Three-Level NOSAS Preconditioners

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1 The standard NOSAS preconditioners

The nonoverlapping spectral additive Schwarz methods (NOSAS) were first introduced as two-level domain decomposition preconditioners [5, 6] designed to solve symmetric positive definite and sparse linear system $Ax = b$ arising from highly heterogeneous coefficients. NOSAS are of the nonoverlapping Schwarz type, and the subdomain interactions are via the coarse problem. NOSAS preconditioners have the following form

$$M_{\text{NOSAS}}^{-1} = \underbrace{R_0^T A_0^{-1} R_0}_{\text{Coarse Level}} + \underbrace{\sum_{i=1}^{N} R_i^T A_i^{-1} R_i}_{\text{First Level}}, \tag{1}$$

where R_i are restriction matrices from Ω to the nonoverlapping open subdomains Ω_i. We require each subdomain to be the union of elements with nodes on the boundaries of neighboring subdomains matching across the interface. $A_i = R_i A R_i^T$ are the local Dirichlet solvers on Ω_i, and A_0 is the coarse matrix corresponding to the global coarse bilinear form on the interface $\Gamma := \cup_{i=1}^{N} \Gamma_i = \cup_{i=1}^{N} (\partial\Omega_i \backslash \partial\Omega)$. A_0 can be constructed as "exact" with $A_0 = R_0 A R_0^T$ or as "inexact" with different choices of $B_{\Gamma\Gamma}^{(i)}$ to obtain better scalability property; see [6]. R_0^T is the global extension operator, which is the sum of a discrete a-harmonic extension of the low-frequency eigenfunctions and a low-cost extension for the high-frequency eigenfunctions inside each subdomain. The eigenfunctions are obtained locally and in parallel from the following generalized eigenvalue problem in each subdomain, (Cf. eq. (3.7) in [1] and eq. (7.3) in [2])

$$S^{(i)}\xi := (A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} (A_{II}^{(i)})^{-1} A_{I\Gamma}^{(i)})\xi = \lambda B_{\Gamma\Gamma}^{(i)} \xi, \tag{2}$$

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where $S^{(i)}$ is the local Schur complement, $A^{(i)}_{\Gamma\Gamma}, A^{(i)}_{\Gamma I}, A^{(i)}_{I\Gamma}, A^{(i)}_{II}$ are obtained from the local Neumann matrices $A^{(i)}$, where the subscripts Γ and I denote the part associated with the interface and interior of the subdomain, respectively. The right hand side $B^{(i)}_{\Gamma\Gamma}$ is positive definite, and we have several choices [8, 7]. We can choose $B^{(i)}_{\Gamma\Gamma} = A^{(i)}_{\Gamma\Gamma}$, which is the energy of the zero extension, or $B^{(i)}_{\Gamma\Gamma} = \hat{A}^{(i)}_{\Gamma\Gamma}$ as the diagonal or block diagonal of $A^{(i)}_{\Gamma\Gamma}$. Next, a threshold η_1 is set up to decompose the space of degrees of freedom on Γ orthogonally with respect to $S^{(i)}$ and $B^{(i)}_{\Gamma\Gamma}$ into two subspaces, the low-frequency (eigenvalues smaller or equal to η_1) eigenfunctions and the high-frequency (eigenvalues larger than η_1) eigenfunctions. This decomposition defines naturally the extension R_0^T, which is the discrete a-harmonic extension for the low-frequency eigenfunctions and zero extension for the high-frequency eigenfunctions. Using $B^{(i)}_{\Gamma\Gamma}$ and low-frequency eigenfunctions we can construct a coarse bilinear form $a_0(\cdot,\cdot)$ and its corresponding matrix form A_0. $a_0(\cdot,\cdot)$ can also be interpreted as the sum of the energy of low-frequency eigenfunctions with respect to $S^{(i)}$ and the energy of high-frequency eigenfunctions with respect to $B^{(i)}_{\Gamma\Gamma}$. To obtain a better convergence rate and a smaller global problem, instead of using the zero extension for the high-frequency eigenfunctions in R_0^T, we can also use $\mathcal{H}^{(i)}_{\delta,D}$, which is the minimum a-energy extension inside a δ layer of Γ_i and zero Dirichlet condition elsewhere inside the subdomain. For this specific R_0^T, the right-hand side $B^{(i)}_{\Gamma\Gamma}$ can be chosen as $S^{(i)}_{\delta,D}$, which is the Schur complement corresponding to $\mathcal{H}^{(i)}_{\delta,D}$; or choose $B^{(i)}_{\Gamma\Gamma}$ as $S^{(i)}_F$, which is a block matrix constructed from the zero extension of vertices and Schur complement of each edge/face in a δ layer. The latter choice has excellent parallel scalability since, then, the assembling $B_{\Gamma\Gamma}$ are block diagonal with blocks related to the edges/faces only. In order to decrease the complexity of the generalized eigenvalue problems, there is also an economic version [8] by replacing the left-hand side $S^{(i)}$ with $S^{(i)}_{\delta,N}$, which is the Schur complement of the discrete a-harmonic extension in a δ layer and with zero Neumann condition inside.

For the "exact" A_0, i.e., choosing $B^{(i)}_{\Gamma\Gamma} = A^{(i)}_{\Gamma\Gamma}$ or $B^{(i)}_{\Gamma\Gamma} = S^{(i)}_{\delta,D}$, the size of the coarse problem is equal to the degrees of freedom (DOF) on the interface. However, for the "inexact" A_0, i.e., choosing $B^{(i)}_{\Gamma\Gamma} = \hat{A}^{(i)}_{\Gamma\Gamma}$ or $B^{(i)}_{\Gamma\Gamma} = S^{(i)}_F$ the coarse problem can be separated into local and global interactions by using the Sherman-Morrison-Woodbury formula [6]. The local part is based on the uncoupled $B^{(i)}_{\Gamma\Gamma}$, which corresponds to solving a small Dirichlet problem in a thin region near the edges/faces of the subdomains. The global part is based on coupled low-frequency modes across the subdomains, which are built from generalized eigenfunctions on the subdomains. The global part is designed to guarantee the robustness of the preconditioner to any ill-conditioned positive definite matrix A, and the size of the global problem is equal to the total number of selected eigenfunctions.

Now, let $V_h(\Omega)$ be any finite element space on a bounded polygonal (polyhedral) domain Ω, and the condition number of NOSAS preconditioners satisfies the following theorem. For a detailed proof, see [7, 8].

Theorem 1 *For any $u_h \in V_h(\Omega)$ the following holds:*

$$\frac{\eta_1}{C_1+1} a(u_h,u_h) \leq a(M_{NOSAS}^{-1} A u_h, u_h) \leq (C_1+1) a(u_h,u_h),$$

where C_1 is a constant based on different choices of $B_{\Gamma\Gamma}^{(i)}$.

For $B_{\Gamma\Gamma}^{(i)} = A_{\Gamma\Gamma}^{(i)}$ or $B_{\Gamma\Gamma}^{(i)} = S_{\delta,D}^{(i)}$, we have $C_1 = 1$. For $B_{\Gamma\Gamma}^{(i)} = \hat{A}_{\Gamma\Gamma}^{(i)}$, we have $A_{\Gamma\Gamma}^{(i)} \leq C_1 B_{\Gamma\Gamma}^{(i)}$ for $1 \leq i \leq N$ with $C_1 = 3$ in two dimensions and $C_1 = 4$ in three dimensions. For $B_{\Gamma\Gamma}^{(i)} = S_F^{(i)}$, we have $S_{\delta,D}^{(i)} \leq C_1 B_{\Gamma\Gamma}^{(i)}$ for $1 \leq i \leq N$ with $C_1 = 3$ in two dimensions and $C_1 = 5$ in three dimensions when $\delta < \frac{H}{2}$. Here we denote H as the size of the nonoverlapping subdomain and h as the size of the finite element. The threshold η_1 is usually chosen to be $O(\frac{h}{H})$ so that the preconditioned system has condition number $O(\frac{H}{h})$. Furthermore, the number of eigenfunctions we choose is only related to the geometry of the heterogeneous coefficients, and we give quantitative results in [6].

A unique feature of the NOSAS preconditioner is that no weighting is required to average the local solutions. This is different from methods like BDD, BDDC, FETI, and FETI-DP, which require expensive deluxe weighting for some highly heterogeneous problems. Moreover, the global matrix of the NOSAS has better sparsity than the coarse matrix of BDD or FETI, with zero blocks corresponding to the eigenfunctions in the subdomains that are not adjacent. NOSAS is constructed purely algebraically from unassembled Neumann matrices $A^{(i)}$, which facilitates the construction of the three-level NOSAS preconditioner, where we use the NOSAS idea recursively on the coarse level.

2 The three-level NOSAS preconditioners

We note that the size of the global problem for NOSAS is the total number of eigenfunctions we choose in all subdomains. Therefore, for a large number of subdomains, the coarse problem can become a bottleneck. The motivation of a three-level extension of the NOSAS methods is to approximate the coarse problem by replacing the direct solver with a new preconditioner; see the three-level BDDC method [4] and the three-level GDSW preconditioner [3]. We can also further recursively apply the preconditioners to new levels, which is algorithmically straightforward leading to multilevel extensions.

We first introduce some notations to define our three-level NOSAS. We decompose Ω into N_0 nonoverlapping open polygonal subregions $\Omega_{j,0}$ of size $O(H_0)$. We denote $W_{jk} = \overline{\Omega}_{j,0} \cap \overline{\Omega}_{k,0}$, which is the common vertex/edge/face of two adjacent subregions when not empty. We further decompose each subregion $\Omega_{j,0}$ into some subdomains Ω_i of size $O(H)$. We define $\Gamma_{j,0}$ as the interface of subregion $\Omega_{j,0}$, and define $\Gamma_{j,I}$ as the union of all subdomain interfaces Γ inside $\Omega_{j,0}$ without touching $\Gamma_{j,0}$. The global interface Γ_0 and the global interface interior Γ_I are the union of $\Gamma_{j,0}$ and $\Gamma_{j,I}$, respectively; see Figure 1 as an illustration. Therefore, we have that $\Gamma = \Gamma_0 \oplus \Gamma_I$. Unless otherwise specified, we use $V_h(D)$ to denote $\{v|_D : v \in V_h(\Omega)\}$, where D is a set in Ω.

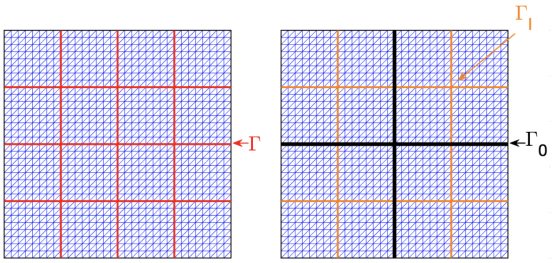

Fig. 1 Comparison of a two-level mesh (left) and a three-level mesh (right) with $h = 1/32$, $H = 1/4$, $H_0 = 1/2$.

We need to decompose the coarse space $V_0 := \{v|_\Gamma : v \in V_h(\Omega)\}$ into some new local spaces and a coarser space. Following the procedure of additive Schwarz methods, we define the local spaces $V_{j,0}$ $(1 \le j \le N_0)$ as the restriction of V_0 on $\Gamma_{j,I}$ and vanishing on $\Gamma_{j,0}$. The coarser space $V_{0,0}$ is the restriction of V_0 on Γ_0. We also define extrapolation operators $R_{j,0}^T : V_{j,0} \to V_0$ as the extension by zero outside of $\Omega_{j,0}$ for $1 \le j \le N_0$, and $R_{0,0}^T : V_{0,0} \to V_0$ will be defined later. Then V_0 admits the following direct sum decomposition

$$V_0 = R_{0,0}^T V_{0,0} \oplus R_{1,0}^T V_{1,0} \oplus \cdots \oplus R_{N_0,0}^T V_{N_0,0}.$$

Next, we follow the procedure of the NOSAS methods to define $R_{0,0}^T$. We will use only the Neumann matrices $A_0^{(i)}$ associated with the bilinear form $a_0^{(i)}(\cdot,\cdot)$. In the subregion $\Omega_{j,0}$ $(1 \le j \le N_0)$, we define the corresponding coarse bilinear form

$$a_{0,0}^{(j)}(\cdot,\cdot) = \sum_{i \in \mathcal{N}(j)} a_0^{(i)}(\cdot,\cdot),$$

where $\mathcal{N}(j)$ is the set of indices of subdomains Ω_i contained in the subregion $\Omega_{j,0}$.

Let Neumann matrices $A_{0,0}^{(j)}$ be associated with the bilinear form $a_{0,0}^{(j)}(\cdot,\cdot)$ defined above. Then, $A_{0,0}^{(j)}$ can be decomposed and written as the block matrix $\begin{bmatrix} A_{\Gamma_0\Gamma_0}^{(j)} & A_{\Gamma_0\Gamma_I}^{(j)} \\ A_{\Gamma_I\Gamma_0}^{(j)} & A_{\Gamma_I\Gamma_I}^{(j)} \end{bmatrix}$, where subscripts Γ_0, Γ_I denote the parts associated with Γ_0 and Γ_I, respectively. We consider the following local generalized eigenvalue problem in each subregion $(1 \le j \le N_0)$ separately

$$S_0^{(j)} \phi_k^{(j)} := (A_{\Gamma_0\Gamma_0}^{(j)} - A_{\Gamma_0\Gamma_I}^{(j)} (A_{\Gamma_I\Gamma_I}^{(j)})^{-1} A_{\Gamma_I\Gamma_0}^{(j)}) \phi_k^{(j)} = \mu_k^{(j)} B_{\Gamma_0\Gamma_0}^{(j)} \phi_k^{(j)}, \quad (k = 1, \cdots, n_{j,0}) \tag{3}$$

where $S_0^{(j)}$ is the Schur complement of $A_{0,0}^{(j)}$ and $n_{j,0}$ is the number of DOFs on $\Gamma_{j,0}$. Similar to the two-level NOSAS, we have the following choices for the right-hand side $B_{\Gamma_0\Gamma_0}^{(j)}$:

1. $B_{\Gamma_0\Gamma_0}^{(j)} = A_{\Gamma_0\Gamma_0}^{(j)}$;
2. $B_{\Gamma_0\Gamma_0}^{(j)} = \hat{A}_{\Gamma_0\Gamma_0}^{(j)}$, which is the diagonal or block diagonal version of $A_{\Gamma_0\Gamma_0}^{(j)}$;

3. $B^{(j)}_{\Gamma_0\Gamma_0} = S^{(j)}_{\delta_0,D}$, which is the Schur complement defined as follows
$$v_j^T S^{(j)}_{\delta_0,D} u_j = a^{(j)}_{0,0}(\mathcal{H}^{(j)}_{\delta_0,D} u_j, \mathcal{H}^{(j)}_{\delta_0,D} v_j), \quad \text{for all } u_j, v_j \in V_h(\Gamma_{j,0}).$$
Here, we set $\delta_0 = lH$ with some integer l, and $\mathcal{H}^{(j)}_{\delta_0,D}$ is defined as the minimum $a^{(j)}_{0,0}$-energy extension from $V_h(\Gamma_{j,0})$ to $V_h(\Gamma_{j,0} \bigcup \Gamma_{j,I})$ with a zero Dirichlet condition outside of a δ_0 layer from $\Gamma_{j,0}$;
4. $B^{(j)}_{\Gamma_0\Gamma_0} = S^{(j)}_W$, which is the block diagonal of $\{A^{(j)}_{W_{jk},W_{jk}}\}_{W_{jk}\in\Gamma_{j,0}}$ if W_{jk} is a vertex in 2D (vertex and edge in 3D) and $\{S^{(j)}_{W_{jk},\delta_0}\}_{W_{jk}\in\Gamma_{j,0}}$ if W_{jk} is an edge in 2D (face in 3D), where $A^{(j)}_{W_{jk},W_{jk}}$ is the submatrix of $A^{(j)}_{0,0}$ relative to W_{jk}, and
$$v_{jk}^T S^{(j)}_{W_{jk},\delta_0} u_{jk} = a^{(j)}_{0,0}(\mathcal{H}^{(j)}_{W_{jk},\delta_0} u_{jk}, \mathcal{H}^{(j)}_{W_{jk},\delta_0} v_{jk}), \quad \text{for all } u_{jk}, v_{jk} \in V_h(W_{jk}),$$
where $\mathcal{H}^{(j)}_{W_{jk},\delta_0}$ is defined as the minimum $a^{(j)}_{0,0}$-energy extension from $V_h(W_{jk})$ to $V_h(\Gamma_{j,0} \bigcup \Gamma_{j,I})$ with zero Dirichlet condition outside of a δ_0 layer from W_{jk}.

For a detailed comparison of the choices above; see [7]. Next, we solve the local generalized eigenvalue problem (3) and fix a threshold $\eta_0 < 1$. We pick the smallest k_j eigenvalues less than η_0 and their corresponding eigenvectors to construct the space $Q_0^{(j)}$ and the local orthogonal projection $\Pi^{(j)}_{0,S}\colon V_h(\Gamma_{j,0}) \to Q_0^{(j)}$ with respect to $B^{(j)}_{\Gamma_0\Gamma_0}$ norm as follows
$$Q_0^{(j)} = [\phi_1^{(j)}, \phi_2^{(j)}, \cdots, \phi_{k_j}^{(j)}], \quad \Pi^{(j)}_{0,S} = Q_0^{(j)}(Q_0^{(j)^T} B^{(j)}_{\Gamma_0\Gamma_0} Q_0^{(j)})^{-1} Q_0^{(j)^T} B^{(j)}_{\Gamma_0\Gamma_0}.$$
We also denote $\Pi^{(j)^\perp}_{0,S} = I_0^{(j)} - \Pi^{(j)}_{0,S}$, where $I_0^{(j)}\colon V_h(\Gamma_{j,0}) \to V_h(\Gamma_{j,0})$ is the identity mapping. Based on different choices of $B^{(j)}_{\Gamma_0\Gamma_0}$, we have the following choices for $R^{(j)^T}_{0,0}\colon V_h(\Gamma_{j,0}) \to V_h(\Gamma_{j,0} \bigcup \Gamma_{j,I})$:

i. $\mathcal{H}_0^{(j)}\Pi^{(j)}_{0,S} + \mathcal{E}_0^{(j)}\Pi^{(j)^\perp}_{0,S}$,
ii. $\mathcal{H}_0^{(i)}\Pi^{(j)}_{0,S} + \sum_{W_{jk}\in\Gamma_{j,0}} \mathcal{H}^{(j)}_{\delta_0,D}\Pi^{(j)^\perp}_{0,S}$,

where $\mathcal{H}_0^{(j)}$ and $\mathcal{E}_0^{(j)}$ are the minimum $a^{(j)}_{0,0}$-energy extension and zero extension from $V_h(\Gamma_{j,0})$ to $V_h(\Gamma_{j,0} \bigcup \Gamma_{j,I})$, respectively. For simplicity, we choose 1. and 2. for $B^{(j)}_{\Gamma_0\Gamma_0}$ and their corresponding option i. in $R^{(j)^T}_{0,0}$ for the rest of the paper. Therefore, $\forall u_{\Gamma_0} \in V_{0,0}$, we define $R^T_{0,0}\colon V_{0,0} \to V_0$ as
$$R^T_{0,0} u_{\Gamma_0} = \begin{bmatrix} u_{\Gamma_0} \\ \sum_{j=1}^{N_0} - R^T_{I_jI_0}(A^{(j)}_{\Gamma_I\Gamma_I})^{-1} A^{(j)}_{\Gamma_I\Gamma_0} \Pi^{(j)}_{0,S} R_{\Gamma_j\Gamma_0} u_{\Gamma_0} \end{bmatrix},$$
where $R_{\Gamma_j\Gamma_0}\colon V_{0,0} \to V_h(\Gamma_{j,0})$ and $R^T_{I_jI_0}\colon V_h(\Gamma_{j,I}) \to V_h(\Gamma_I)$ are the trivial restriction and extension operators, respectively.

Then, we define $\hat{A}_{0,0}$ corresponding to the following bilinear form

$$\hat{a}_{0,0}(u_{\Gamma_0}, v_{\Gamma_0}) = v_{\Gamma_0}^T \sum_{j=1}^{N_0} \Big((\Pi_{0,S}^{(j)} R_{\Gamma_j\Gamma_0})^T S_0^{(j)} (\Pi_{0,S}^{(j)} R_{\Gamma_j\Gamma_0}) + (\Pi_{0,S}^{(j)^\perp} R_{\Gamma_j\Gamma_0})^T B_{\Gamma_0\Gamma_0}^{(j)} (\Pi_{0,S}^{(j)^\perp} R_{\Gamma_j\Gamma_0}) \Big) u_{\Gamma_0}$$

$$= v_{\Gamma_0}^T \sum_{j=1}^{N_0} R_{\Gamma_j\Gamma_0}^T \big(B_{\Gamma_0\Gamma_0}^{(j)} - B_{\Gamma_0\Gamma_0}^{(j)} Q_0^{(j)} D_0^{(j)} (Q_0^{(j)^T} B_{\Gamma_0\Gamma_0}^{(j)} Q_0^{(j)})^{-1} Q_0^{(j)^T} B_{\Gamma_0\Gamma_0}^{(j)} \big) R_{\Gamma_j\Gamma_0} u_{\Gamma_0} \quad \forall u_{\Gamma_0}, v_{\Gamma_0} \in V_{0,0},$$

where $D_0^{(j)} = \text{diagonal}(1-\mu_1^{(j)}, 1-\mu_2^{(j)}, \cdots, 1-\mu_{k_j}^{(j)})$ and $\mu_k^{(j)}$ are the generalized eigenvalues corresponding to $\phi_k^{(j)}$.

Then, the three-level NOSAS preconditioners have the following form

$$M_{\text{3NOSAS}}^{-1} = \underbrace{R_0^T \big(\overbrace{R_{0,0}^T \hat{A}_{0,0}^{-1} R_{0,0}}^{\text{Third Level}} + \overbrace{\sum_{j=1}^{N_0} R_{j,0}^T \hat{A}_{j,0}^{-1} R_{j,0}}^{\text{Second Level}} \big) R_0}_{\text{Coarse Level}} + \underbrace{\sum_{i=1}^{N} R_i^T A_i^{-1} R_i}_{\text{First Level}}. \tag{4}$$

For the first level, $A_i = R_i A R_i^T, (1 \le i \le N)$ are the matrices corresponding to the exact local bilinear form of A, the same as in the two-level method. For the second level, $\hat{A}_{j,0} = R_{j,0} A_0 R_{j,0}^T, (1 \le j \le N_0)$ are the matrices form corresponding to the following exact local bilinear form of A_0

$$\hat{a}_{j,0}(u_{j,0}, v_{j,0}) = a_0(R_{j,0}^T u_{j,0}, R_{j,0}^T v_{j,0}), \quad \forall u_{j,0}, v_{j,0} \in V_{j,0}.$$

For the third level, $\hat{A}_{0,0}$ is the matrix form of $\hat{a}_{0,0}(\cdot,\cdot)$ defined above.

To show the condition number of three-level NOSAS preconditioners, we first focus on the preconditioner in the coarse level and define B_0^{-1} as

$$B_0^{-1} = R_{0,0}^T \hat{A}_{0,0}^{-1} R_{0,0} + \sum_{j=1}^{N_0} R_{j,0}^T \hat{A}_{j,0}^{-1} R_{j,0}.$$

We note that B_0 can be seen as an approximation of A_0, and B_0^{-1} is a two-level NOSAS preconditioner of A_0. Therefore, similar to the two-level methods, we should also consider the relation of $B_{\Gamma_0\Gamma_0}^{(j)}$ with $A_{\Gamma_0\Gamma_0}^{(j)}$. For different choices of $B_{\Gamma_0\Gamma_0}^{(j)}$, let C_0 be the constant such that $A_{\Gamma_0\Gamma_0}^{(j)} \le C_0 B_{\Gamma_0\Gamma_0}^{(j)}$ for $1 \le j \le N_0$. For $B_{\Gamma_0\Gamma_0}^{(j)} = A_{\Gamma_0\Gamma_0}^{(j)}$, we have $C_0 = 1$. For $B_{\Gamma_0\Gamma_0}^{(j)} = \hat{A}_{\Gamma_0\Gamma_0}^{(j)}$, we have $C_0 = 3$ in two dimensions and $C_0 = 4$ in three dimensions. Then, using the NOSAS methods property we have shown in Theorem 1, we have $\forall u_\Gamma \in V_0$,

$$\frac{\eta_0}{C_0+1} u_\Gamma^T A_0^{-1} u_\Gamma \le u_\Gamma^T B_0^{-1} u_\Gamma \le (C_0+1) u_\Gamma^T A_0^{-1} u_\Gamma.$$

Since A_0 and B_0 are symmetric and positive definite matrices, it is equivalent to

$$\frac{1}{C_0+1} u_\Gamma^T A_0 u_\Gamma \le u_\Gamma^T B_0 u_\Gamma \le \frac{C_0+1}{\eta_0} u_\Gamma^T A_0 u_\Gamma.$$

Using the above property of B_0 and combining it with the abstract theory of the additive Schwarz method, we can obtain the following condition number for the three-level NOSAS methods.

Theorem 2 *For any $u_h \in V_h(\Omega)$ the following holds:*

$$\Big(\frac{C_1}{\eta_1}+\frac{C_0+1}{\eta_1\eta_0}\Big)^{-1} a(u_h,u_h) \le a(M_{3NOSAS}^{-1}Au_h,u_h) \le (1+C_1+C_1C_0)a(u_h,u_h).$$

3 Numerical experiments

We present numerical results for the variational formulation of $\int_\Omega \rho(x)\nabla u\cdot\nabla v dx = \int_\Omega f v dx$ with $f=1$ and a heterogeneous coefficient function $\rho(x)$. We choose four specific $\rho(x)$ from the SPE10 model problems, Kxx_06 and Kxx_85, with the computational domain $\Omega=(0,22)\times(0,6)$. We decompose Ω into 33 congruent square subregions of size $H_0=2$, and 528 congruent square subdomains of size $H=1/2$. We further decompose each square subdomain into $(H/h)^2$ congruent small squares of size $h=0.1$. The shape-regular partition $\mathcal{T}_h$ is obtained by dividing each of these small squares into two right triangle elements. $V_h(\Omega)$ are the piecewise linear basis functions on the triangulation $\mathcal{T}_h$. We impose a zero Dirichlet boundary condition on $\partial\Omega$ and use the PCG method for the preconditioned system with the relative residual error 10^{-6} in the l^2 norm.

Table 1 The two-level NOSAS preconditioners applied to four SPE10 model problems with different η_1.

η_1	Iter.	Cond.	Size of global problem
0.025	46	34.45	431
0.05	39	24.47	443
0.1	32	15.52	486
0.2	23	6.82	735
0.4	16	4.05	1674

Kxx_06

η_1	Iter.	Cond.	Size of global problem
0.025	46	29.43	480
0.05	42	27.71	513
0.1	33	13.92	600
0.2	25	8.61	833
0.4	18	4.57	1535

Kxx_85

Table 2 The three-level NOSAS preconditioners applied to four SPE10 meshes with $\eta_0=0.25$ and different η_1.

η_1	Iter.	Cond.	Size of global problem
0.025	57	48.62	123
0.05	49	31.16	124
0.1	45	31.19	130
0.2	34	14.94	163
0.4	26	9.36	236

Kxx_06

η_1	Iter.	Cond.	Size of global problem
0.025	62	61.18	169
0.05	54	38.68	174
0.1	45	27.55	191
0.2	39	23.88	226
0.4	28	11.93	276

Kxx_85

The scalability of NOSAS methods with the "inexact" solver is shown in [5, 6]. The main focus of our numerical experiments is to compare two-level NOSAS preconditioners with three-level NOSAS preconditioners, and show that three-level NOSAS have a smaller size of the global problem while maintaining a good iteration and condition number. For the two-level NOSAS preconditioners, we choose $B_{\Gamma\Gamma}^{(i)} = S_F^{(i)}$ with $\delta = 2h$, and choose R_0^T the discrete a-harmonic extension for the low-frequency eigenfunctions and $\mathcal{H}_{\delta,D}^{(i)}$ for the high-frequency eigenfunctions in (2). For the three-level NOSAS preconditioners, $B_{\Gamma\Gamma}^{(i)}$, R_0^T are the same as the two-level preconditioners. Then we choose $B_{\Gamma_0\Gamma_0}^{(j)} = \hat{A}_{\Gamma_0\Gamma_0}^{(j)}$ as the diagonal of $A_{\Gamma_0\Gamma_0}^{(j)}$, and $R_{0,0}^T$ as the discrete a-harmonic extension for the low-frequency eigenfunctions and zero extension for the high-frequency eigenfunctions in (3). Note that $\frac{h}{H} = 0.2$ and Table 1 shows the performance of the two-level NOSAS with different thresholds η_1 for the SPE10 model problems. For the three-level NOSAS, we choose a fixed $\eta_0 = \frac{H}{H_0}$ and show the corresponding results for different thresholds η_1 in Table 2. All our test results support the theoretical condition number bound in Theorem 1 and Theorem 2. In addition, we observe a much smaller condition number numerically. The reason is that numerically, the constant C_0 is close to 1.8 and C_1 is close to 1.5 for the "inexact" solver.

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