Adaptive Schwarz Method for a Non-Conforming Crouzeix-Raviart Discretization of a Multiscale Elliptic Problem

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1 Introduction

In many physical or engineering practical applications, we see a heterogeneity of coefficients; e.g., in ground flow problems in heterogeneous media. We also see that many models of those phenomena are differential ones, i.e. the physical phenomenon is modeled by partial differential equations. Then, if those PDEs models are discretized by a finite element method, one gets the discrete system which is quite often very hard to solve by standard iterative methods without a proper precondition; see, e.g., [21].

The Domain Decomposition Methods (DDMs) approach, in particular, Schwarz methods; see, e.g., [24], allow us to construct a large class of parallel and effective preconditioners. A very important role in such construction is taken by a carefully defined coarse space. The classical DDMs constructed in the 1990s and 2000s are well suited only for problems with coefficients that are constant or slightly varying in subdomains. However, those 'classical' methods are not effective when the coefficients may be highly varying and discontinuous almost everywhere. Since the classical coarse spaces of Schwarz methods do not give us efficient and robust solvers for multiscale problems with heterogeneous coefficients we will propose a way of enrichment of the coarse spaces which made DDMs effective for heterogeneous problems. That gives us new adaptive coarse spaces which are independent or robust for the jumps of the coefficients, i.e., the convergence of the constructed DDM is independent of the distribution and the magnitude of the coefficients of the original

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problem. We refer to [9], [23] and the references therein for similar earlier works on domain decomposition methods used adaptively in the construction of coarse spaces.

In recent years there appeared many new research results on this topic; see, e.g., [3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20] and many others.

In our paper, we consider a minimal overlap Schwarz method for the nonconforming Crouzeix-Raviart (CR) element discretization, also called the nonconforming P_1 element discretization; see, e.g., [1]. We extend the results from [10] where the conforming P_1 element is considered to the case of the CR non-conforming discretization applied to highly heterogeneous coefficients.

The remainder of the paper is organized as follows: in Section 2 we introduce our differential problem and its CR discretization. In Section 3 a classical overlapping Additive Schwarz method is presented and the theoretical bound for the condition number of the resulting system is given.

2 Discrete problem

Our model differential problem is the following elliptic second order boundary value problem: Find $u^* \in H_0^1(\Omega)$ such that

$$\int_\Omega \alpha(x)\nabla u^* \nabla v \, dx = \int_\Omega f v \, dx, \qquad \forall v \in H_0^1(\Omega),$$

where Ω is a polygon in $\mathbb{R}^2$, $0 < \alpha_0 \le \alpha(x) \le \alpha_1$ is a coefficient, α_0, α_1 are positive constant, and $f \in L^2(\Omega)$.

We need a quasi-uniform triangulation $\mathcal{T}_h = \{K\}$ of Ω consisting of open triangles such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$. Let further, h_K be the diameter of $K \in \mathcal{T}_h$, and we define $h = \max_{K\in\mathcal{T}_h} h_K$ as the triangulation diameter.

We also introduce a coarse non-overlapping partitioning of Ω (see, Fig. 1) into open, connected Lipschitz polygonal subdomains (substructures) Ω_i such that

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i,$$

which are aligned to the fine triangulation, i.e. we have that any fine triangle $K \in \mathcal{T}_h$ is contained in a coarse substructure Ω_k. Thus each substructure Ω_j has its local triangulation $\mathcal{T}_h(\Omega_j)$ of triangles from $\mathcal{T}_h$ which are contained in $\overline{\Omega}_j$. For the simplicity of presentation, we further assume that these substructures form a coarse triangulation of the domain which is shape-regular in the sense of [2] and let $H = \max_j \operatorname{diam}(\Omega_j)$ be its coarse parameter. Let Γ_{ij} denote the open edge common to subdomains Ω_i and Ω_j not in $\partial\Omega$ and let Γ be the union of all $\partial\Omega_k \setminus \partial\Omega$.

However, it is good to note that the theory of this paper holds also for the case when the coarse partition is obtained by a mesh partitioner. Then naturally an edge

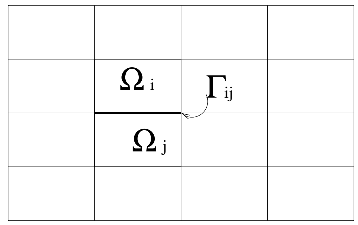

Fig. 1 An example of a coarse partition of Ω, where Γ_{ij} is an edge on the interface.

(interface) Γ_{ij} is not a straight segment but a 1D curve made of respective edges of some fine triangles.

Let Ω_h^{CR}, $\partial\Omega_h^{CR}$, $\Omega_{i,h}^{CR}$, $\partial\Omega_{i,h}^{CR}$, and $\Gamma_{ij,h}^{CR}$ be defined as the sets of midpoints of fine edges of the elements of $\mathcal{T}_h$, contained in Ω, $\partial\Omega$, Ω_i, $\partial\Omega_i$, and Γ_{ij}, respectively. We call those sets the CR nodal points of the respective sets.

The discrete solution space is the Crouzeix-Raviart finite element space, (see, e.g., [1]), or nonconforming P_1 element space defined as:

$$V_h(\Omega) = V_h = \{v \in L^2(\Omega) : v_{|K} \in P_1(K),\ v \text{ continuous at } \Omega_h^{CR},\\ v(m) = 0 \quad m \in \partial\Omega_h^{CR}\},$$

where $P_1(K)$ is the space of linear polynomials defined on K.

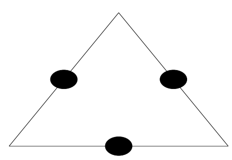

Fig. 2 The CR nodal points, i.e., the degrees of freedom of the Crouzeix-Raviart finite element space on a fine triangle.

The degrees of freedom of a CR finite element function u on a triangle K with the three edges e_k $k = 1, 2, 3$, are: $\{u(m_{e_k})\}_{k=1,2,3}$, where m_{e_k} is the midpoint of the fine edge e_k; see, Fig. 2. Note that a function in V_h is multivalued on boundaries of all fine triangles of $\mathcal{T}_h$ except the midpoints of the edges (CR nodal points). Thus $V_h \not\subset H_0^1(\Omega)$ is a space of discontinuous functions. V_h is only a subspace of $L^2(\Omega)$, and in this lies the non-conformity of this discretization.

We introduce the following Crouzeix-Raviart discrete problems: find $u_h^* \in V_h$ such that :

$$a_h(u_h^*.v) = f(v) \qquad \forall v \in V_h, \tag{1}$$

where the broken bilinear form $a_h : (V_h \cup H_0^1(\Omega)) \times (V_h \cup H_0^1(\Omega)) \to \mathbb{R}$ is defined as $a_h(u, v) = \sum_{K\in\mathcal{T}_h} \int_K \alpha_{|K}(x)\nabla u \nabla v\, dx$. It is easy to see that the broken form is V_h elliptic; see, e.g., [1], and we see that our discrete problem has a unique solution.

We see that ∇u_h for $u_h \in V_h$ is constant over any fine triangle $K \in \mathcal{T}_h$, thus

$$\int_K \alpha \nabla u \nabla v \, dx = (\nabla u)_{|K} (\nabla v)_{|K} \int_K \alpha(x) \, dx.$$

Hence, we can further assume that α is piecewise constant function over the elements of $\mathcal{T}_h$.

3 Additive Schwarz method (ASM)

In this section, we present our Schwarz method for solving (1) which is based on the abstract Additive Schwarz Method framework; see, e.g., [24]. Our method is of minimal overlap, however, the same estimates hold if we introduce a more generous overlap. In the abstract scheme of ASM one has to introduce a decomposition of the discrete space into subspaces, usually, a coarse space and local subspaces. We also need local bilinear forms defined on those subspaces respectively. In our case for simplicity of presentation, all bilinear forms are taken as equal to the original broken-form $a_h(u, v)$.

The local spaces are defined as:

$$V_i = \{v \in V_h : v(m) = 0 \quad m \notin \overline{\Omega}_{i,h}^{CR}\},$$

i.e. V_i is formed by all discrete CR FEM functions which are zero at all CR nodes not in $\overline{\Omega}_i$. Thus, it is a minimal overlap subspace since a function $u \in V_i$ can be nonzero on $\overline{\Omega}_i$ and the fine triangles which have an edge on the boundary of Ω_i. We see that $V_h = \sum_{i=1}^N V_i$.

In our case, the coarse space will be a harmonically enriched CR version of the multiscale coarse space introduced in [11] for standard conforming linear finite element space. Let $\mathcal{T}_h(\Omega_k)$ be a local triangulation of Ω_k inherited from $\mathcal{T}_h$. We now

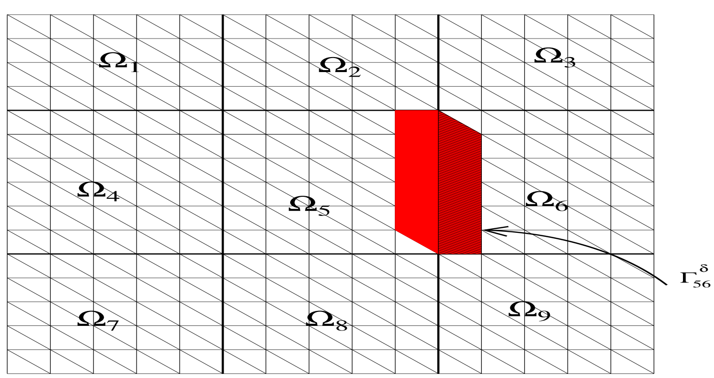

Fig. 3 An edge patch.

introduce a patch around a coarse interface Γ_{kl} the common edge of Ω_k, Ω_l. We define $\overline{\Gamma}^{\delta}_{kl}$ as the closure of the boundary patch around Γ_{kl} the union of all closed fine triangles, such that each fine triangle of the patch has a vertex on Γ_{kl}. The open patch Γ^{δ}_{kl} is then defined as the interior of $\overline{\Gamma}^{\delta}_{kl}$; see, Fig. 3.

For simplicity of presentation, we assume that if two edges Γ_{kl}, Γ_{kj} which have a common vertex (crosspoint - a common vertex of $\Omega_k, \Omega_j, \Omega_l$) then the patches $\Gamma^{\delta}_{kl}, \Gamma^{\delta}_{kj}$ are disjoint. Each patch Γ^{δ}_{kl} can be split into two subpatches – the respective subsets contained in one of two subdomains:

$$\Gamma^{\delta,i}_{kl} = \Gamma^{\delta}_{kl} \cap \Omega_i, \quad i = k, l.$$

Naturally, we have that $\overline{\Gamma^{\delta}_{kl}} = \overline{\Gamma^{\delta,l}_{kl} \cup \Gamma^{\delta,k}_{kl}}$. We next introduce the interior boundary layer of Ω_k:

$$\Omega^{in,\delta}_k = \bigcup_{\Gamma_{kl} \subset \partial\Omega_k \cap \Gamma} \Gamma^{\delta,k}_{kl}.$$

We also define the local subspaces: let $V_{h,k}$ be formed by the restrictions to $\overline{\Omega}_k$ of the functions from V_h, i.e.

$$V_{h,k} = \{v \in L^2(\Omega_k) : v_{|K} \in P_1(K), K \in \mathcal{T}_h(\Omega_k), \\ v - \text{continuous at CR nodes}, v_{|\partial\Omega^{CR}_h} = 0\}$$

Let $V^0_{h,k} \subset V_{h,k}$ be space of functions that are zero at $\partial\Omega^{CR}_{k,h}$ and at the CR nodes in the interior $\Omega^{in,\delta}_k$. Any $u \in V^0_{h,k}$ can be extended by zero to the whole Ω and we will further identify $V^0_{h,k}$ with the subspace of V_h formed by such zero extensions of functions in this local space.

Let $\mathcal{P}_k : V^h \to V^0_{h,k}$ be the orthogonal projection:

$$a_{k,h}(\mathcal{P}_k u, v) = a_h(\mathcal{P}_k u, v) = a_h(u, v) \quad \forall v \in V^0_{h,k}, \tag{2}$$

where $a_{k,h}(u, v)$ is the local bilinear form defined as the restriction of the broken form to Ω_k. Let $\mathcal{P}u = \sum_{k=1}^N \mathcal{P}_k u$, ($\mathcal{P}_k u$ extended by zero to $\overline{\Omega}$). Then the discrete harmonic operator is set as $\mathcal{H} = I - \mathcal{P}$ and we say that $u \in V_h$ is discrete harmonic if:

$$u = \mathcal{H}u. \tag{3}$$

Next, we need to set a local edge related space $V_{kl} \subset V_h$:

$$V_{kl} = \{v \in V_h : v(m) = 0 \quad m \notin \overline{\Gamma}^{\delta,CR}_{kl,h}\}.$$

The support of any function $u \in V_{kl}$ is not contained in the patch Γ^{δ}_{kl}.

We also need a subspace of V^v_{kl} defined as:

$$V^v_{kl} = \{v \in V_{kl} : v(m) = 0 \quad m \in \mathcal{V}(\Gamma_{kl})\},$$

where $\mathcal{V}(\Gamma_{kl}) \subset \Gamma_{kl,h}^{CR}$ comprise the two CR nodes of the edge which are next to the ends of this edge.

Let $V_0^{msc} \subset V_h$ be the multiscale part of the coarse space (analogous to the one in [11]), i.e., the space of discrete harmonic functions; see, (3), which satisfy

$$a_{kl,h}(u,v) = 0 \quad \forall v \in V_{kl}^{v}, \tag{4}$$

where $a_{kl,h}(u,v) = \sum_{K \subset \Gamma_{kl}^{\delta}} \int_K \alpha_{|K} \nabla u \nabla v \, dx$ for any edge $\Gamma_{kl} \subset \Gamma$.

Let us introduce the local generalized eigenvalue problem, which is to find all eigenpairs: $(\lambda_i^{kl}, \psi_i^{kl}) \in \mathbb{R}_+ \times V_{kl}^{v}$ such that

$$a_{kl,h}(\psi_i^{kl}, v) = \lambda_i^{kl} b_{kl}(\psi_i^{kl}, v), \qquad \forall v \in V_{kl}^{v}, \tag{5}$$

where $b_{kl}(u,v) = h^{-2} \int_{\Gamma_{kl}^{\delta}} \alpha u v \, dx$.

Any eigenfunction ψ_j^{kl} can be extended further onto other patches as zero and then, further to the interiors of all subdomains as a discrete harmonic function. Then, we will further denote it by Ψ_j^{kl}. We can number the eigenvalues in increasing order: $0 < \lambda_1^{kl} \leq \lambda_2^{kl} \leq \ldots \leq \lambda_{M_{kl}}^{kl}$ for $M_{kl} = \dim(V_{kl}^{v})$. Next, we introduce the local spectral component of the coarse space for all Ω_j:

$$V_{kl}^{eig} = \mathrm{Span}(\Psi_i^{kl})_{i=1}^{n_{kl}}, \tag{6}$$

where $0 \leq n_{kl} \leq M_{kl}$ can be pre-selected by the user. It can be decided using the experience or by some rule; e.g., one can include all eigenfunctions for which related eigenvalues are below a certain threshold. The coarse space V_0 is introduced as:

$$V_0 = V_0^{msc} + \sum_{\Gamma_{kl} \subset \Gamma}^{N} V_{kl}^{eig}.$$

Next, we define the projection operators $T_i \colon V_h \to V_i$ for $i = 0, \ldots, N$ as

$$a_h(T_i u, v) = a_h(u,v), \quad \forall v \in V_i;$$

see, e.g., [22]. Note that to compute the $T_i u$, $i = 1, \ldots, N$ we have to solve N independent local problems.

Let $T := \sum_{i=0}^{N} T_i$, be the additive Schwarz operator; see, e.g., [22]. We further replace (1) by the following equivalent problem: Find $u_h^* \in V_h$ such that

$$T u_h^* = g,$$

where $g = \sum_{i=0}^{N} g_i$ and $g_i = T_i u_h^*$. The functions g_i may be computed without knowing the solution u_h^* of (1); see, e.g., [22].

The following theoretical estimate of the condition number can be obtained:

Theorem 1 *For all $u \in V_h$, the following holds,*

$$c\left(1+\max_{\Gamma_{kl}\subset\Gamma}\left(\lambda^{kl}_{n_{kl}+1}\right)^{-1}\right)^{-1} a_h(u,u) \le a_h(Tu,u) \le C\, a_h(u,u),$$

where C and c are positive constants independent of the coefficient α, the mesh parameter h and the subdomain size H, and $\lambda^{kl}_{n_{kl}+1}$ is defined in (5) for both types of the coarse space.

Below we give a very brief sketch of the proof, which is based on the standard abstract ASM Method framework; see, [24]. We have to prove three key assumptions, the most technical is the stable splitting ass., namely, we show that for any $u \in V_h$ there exists: $u_j \in V_j$ $j = 0,\dots,N$ such that $\sum_{j=0}^{N} a_h(u_j,u_j) \le c^{-1}\left(1+\max_{\Gamma_{kl}}\left(\lambda^{kl}_{n_{kl}+1}\right)^{-1}\right) a(u,u)$. The two others assumptions are easy to verify.

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