An Adaptive Overlapping Schwarz Algorithm for Isogeometric Analysis

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1 Introduction

The algorithm considered in this paper is known as the RAGDSW – the Reduced Adaptive Generalized Dryja-Smith-Widlund – method. Following an idea of Clark Dohrmann, the GDSW algorithms were introduced in order to avoid the need for coarser meshes for overlapping Schwarz algorithms, see [4]; for a general introduction to Schwarz method see [9, Chap. 2–3]. This was accomplished by borrowing coarse spaces from another family of domain decomposition algorithms namely the iterative substructuring methods, see [9, Chap. 4–6]. These algorithms were later further refined decreasing the dimension of the coarse spaces, see [5, 6]. We note that the GDSW methods, without adaptation, can be used for problems for which only fully assembled stiffness matrices are available. Unfortunately, the adaptive variants require access to the stiffness matrices for individual subdomains, matrices that cannot be recovered from fully assembled matrices.

The purpose of our present work is to extend previous work on low order finite elements to isogeometric analysis (IgA) based on B-splines and NURBS (nonuniform rational B-splines) of arbitrary order p; for an introduction to IgA, see, e.g., [1]. Our elliptic problems are scalar elliptic problems and compressible linear elasticity in two

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or three dimensions. We note that the NURBS are commonly used in computer aided design and we always assume that the domains of the elliptic problems considered can be represented exactly using this class of functions. With B-splines of order p and smoothness $k = p - 1$, there are p one-dimensional (1D) B-spline basis functions which differ from zero at any fixed internal knot. This is the origin of our *fat interfaces*; see further Section 3.

New coarse spaces for overlapping Schwarz methods are generated adaptively by solving generalized eigenvalue problems on subsets of the fat interface which subdivide the domain of the elliptic problem into subdomains. We call these subsets *eigensets*. The resulting eigenvectors with eigenvalues less than or equal to a tolerance are then extended by zero to the rest of the fat interface to provide Dirchlet data for the computation of basis functions, of minimum energy, for the coarse subspace of our Schwarz algorithms. We note that the AGDSW algorithms considered in [7] use eigensets of the interface based of equivalence classes directly related to subdomain vertices, edges, and faces while the RAGDSW algorithms of [8] use only one type of eigensets each associated with a vertex of the interface. The latter choice leads to considerably much smaller coarse subspaces. The two papers just cited, which have provided a foundation for our work, are for low order finite elements. These new algorithms improve the rate of convergence of the overlapping Schwarz algorithms, in particular, for cases when the material coefficients of our problems vary considerably; the work by Heinlein et al. have very strong results of that kind.

Our theoretical result provides an estimate of the condition number for our preconditioned conjugate gradient methods in terms of the tolerance used in the selection of eigenvectors of the generalized eigenvalue problems. Our experimental work with our algorithm is still in progress and will further be reported in a forthcoming paper, which will also provide complete proofs of our theoretical results.

2 The discrete problems

In this paper, the coarse space of the two-level additive Schwarz methods is given in terms of a coarse partition of the domain into non-overlapping subdomains $\{\Omega_k\}$. The union of the intersections of the boundaries of these subdomains form the interface Γ. In a reference domain, each non-overlapping subdomains, $\widehat{\Omega}_k$, is a preimage of Ω_k and is a square with a side length H each of which is partitioned into elements, with a side length of about h, by B-spline knots which we assume to form a quasi-uniform mesh. The coarse space of the pioneering paper [2] is associated with B-spline elements given in terms of the reference subdomains and of the same degree as those on the fine decomposition into small elements. In order to keep the dimension of the coarse space small, maximal smoothness of the B-splines, $k = p-1$, is chosen in that paper and in our work and this assumption also assures us that the coarse space is contained in the global space on the fine mesh which is also chosen to be of maximal smoothness. In the reference subdomains, tensor-product B-splines, $B_i^p(x)B_j^p(y)$ and $B_i^p(x)B_j^p(y)B_k^p(z)$, of order p in all coordinates in two and three dimensions

(2D and 3D), respectively, are used. The physical subdomains are the images of the reference subdomains under a mapping using NURBS. As already indicated the coarse spaces of our algorithm is chosen differently. Our local subproblems are given by Dirichlet problems on subdomains which share at least one layer of knots with all their neghbors. This overlap is measured by a parameter $r \geq 0$ where $r = 0$ represents minimal overlap and a value of $r > 0$ indicates that r layers of knots are shared between the neighboring local problems.

3 Equivalence classes, related subspaces, and preconditioners

For 2D, the subdomain vertices and edges of Γ are associated with equivalence classes of the knots of the fine mesh. The pairs of indices, (i, j), associated with the *fat interface*, Γ^{fat}, are determined by the set of 2D B-splines with values which differ from zero on part of the interface Γ. This fat interface, in turn, is divided into equivalence classes of *fat vertices* and *fat edges*.

Each interior subdomain vertex, and for $k = p - 1$, is associated with a fat vertex set of p^2 knots with p^2 B-spline basis functions that do not vanish at that vertex. Similarly, each interior subdomain edge is associated with a fat edge with basis functions which vanish at the two vertices at the end of the edge but differ from zero on part of a particular edge of Γ. Each fat edge can be viewed as being built from p thin edges of knots parallel to the subdomain edge in question in the reference domain. There is also an equivalence class of knots of the interior of any subdomain basis functions which vanish identically on the interface Γ. In 3D, there are also fat faces and the subspaces associated with fat vertices, fat edges, fat faces, and interiors of subdomains which are defined very similarly to the 2D case.

The eigensets of the RAGDSW algorithms are only of one type each associated with a subdomain vertex V; see Fig. 1. We denote such an eigenset by R and by Ω_R the interior of the union of the closure of four or eight subdomains which share the vertex at the center of this eigenset for 2D and 3D, respectively. In Fig. 1, the dots represent the locations of the maxima of the different B-spline basis functions associated with the fat interfaces. In the 2D case, the dots of R are those of the fat vertex and of the parts of the fat edges which are closer to V than to any other subdomain vertex. We note that for an even value of p, none of these dots would fall on the interface Γ. The figures would also look different if the parameter $k < p - 1$.

To decrease the cost of the computation of the elements of the matrices of the generalized eigenvalue problems (1), we can also use a subset of Ω_R making sure that all the basis functions associated with the set R are supported in the set that replaces Ω_R. We note that the theory, which we have developed, is equally valid in this case.

Considering the 3D case, we can now provide details on the construction of the set R. The B-spline knots and tensor-product B-splines associated with one of these eigensets are those of its fat vertex and the halves of the fat edges closest to the vertex and the nearest quarters of the adjacent fat faces; see Fig. 1. In case the

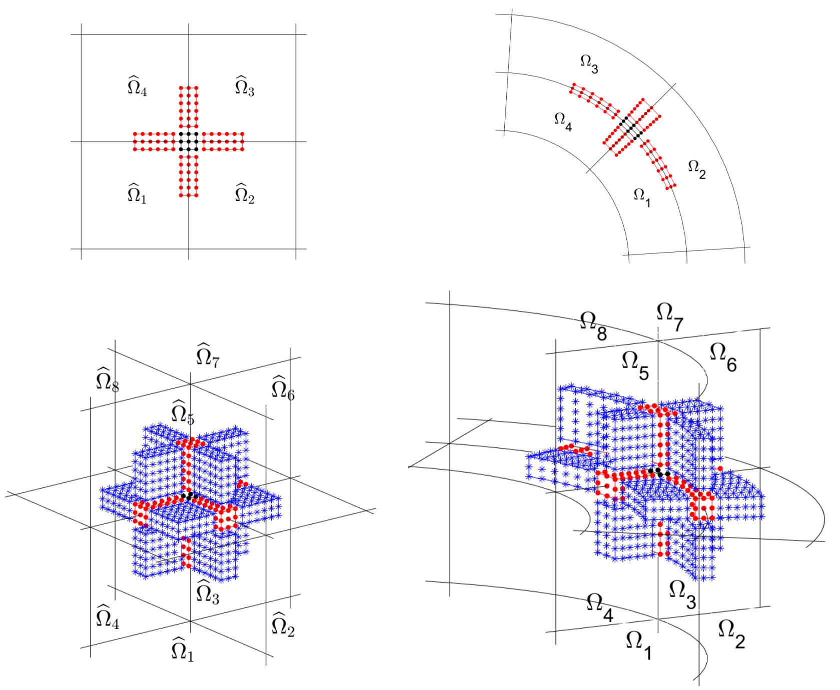

Fig. 1 RAGDSW eigensets for $p = 3, k = 2$. Top left: A 2D eigenset consisting of the union of a fat vertex (black dots) and adjcent halves of fat edges edges (red dots). Top right: the eigenset on a quarter-ring domain. Bottom left: A 3D eigenset consisting of the union of a fat vertex (black dots), halves of the adjacent fat edges (red dots), and quarters of adjacent fat faces (blue asterisks). Bottom right: The eigenset in a thick quarter-ring domain.

number of knots on a subdomain edge is not even, we allocate the knots furthest away from the relevant fat vertices to any of its closest eigensets. We also make similar minor modifications of the set of knots of the fat subdomain faces. The union of these eigensets, which do not overlap, covers the entire fat interface. We note that the knots originating from the fat vertex and subsets of fat edges and fat faces are displayed with different symbols and colors; we will need to partition this eigenset R, accordingly, when constructing the generalized eigenvalue problems.

The generalized eigenvalue problem associated with such an eigenset is of the form

$$S_{RR}^{\Omega_R}\,\tau_{\star,R} = \lambda_{\star,R}\,K_{RR}^{\Omega_R}\,\tau_{\star,R}, \tag{1}$$

defined by the Schur complement $S_{RR}^{\Omega_R}$ generated from the stiffness matrix K^{Ω_R} of the Neumann problem on Ω_R built from the four or eight subdomains sharing the eigenset or from a subset of Ω_R as indicated above. The Schur complement is generated by eliminating all degrees of freedom except those of R. The other matrix, $K_{RR}^{\Omega_R}$, is the principal minor, associated with R, of the same stiffness matrix after eliminating the off-diagonal blocks that represent coupling between the sets from which the set R is constructed.

It is easy to see that the Schur complement is singular with one constant null vector for any scalar elliptic problem. For elasticity, there are three- and six-dimensional null spaces for 2D and 3D, respectively; they originate from the rigid body modes. This fact shows that the null space condition will always be satified. For elasticity and the case $p = 1$, the other matrix of the generalized eigenvalue problems can also be singular but this issue does not arise if $p \geq 2$.

Working with a tolerance $tol_R \geq 0$, we select all eigenvectors with $\lambda_{\star,R} \leq tol_R$, extend their values by zero to the rest of the fat interface and then compute their minimal energy extensions to find the coarse space elements $v_{\star,R}$ associated with R. For each eigenvector, this requires the solutions of a Dirichlet problem for each of the subdomains of Ω_R with a zero right hand side. Thus, any such basis function has values in the interior of the subdomains obtained by a minimal energy extension.

4 A theoretical result

Our theoretical result is an estimate of the condition number of the two-level additive Schwarz algorithm using the coarse space obtained from the coarse basis functions introduced above and one local subspace for each sudomain Ω_k. Any such local subspace is associated with all the knots of the subdomain and the part of the fat interface adjacent to the subdomain. Currently our proof does not work for smaller overlaps.

Theorem 1 *There are constants C_1 and C_2 such that the condition number of the two-level additive Schwarz operator P_{add} satifies*

$$\kappa(P_{add}) \leq C_1(1 + C_2/tol). \tag{2}$$

Here tol is the smallest tolerance tol_R used for the generalized eigenvalue problems and C_1 and C_2 are computable constants independent of the number of subdomains, the dimension of the subprobems, and the coefficients of our elliptic problems.

Our proof relies to a large extent on the work reported in the two papers by Heinlein et al.

5 Numerical results

In this section, we report on numerical experiments with the isogeometric RAGDSW preconditioner for the 2D Poisson equation on a quarter-ring domain, discretized by isogeometric NURBS spaces with mesh size h, polynomial degree p, regularity $k = p - 1$, and the overlap parameter r. The domain is decomposed into N nonoverlapping subdomains of characteristic size H. The linear systems of equations arising from the discretizations are solved by the PCG algorithm accelerated by the isogeometric RAGSW preconditioner, with a zero initial guess and a stopping criterion of a 10^{-6} reduction of the Euclidean norm of the PCG residual. In the tests, we study how the convergence rate of the RAGDSW preconditioner depends on the parameters h, N, p, and r The numerical tests have been performed with a MATLAB code based on the GeoPDEs library [3]. We expect to be able to show results for linear elasticity and much larger 3D problems in a forthcoming paper.

Table 1 RAGDSW preconditioner in 2D quarter-ring domain: condition number κ_2, iteration count, it, and coarse problem size N_Π as a function of the number of subdomains N and mesh size h. Fixed spline parameters $p = 2$, $k = 1$, minimal overlap parameter $r = 0$, $tol = 0.1$.

RAGDSW preconditioner, quarter-ring domain $p = 2, k = 1, r = 0, tol = 0.1$															
	$1/h = 8$			$1/h = 16$			$1/h = 32$			$1/h = 64$			$1/h = 128$		
N	κ_2	it.	N_Π	κ_2	it.	N_Π	κ_2	it.	N_Π	κ_2	it.	N_Π	κ_2	it.	N_Π
2×2	3.60	10	1	6.69	14	2	9.23	17	3	16.10	22	4	30.60	29	4
4×4				7.63	16	9	8.77	18	18	16.16	24	18	16.31	23	36
8×8							10.85	20	49	10.38	19	98	18.90	25	98
16×16										13.73	22	225	11.40	19	450
32×32													16.23	24	961

Table 2 RAGDSW preconditioner in 2D quarter-ring domain: condition number κ_2, iteration counts it. and coarse problem size N_Π as a function of the number of subdomains N and mesh size h, Fixed spline parameters $p = 3$, $k = 2$, minimal overlap parameter $r = 0$, $tol = 0.1$.

RAGDSW preconditioner, quarter-ring domain $p = 3, k = 2, r = 0, tol = 0.1$															
	$1/h = 8$			$1/h = 16$			$1/h = 32$			$1/h = 64$			$1/h = 128$		
N	κ_2	it.	N_Π	κ_2	it.	N_Π	κ_2	it.	N_Π	κ_2	it.	N_Π	κ_2	it.	N_Π
2×2	7.15	14	1	9.72	15	2	16.87	18	3	20.10	20	4	26.63	22	8
4×4				11.79	22	9	14.41	23	18	17.34	24	27	22.40	24	36
8×8							17.16	27	49	14.53	23	98	17.55	23	147
16×16										22.15	29	225	14.91	23	450
32×32													26.42	30	961

5.1 Scalability in N and quasi-optimality in H/h

The condition number κ_2 of the RAGDSW preconditioned system and the conjugate gradient iteration count, it, are reported in Tables 1 and 2 as a function of the number of subdomains N and the mesh size h for $p = 2$ and $p = 3$, respectively. In both cases, we consider the maximal regularity $k = p - 1$. We set the adaptive tolerance to $tol = 0.1$. The results show that the proposed preconditioner is scalable, since, moving along the diagonals of each table, both the condition number and iteration count exhibit a moderate increase that seems to level off and approach a constant

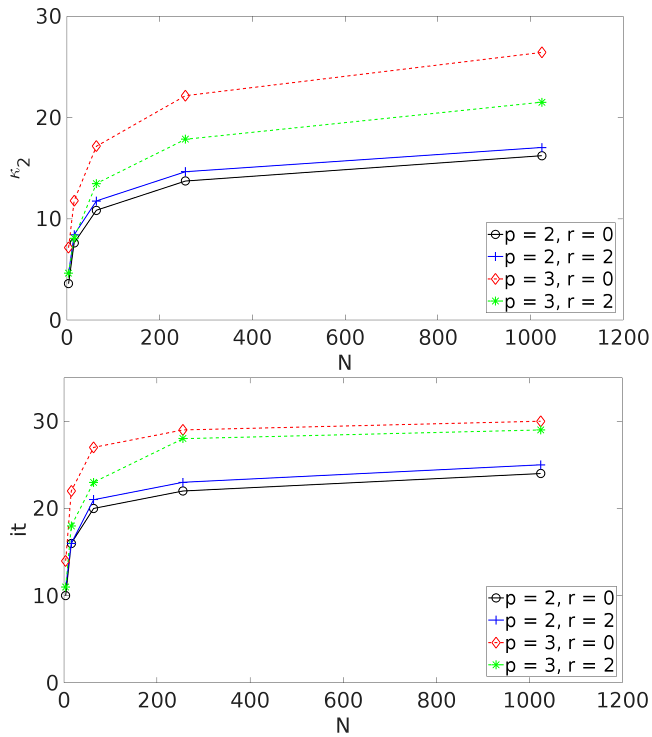

Fig. 2 Scalability of RAGDSW preconditioner in 2D quarter-ring domain: condition number κ_2 (top) and iteration counts it. (bottom) as a function of the number of subdomains N, fixed ratio $H/h = 4$, overlap parameter $r = 0$ and $r = 2$, $tol = 0.1$.

value; see also Fig. 2. We note that for the largest problems of these tables, the dimension of the coarse space appears to increase about four times when the number of subdomains increases by four.

5.2 Dependence on *p*

In this test, we study the robustness of the RAGDSW preconditioner with respect to the spline polynomial degree p. The quarter-ring domain is discretized with a mesh size $h = 1/64$ and $N = 4 \times 4$ subdomains, while the degree p varies from 2 to 8 and the regularity $k = p - 1$ is always maximal. The results reported in Table 3 show that the condition numbers and iteration counts exhibit a moderate increase up to $p = 5$. They then start to increase, more slowly when the adaptive tolerance tol is large and the coarse space sufficiently rich. We note that the condition numbers without preconditioning – not reported – grows very rapidly with p.

Table 3 RAGDSW preconditioner in 2D quarter-ring domain: condition number κ_2 and iteration counts it. as a function of the spline polynomial degree p and adaptive tolerance parameter tol, with maximal regularity $k = p - 1$, fixed number of subdomains $N = 4 \times 4$, $1/h = 64$, $r = 0$. N_Π denotes the dimension of the coarse space.

RAGDSW prec., quarter-ring domain $N = 4 \times 4$, $1/h = 64$, $r = 0$													
		$tol = 0.05$			$tol = 0.1$			$tol = 0.2$			$tol = 0.5$		
p	dofs	κ_2	it	N_Π	κ_2	it	N_Π	κ_2	it	N_Π	κ_2	it	N_Π
2	4356	16.16	24	18	16.16	24	18	10.76	18	36	10.76	18	36
3	4489	22.21	27	18	17.34	24	27	12.20	20	36	8.56	17	81
4	4624	17.56	25	18	14.98	21	27	10.60	19	63	9.24	17	144
5	4761	29.15	31	18	20.05	26	63	13.51	22	108	11.48	20	225
6	4900	52.77	40	54	31.18	32	99	26.33	29	288	24.09	27	324
7	5041	35.19	41	99	26.17	33	252	25.14	31	441	25.14	31	441
8	5184	135.56	73	243	89.68	57	513	82.49	55	576	82.49	55	576

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