0_O 0_{\Gamma_1} 0_{\Theta_2}\right]^T,\ E_{\Gamma_1}^{\Omega_2} := \left[0_{\Gamma_2} 0_O I_{\Gamma_1} 0_{\Theta_2}\right]^T,\ E_{\Theta_2}^{\Omega_2} := \left[0_{\Gamma_2} 0_O 0_{\Gamma_1} A_{\Theta_2,\Gamma_1}\right]^T,$$

we can write

$$\tilde{A}_{\Omega_i} = A_{\Omega_i} + E_{\Gamma_i}^{\Omega_i} D \left(E_{\Gamma_i}^{\Omega_i}\right)^T, \quad i = 1, 2,$$

and formulate a convergence result for MRAS, analogue to [2, Theorem 3.2].

Theorem 1 ([2, Section 3])
The MRAS iteration matrix T in (5) *has the structure*

$$T = \begin{bmatrix} 0 & K \\ L & 0 \end{bmatrix}, \quad \begin{aligned} K &:= A_{\Omega_1}^{-1} E_{\Gamma_1}^{\Omega_1} \left[I + D(A_{\Omega_1}^{-1})_{\Gamma_1,\Gamma_1}\right]^{-1} \left(-D(E_{\Gamma_1}^{\Omega_2})^T + (E_{\Theta_2}^{\Omega_2})^T\right), \\ L &:= A_{\Omega_2}^{-1} E_{\Gamma_2}^{\Omega_2} \left[I + D(A_{\Omega_2}^{-1})_{\Gamma_2,\Gamma_2}\right]^{-1} \left(-D(E_{\Gamma_2}^{\Omega_1})^T + (E_{\Theta_1}^{\Omega_1})^T\right). \end{aligned} \tag{6}$$

Moreover, the asymptotic convergence factor of MRAS is bounded by

$$\begin{aligned} &\sqrt{\|M_1 B_1\|_2 \cdot \|M_2 B_2\|_2}, \\ M_1 &:= \left[I + D(A_{\Omega_1}^{-1})_{\Gamma_1,\Gamma_1}\right]^{-1} \left(-D - A_{\Gamma_1,\Theta_2} A_{\Theta_2}^{-1} A_{\Theta_2,\Gamma_1}\right),\ B_1 := (A_{\Omega_2}^{-1})_{\Gamma_1,\Gamma_2}, \\ M_2 &:= \left[I + D(A_{\Omega_2}^{-1})_{\Gamma_2,\Gamma_2}\right]^{-1} \left(-D - A_{\Gamma_2,\Theta_1} A_{\Theta_1}^{-1} A_{\Theta_1,\Gamma_2}\right),\ B_2 := (A_{\Omega_1}^{-1})_{\Gamma_2,\Gamma_1}. \end{aligned} \tag{7}$$

Due to the symmetry of the model problem and the method we have $B := B_1 = B_2$ and $M := M_1 = M_2$, which in turn simplifies the bound in (7) *to $\|MB\|_2$.*

2 Analysis of the MRAS bound and its reformulation

First, we recall the sine series expansion in the y direction $\mathcal{F}_y$, so that we have

$$u(x, y) = \sum_{k=1}^{+\infty} \mathcal{F}_y u(x, k) \sin\left(\frac{k\pi}{b} y\right) \equiv \sum_{k=1}^{+\infty} \hat{u}(x, k) \sin\left(\frac{k\pi}{b} y\right),$$

with[2] $\mathcal{F}_y u := \int_0^b u(x, y) \sin(k\pi y/b) \mathrm{d}y$. Next, we factor out $(A_{\Omega_1}^{-1})_{\Gamma_1,\Gamma_1}$ and $(A_{\Omega_2}^{-1})_{\Gamma_2,\Gamma_2}$ on the left from $M_{1,2}$, so that instead of (7) we focus on the asymptotically equivalent

[2] Using the sine series relies on the Dirichlet boundary conditions (BCs) along $\{y = 0\}$ and $\{y = b\}$ in (1); for different BCs see [4].

$$MB := \underbrace{\left[\left((A_{\Omega_1}^{-1})_{\Gamma_1,\Gamma_1}\right)^{-1} + D\right]^{-1}}_{\left(T^{\text{Denom}}\right)^{-1}} \underbrace{\left(-D - A_{\Gamma_1,\Theta_2}A_{\Theta_2}^{-1}A_{\Theta_2,\Gamma_1}\right)}_{T^{\text{Numer}}} \underbrace{(A_{\Omega_2}^{-1})_{\Gamma_1,\Gamma_2}\left((A_{\Omega_2}^{-1})_{\Gamma_2,\Gamma_2}\right)^{-1}}_{T^{\text{Over}}}. \quad (8)$$

The *key* question is whether the bound (7), which now becomes $\|MB\|$, is the discrete analogue of (3) – piece by piece. Linking each of the blocks in (8) to a discrete linear operator with a continuous counterpart, we analyze it using the Fourier series expansion. Taking $\mathbf{b} \in \mathbb{R}^{N_r-1}$ and interpolating it to a function $\gamma\colon \Gamma_1^h \to \mathbb{R}$, the following problems are equivalent up to the FD discretization:

$$A_{\Omega_1}\mathbf{u} = -\frac{1}{h^2}E_{\Gamma_1}^{\Omega_1}\mathbf{b} \qquad \text{and} \qquad \begin{array}{c} \Delta u = 0 \quad \text{in } \Omega_1^h, \\ u = 0 \quad \text{on } \partial\Omega_1^h\backslash\Gamma_1^h, \quad \text{and} \quad u = \gamma \quad \text{on } \Gamma_1^h. \end{array} \quad (9)$$

Defining the solution operator by $\mathcal{S}_1(\gamma) = u\big|_{\Gamma_1}$ where u is the solution of (9), we have (up to the FD discretization) the equivalence of the linear operators $-1/h^2(A_{\Omega_1}^{-1})_{\Gamma_1,\Gamma_1}$ and $\mathcal{S}_1$. To calculate $\mathcal{S}_1$ we expand in the y variable using $\mathcal{F}_y$, simplifying the continuous problem in (9) to the semi-discrete problem

$$\begin{aligned} &\left(\partial_{xx} - \left(\tfrac{k\pi}{b}\right)^2\right)\hat{u}(x,k) = 0 \quad \text{for } x \in (-a, L/2+h) \text{ and } k \in \mathbb{N}, \\ &\hat{u}(-a,k) = 0 \quad \text{and} \quad \hat{u}(L/2+h,k) = \hat{\gamma}(k) \quad \text{for } k \in \mathbb{N}, \end{aligned} \quad (10)$$

and denote by $\hat{\mathcal{S}}_1 := \mathcal{F}_y\mathcal{S}_1$ the Fourier symbol of $\mathcal{S}_1$. A direct calculation yields

$$\hat{u}(x,k) = \frac{\sinh\left(\frac{k\pi}{b}(a+x)\right)\hat{\gamma}(k)}{\sinh\left(\frac{k\pi}{b}(a+L/2+h))\right)}, \quad \hat{\mathcal{S}}_1\hat{\gamma}(k) = \frac{\sinh\left(\frac{k\pi}{b}(a+L/2)\right)}{\sinh\left(\frac{k\pi}{b}(a+L/2+h))\right)}\hat{\gamma}(k).$$

Therefore, the eigenvalues of the linear operator $-1/h^2(A_{\Omega_1}^{-1})_{\Gamma_1,\Gamma_1}$ approximate the modes $k = 1, \ldots, N_r - 1$ of $\hat{\mathcal{S}}_1$ given above, as we see in Figure 2. The rest of the blocks in (8) are summarized in Table 1 and illustrated in Figure 2, see [4] for detailed calculations. We see that the approximation is very accurate for the low-frequency modes but not quite accurate for the high-frequency ones. If D diagonalizes in the same basis as the rest of the blocks and we denote its eigenvalues by $\delta_1, \ldots, \delta_{N_r-1}$, then the eigenvalues of $T^{\text{Denom}}, T^{\text{Numer}}, T^{\text{Over}}$ approximate certain discrete (truncated) Fourier symbols we present in Table 2 and illustrate in Figure 3. We see that the inaccuracy on the high frequencies is still present. More importantly, comparing Table 2 with (3) shows that the contraction factor due to the domain overlap in (3) matches exactly θ_k for each k, i.e., the one due to the continuous representation of T^{Over}. However, this is clearly *not* the case for the contraction factor due to the transmission condition induced by D. The ratio η_k/ζ_k shows that choosing $\delta_k = p$ (the naive choice) is *not* the correct one (see [4] for more details) and we continue by reformulating Theorem 1 to reflect also the transmission part of (3).

Table 1 The blocks and corresponding linear operators (LO) from (8).

block	discrete LO	continuous LO	Fourier symbol
$(A^{-1}_{\Omega_1})_{\Gamma_1,\Gamma_1}$	$-\frac{1}{h^2}(A^{-1}_{\Omega_1})_{\Gamma_1,\Gamma_1}$	$\mathcal{S}_1 : \gamma \mapsto u\big\rvert_{\Gamma_1}$	$\hat{\mathcal{S}}_1 = \frac{\sinh\left(\frac{k\pi}{b}(a+L/2)\right)}{\sinh\left(\frac{k\pi}{b}(a+L/2+h)\right))}$
$A_{\Gamma_1,\Theta_2}A^{-1}_{\Theta_2}A_{\Theta_2,\Gamma_1}$	$-h^2 A_{\Gamma_1,\Theta_2}A^{-1}_{\Theta_2}A_{\Theta_2,\Gamma_1}$	$\mathcal{S}_2 : \gamma \mapsto u\big\rvert_{\Gamma_1}$	$\hat{\mathcal{S}}_2 = \frac{\sinh\left(\frac{k\pi}{b}(a-L/2-h)\right)}{\sinh\left(\frac{k\pi}{b}(a-L/2)\right))}$
$(A^{-1}_{\Omega_2})_{\Gamma_1,\Gamma_2}$	$-\frac{1}{h^2}(A^{-1}_{\Omega_2})_{\Gamma_1,\Gamma_2}$	$\mathcal{S}_3 : \gamma \mapsto u\big\rvert_{\Gamma_1}$	$\hat{\mathcal{S}}_3 = \frac{\sinh\left(\frac{k\pi}{b}(a-L/2)\right)}{\sinh\left(\frac{k\pi}{b}(a+L/2+h)\right))}$
$(A^{-1}_{\Omega_2})_{\Gamma_2,\Gamma_2}$	$-\frac{1}{h^2}(A^{-1}_{\Omega_2})_{\Gamma_2,\Gamma_2}$	$\mathcal{S}_4 : \gamma \mapsto u\big\rvert_{\Gamma_2}$	$\hat{\mathcal{S}}_4 = \frac{\sinh\left(\frac{k\pi}{b}(a+L/2)\right)}{\sinh\left(\frac{k\pi}{b}(a+L/2+h)\right))}$

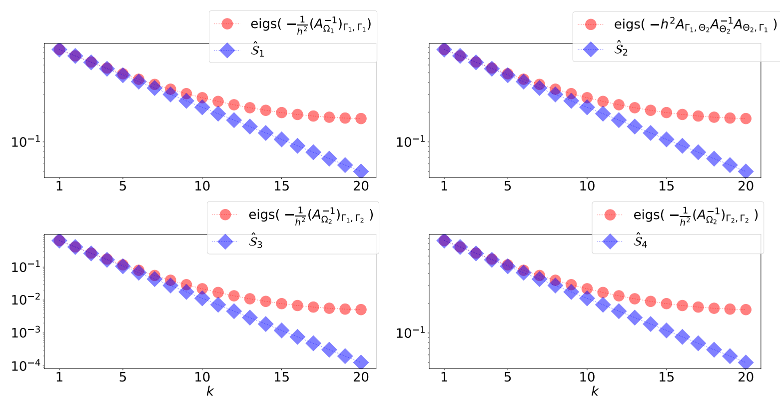

Fig. 2 Results obtained for the parameters $a = b = 1$, $L = 2h$, $N_r = 22$.

Table 2 The matrices and their corresponding (truncated) Fourier symbols.

$\left(T^{\text{Denom}}\right)^{-1}$	T^{Numer}	T^{Over}
$\eta_k := \delta_k - \frac{1}{h^2}\frac{\sinh\left(\frac{k\pi}{b}(a+L/2+h)\right)}{\sinh\left(\frac{k\pi}{b}(a+L/2)\right)}$	$\zeta_k := -\delta_k + \frac{1}{h^2}\frac{\sinh\left(\frac{k\pi}{b}(a-L/2-h)\right)}{\sinh\left(\frac{k\pi}{b}(a-L/2)\right))}$	$\theta_k := \frac{\sinh\left(\frac{k\pi}{b}(a-L/2)\right)}{\sinh\left(\frac{k\pi}{b}(a+L/2)\right))}$
$\left(\overline{T}^{\text{Denom}}\right)^{-1}$	$\overline{T}^{\text{Numer}}$	$\overline{T}^{\text{Over}}$
$\overline{\eta}_k := -\frac{1}{h}\frac{k\pi}{b}\coth\left(\frac{k\pi}{b}(a+L/2)\right) - \lambda_k$	$\overline{\zeta}_k := -\frac{1}{h}\frac{k\pi}{b}\coth\left(\frac{k\pi}{b}(a-L/2)\right) + \lambda_k$	$\overline{\theta}_k := \frac{\sinh\left(\frac{k\pi}{b}(a-L/2)\right)}{\sinh\left(\frac{k\pi}{b}(a+L/2)\right))}$

The main tool used to obtain Theorem 1 is the Sherman-Morrison-Woodbury formula for the inverse of a low-rank updated matrix, here the update was the corner block D. We now show that using the same formula for a slightly different block gives the "correct" result. We split the interface blocks as in [3, Section 5.2] and write $A_{\Gamma_1} = A^L_{\Gamma_1} + A^R_{\Gamma_1}$ and $A_{\Gamma_2} = A^L_{\Gamma_2} + A^R_{\Gamma_2}$ so that we have

$$\begin{aligned} -h(A_{\Gamma_1,O}\mathbf{u}_O + A^L_{\Gamma_1}\mathbf{u}_{\Gamma_1}) &\approx u_x\big\rvert_{\Gamma_1}, \quad -h(A_{\Gamma_1,\Theta_2}\mathbf{u}_{\Theta_2} + A^R_{\Gamma_1}\mathbf{u}_{\Gamma_1}) \approx -u_x\big\rvert_{\Gamma_1}, \\ -h(A_{\Gamma_2,O}\mathbf{u}_O + A^R_{\Gamma_2}\mathbf{u}_{\Gamma_2}) &\approx -u_x\big\rvert_{\Gamma_2}, \quad -h(A_{\Gamma_2,\Theta_1}\mathbf{u}_{\Theta_1} + A^L_{\Gamma_2}\mathbf{u}_{\Gamma_2}) \approx u_x\big\rvert_{\Gamma_2}. \end{aligned} \tag{11}$$

This is natural for FD and FEM discretizations. Using the so-called *ghost point trick* we get $A^L_{\Gamma_1} = A^R_{\Gamma_1} = \frac{1}{2}A_{\Gamma_1}$, $A^L_{\Gamma_2} = A^R_{\Gamma_2} = \frac{1}{2}A_{\Gamma_2}$. Adopting this we rewrite $\tilde{A}_{aug}$ as

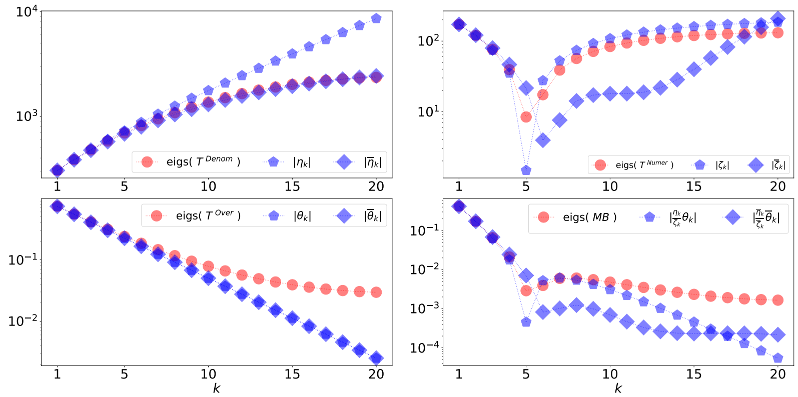

Fig. 3 Results obtained with $a = b = 1$, $L = 2h$, $N_r = 21$ and $D = \mathrm{diag}\left(\pi^2/h\right)$.

$$\overline{A}_{\text{aug}} := \begin{bmatrix} A^L_{\Omega_1} + \overline{A}_{\Omega_1} & \tilde{A}_{\Omega_1,\Omega_2} \\ \tilde{A}_{\Omega_2,\Omega_1} & A^R_{\Omega_1} + \overline{A}_{\Omega_2} \end{bmatrix} := \begin{bmatrix} A_{\Theta_1} & A_{\Theta_1,\Gamma_2} & & & & & & \\ A_{\Gamma_2,\Theta_1} & A_{\Gamma_2} & A_{\Gamma_2,O} & & & & & \\ & A_{O,\Gamma_2} & A_O & A_{O,\Gamma_1} & & & & \\ & & A_{\Gamma_1,O} & A^L_{\Gamma_1} + \overline{A}_{\Gamma_1} & & & \tilde{A}_{\Gamma_1,\Gamma_1} & A_{\Gamma_1,\Theta_2} \\ A_{\Gamma_2,\Theta_1} & \tilde{A}_{\Gamma_2,\Gamma_2} & & & A^R_{\Gamma_2} + \overline{A}_{\Gamma_2} & A_{\Gamma_2,O} & & \\ & & & & A_{O,\Gamma_2} & A_O & A_{O,\Gamma_1} & \\ & & & & & A_{\Gamma_1,O} & A_{\Gamma_1} & A_{\Gamma_1,\Theta_2} \\ & & & & & & A_{\Theta_2,\Gamma_1} & A_{\Theta_2} \end{bmatrix},$$

with the transmission conditions kept the same as in (4) but reorganized with

$$\overline{A}_{\Gamma_1} := A^R_{\Gamma_1} + D, \quad \text{and} \quad \overline{A}_{\Gamma_2} := A^L_{\Gamma_2} + D.$$

As a result, the Sherman-Morrison-Woodbury formula is now used for $\left(A^L_{\Omega_1} + \overline{A}_{\Omega_1}\right)^{-1}$ and $\left(A^R_{\Omega_1} + \overline{A}_{\Omega_2}\right)^{-1}$ and analogously to [2, Lemma 3.1, Theorem 3.2] we obtain Theorem 2 (we take advantage of the symmetry, for the general case see [4]).

Theorem 2 *The MRAS iteration matrix T in* (5) *can also be written as*

$$\overline{T} = \begin{bmatrix} 0 & \overline{K} \\ \overline{L} & 0 \end{bmatrix}, \quad \text{with}$$

$$\overline{K} := \left(A^L_{\Omega_1}\right)^{-1} E^{\Omega_1}_{\Gamma_1} \left((\left(A^L_{\Omega_1}\right)^{-1})_{\Gamma_1,\Gamma_1} \right)^{-1} \left[\left((\left(A^L_{\Omega_1}\right)^{-1})_{\Gamma_1,\Gamma_1} \right)^{-1} + \overline{A}_{\Gamma_1} \right]^{-1} \left(-D (E^{\Omega_2}_{\Gamma_1})^T + (E^{\Omega_2}_{\Theta_2})^T \right),$$
$$\overline{L} := \left(A^R_{\Omega_2}\right)^{-1} E^{\Omega_2}_{\Gamma_2} \left((\left(A^R_{\Omega_2}\right)^{-1})_{\Gamma_2,\Gamma_2} \right)^{-1} \left[\left((\left(A^R_{\Omega_2}\right)^{-1})_{\Gamma_2,\Gamma_2} \right)^{-1} + \overline{A}_{\Gamma_2} \right]^{-1} \left(-D (E^{\Omega_1}_{\Gamma_2})^T + (E^{\Omega_1}_{\Theta_1})^T \right).$$

Moreover, the asymptotic convergence factor of POSM is bounded by

$$\|\overline{MB}\|_2, \quad \text{where} \tag{12}$$

$$\begin{aligned} \overline{M} &:= \left(\overline{T}^{\,\text{Denom}}\right)^{-1}\overline{T}^{\,\text{Numer}} = \left[\left(\left(\left(A^L_{\Omega_1}\right)^{-1}\right)_{\Gamma_1,\Gamma_1}\right)^{-1} + \overline{A}_{\Gamma_1}\right]^{-1}\left(\left(A^R_{\Gamma_1} - A_{\Gamma_1,\Theta_2}A^{-1}_{\Theta_2}A_{\Theta_2,\Gamma_1}\right) - \overline{A}_{\Gamma_1}\right), \\ \overline{B} &:= \overline{T}^{\,\text{Over}} = \left(\left(A^R_{\Omega_2}\right)^{-1}\right)_{\Gamma_1,\Gamma_2}\left(\left(\left(A^R_{\Omega_2}\right)^{-1}\right)_{\Gamma_2,\Gamma_2}\right)^{-1}. \end{aligned} \tag{13}$$

Focusing on the first block in (13), we take $\mathbf{b} \in \mathbb{R}^{N_r-1}$ and interpolating it to a function $\gamma\colon \Gamma_1 \to \mathbb{R}$, the following problems are equivalent up to the FD discretization:

$$A^L_{\Omega_1}\mathbf{u} = -\tfrac{1}{h}E^{\Omega_1}_{\Gamma_1}\mathbf{b} \qquad \text{and} \qquad \begin{aligned} &\Delta u = 0 \quad \text{in } \Omega_1, \\ u = 0 \quad \text{on } \partial\Omega_1\backslash\Gamma_1, \quad &\text{and} \quad \mathbf{n}_1\cdot\nabla u = \gamma \quad \text{on } \Gamma_1. \end{aligned} \tag{14}$$

Setting $\overline{\mathcal{S}}_2(\gamma) = u\big|_{\Gamma_1}$, where u is the solution of (14) we have the equivalence (up to the FD discretization) of $-1/h(A^{-1}_{\Omega_1})_{\Gamma_1,\Gamma_1}$ and $\overline{\mathcal{S}}_2$. Considering

$$\begin{aligned} \left(\partial_{xx} - \left(\tfrac{k\pi}{b}\right)^2\right)\hat{u}(x,k) = 0 \quad &\text{for } x \in (-a, L/2+h) \text{ and } k \in \mathbb{N}, \\ \hat{u}(-a,k) = 0 \quad \text{and} \quad \hat{u}_x(L/2+h,k) = \hat{\gamma}(k) \quad &\text{for } k \in \mathbb{N}, \end{aligned} \tag{15}$$

we set $\hat{\overline{\mathcal{S}}}_1 := \mathcal{F}_y\overline{\mathcal{S}}_1$ and a direct calculation yields the solution of (15) and $\hat{\overline{\mathcal{S}}}_1$ as

$$\hat{u}(x,k) = \frac{\sinh\left(\frac{k\pi}{b}(a+x)\right)}{\frac{k\pi}{b}\cosh\left(\frac{k\pi}{b}(a+L/2)\right)}, \quad \hat{\overline{\mathcal{S}}}_1\hat{\gamma}(k) = \frac{\sinh\left(\frac{k\pi}{b}(a+L/2)\right)}{\frac{k\pi}{b}\cosh\left(\frac{k\pi}{b}(a+L/2)\right)}\hat{\gamma}(k).$$

Therefore, the eigenvalues of $-1/h((A^L_{\Omega_1})^{-1})_{\Gamma_1,\Gamma_1}$ approximate the first $N_r - 1$ modes of $\mathcal{F}_y\overline{\mathcal{S}}_1$ with better accuracy in high-frequencies than we observed with $\mathcal{S}_1$, see Figure 2 and Figure 4. For the other blocks see Table 3 and Figure 4. If $-\overline{A}^R_{\Gamma_1}$ diagonalizes in the Fourier discrete basis with eigenvalues $\lambda_1, \ldots, \lambda_{N_r-1}$, then the eigenvalues of $\overline{T}^{\,\text{Denom}}, \overline{T}^{\,\text{Numer}}, \overline{T}^{\,\text{Over}}$ approximate certain discrete (truncated) Fourier symbols, presented in Table 2 and Figure 3. Notice that at the discrete level we have $MB = \overline{MB}$, i.e., the difference is in the *representation* of the bound (blue markers in Figure 3) as we changed *only* the block organization in the Sherman-Morrison-Woodbury formula. Comparing Table 2 with (3), we get the link between λ_k (and hence also δ_k) and the Robin parameter p in (3). Calculating the optimal p now directly translates to the optimal choice of D by

$$pI = -hW^T\left(A^R_{\Gamma_1} + D\right)W \quad \text{, i. e.,} \quad D = -\frac{p}{h}I - A^R_{\Gamma_1}.$$

Table 3 The blocks and corresponding linear operators (LO) from (8).

block	discrete LO	continuous LO	Fourier symbol
$\left(\left(A^L_{\Omega_1}\right)^{-1}\right)_{\Gamma_1,\Gamma_1}$	$-\frac{1}{h}\left(\left(A^L_{\Omega_1}\right)^{-1}\right)_{\Gamma_1,\Gamma_1}$	$\overline{\mathcal{S}}_1 : \gamma \mapsto u\big\rvert_{\Gamma_1}$	$\hat{\overline{\mathcal{S}}}_1 = \frac{1}{\frac{k\pi}{b}\coth\left(\frac{k\pi}{b}(a+L/2)\right)}$
$\overline{A}^R_{\Gamma_1} - A_{\Gamma_1,\Theta_2} A^{-1}_{\Theta_2} A_{\Theta_2,\Gamma_1}$	$-h\left(\overline{A}^R_{\Gamma_1} - A_{\Gamma_1,\Theta_2} A^{-1}_{\Theta_2} A_{\Theta_2,\Gamma_1}\right)$	$\overline{\mathcal{S}}_2 : \gamma \mapsto \mathbf{n}_1 \cdot \nabla u\big\rvert_{\Gamma_1}$	$\hat{\overline{\mathcal{S}}}_2 = \frac{k\pi}{b}\coth\left(\frac{k\pi}{b}(a-L/2)\right)$
$\left(\left(A^R_{\Omega_2}\right)^{-1}\right)_{\Gamma_1,\Gamma_2}$	$-\frac{1}{h}\left(\left(A^R_{\Omega_2}\right)^{-1}\right)_{\Gamma_1,\Gamma_2}$	$\overline{\mathcal{S}}_3 : \gamma \mapsto u\big\rvert_{\Gamma_1}$	$\hat{\overline{\mathcal{S}}}_3 = \frac{\sinh\left(\frac{k\pi}{b}(a-L/2)\right)}{\frac{k\pi}{b}\cosh\left(\frac{k\pi}{b}(a+L/2)\right)}$
$\left(\left(A^R_{\Omega_2}\right)^{-1}\right)_{\Gamma_2,\Gamma_2}$	$-\frac{1}{h}\left(\left(A^R_{\Omega_2}\right)^{-1}\right)_{\Gamma_2,\Gamma_2}$	$\overline{\mathcal{S}}_4 : \gamma \mapsto u\big\rvert_{\Gamma_2}$	$\hat{\overline{\mathcal{S}}}_4 = \frac{\sinh\left(\frac{k\pi}{b}(a+L/2)\right)}{\frac{k\pi}{b}\cosh\left(\frac{k\pi}{b}(a+L/2)\right)}$

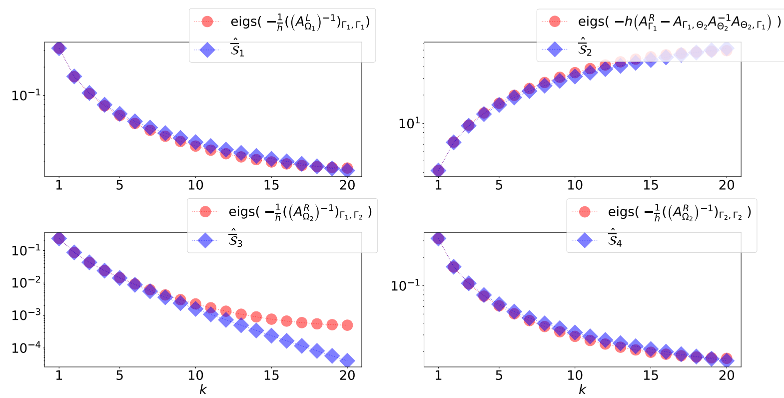

Fig. 4 Results obtained for the parameters $a = b = 1$, $L = 2h$, $N_r = 21$.

References

1. Gander, M. J. On the influence of geometry on optimized Schwarz methods. *SeMA Journal* **53**(1), 71–78 (2013).
2. Gander, M. J., Loisel, S., and Szyld, D. B. An optimal block iterative method and preconditioner for banded matrices with applications to PDEs on irregular domains. *SIAM Journal on Matrix Analysis and Applications* **33**(2), 653–680 (2012).
3. Gander, M. J. and Zhang, H. A class of iterative solvers for the Helmholtz equation: factorizations, sweeping preconditioners, source transfer, single layer potentials, polarized traces, and optimized Schwarz methods. *SIAM Review* **61**(1), 3–76 (2019).
4. Outrata, M. *Schwarz methods, Schur complements, preconditioning and numerical linear algebra*. Ph.D. thesis, University of Geneva, Math Department (2022).