A Variational-Based Multirate Time-Integrator for FETI and Structural Dynamics: Lagrange-Multiplier with Micro-Discretization

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1 Introduction

The FETI-method is well known for its scalability and applicability to nonlinear structural dynamics [3, 4]. In case of models with different fast dynamics, the classical FETI-method with common time-discretizations can become inefficient, as the subdomain with slow dynamics has to be solved more often, than necessary. The PH-method [8] and BGC-macro-method [1] enable subcycling of a macro-time-discretization, but suffer from spurious oscillations and are not variational methods. In literature, a variational framework for multiple time-discretizations has been introduced [7]. In this work, we further extend this approach to a micro-discretization. In section 2, the FETI-method and nonlinear BGC-macro method are introduced. In section 3, the variational-based multirate method is derived with its modifications and in section 4 both methods are compared in numerical experiments.

2 Nonlinear BGC-macro method for the FETI-method

2.1 Model problem and FETI-method

The dynamic behavior over time of a solid elastic body with nonlinear material can be modeled by a nonlinear hyperbolic partial differential equation (PDE). For the solution of such a hyperbolic PDE, consider a geometrical discretization with the Finite Element method and the Finite Element Tearing and Interconnecting (FETI) for the spacial non-overlapping domain decomposition into N_s subdomains. Hence,

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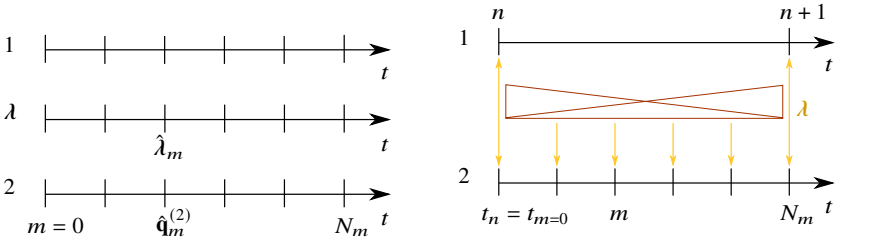
the spacially discretized, time-continuous differential equation of motion of a subdomain s and the compatibility condition for velocities are

$$\mathbf{M}^{(s)}\ddot{\mathbf{q}}^{(s)} + \mathbf{f}_{int}(\mathbf{q}^{(s)}) + \mathbf{B}^{(s)^T}\boldsymbol{\lambda} - \mathbf{f}_{ext}^{(s)}(t) = \mathbf{0}, \quad \sum_{s=1}^{N_s} \mathbf{B}^{(s)}\dot{\mathbf{q}}^{(s)} = \mathbf{0}.$$

Here, $\mathbf{q}^{(s)}$ describes the nodal displacements and its time-derivatives $\dot{\mathbf{q}}^{(s)}$ and $\ddot{\mathbf{q}}^{(s)}$ are the velocities and accelerations. $\mathbf{M}^{(s)}$ is the mass-matrix, $\mathbf{f}_{int}$ are the nonlinear internal forces and $\mathbf{f}_{ext}^{(s)}$ are the external forces of the subdomain. The dual quantity or interface-force is described by $\boldsymbol{\lambda}$ and $\mathbf{B}^{(s)}$ is a signed boolean matrix mapping the subdomain's geometrical degrees of freedom (dof) to interface-dofs. The unknowns $\mathbf{q}^{(s)}$, its derivatives and $\boldsymbol{\lambda}$ are discretized in time, with a common time-step-size Δt and to time-nodal values $\hat{\mathbf{q}}_m^{(s)}$ and $\hat{\boldsymbol{\lambda}}_m$ at a time-step m, as depicted in Fig. 1a. For the time-stepping from time-step m to $m+1$, a time-integration scheme is applied, such as the Newmark-β scheme.



(a) Time-discretization in nodal values at time-steps m.

(b) Subcycling of the time-discretization of two subdomains.

Fig. 1 Time-discretizations for two exemplary subdomains 1 and 2 and the Lagrange-multipliers.

2.2 Multirate with nonlinear BGC-macro method

Having different time-step-sizes in the FETI-method can be achieved by the BGC-macro method [1], later adapted for nonlinear problems and FETI [9]. The time-discretization on the subdomain with the smaller time-step-size, also referred to as the micro-discretization, subcycles subdomains with a larger time-step-size, the macro-discretization, as depicted in Fig. 1b. The Lagrange-multipliers are discretized with the macro-discretization and interpolated linearly on the micro-discretized subdomain. Hence, the dynamic equation of motion and the compatibility condition, which is enforced at the macro-discretization, follow as

$$\mathbf{M}^{(s)}\ddot{\hat{\mathbf{q}}}_m^{(s)} + \mathbf{f}_{int}(\hat{\mathbf{q}}_m^{(s)}) + \mathbf{B}^{(s)^T}\boldsymbol{\lambda}_m - \mathbf{f}_{ext}^{(s)}(t_m) = \mathbf{0} \qquad \sum_{s=1}^{N_s} \mathbf{B}\dot{\hat{\mathbf{q}}}_n^s = \mathbf{0}$$

with the interpolated Lagrange-multiplier

$$\boldsymbol{\lambda}_m = \frac{t_{n+1} - t_m}{t_{n+1} - t_n}\boldsymbol{\lambda}_n + \frac{t_m - t_n}{t_{n+1} - t_n}\boldsymbol{\lambda}_{n+1}.$$

3 Variational multirate method with micro discretization of the dual field

The equation of motion (1) and the well-known Newmark-β time-integration scheme can also be derived from the variational principle for $\gamma = 0.5$ and $\beta = 0.25$, as shown by Kane e.a. [5]. We define the time-continuous kinetic energy of a subdomain as $\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}^{(s)^T}\mathbf{M}\dot{\mathbf{q}}^{(s)}$, the nonlinear potential energy $\mathcal{V}(\mathbf{q}^{(s)})$ and the interface-energy $\mathcal{G} = \mathbf{g}(\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(N_s)})^T\boldsymbol{\lambda}$ with the gap on interface $\mathbf{g}$, corresponding to the Lagrange-multipliers $\boldsymbol{\lambda}$. In case of the FETI-method this gap is $\mathbf{g}(\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(N_s)}) = \sum_{s=1}^{N_s}\mathbf{B}^{(s)}\mathbf{q}^{(s)}$, which has not been explicitly specified in literature [7]. The Lagrangian then follows as

$$\mathcal{L}(\dot{\mathbf{q}}^{(1)}, \mathbf{q}^{(1)}, \ldots, \dot{\mathbf{q}}^{(N_s)}, \mathbf{q}^{(N_s)}, \boldsymbol{\lambda}) = \sum_{s=1}^{N_s}\left(\mathcal{T}(\dot{\mathbf{q}}^{(s)}) - \mathcal{V}(\mathbf{q}^{(s)})\right) + \mathcal{G}. \tag{1}$$

According to Hamilton's principle, the mechanical system will move such that the action integral of this Lagrangian is stationary. Hence, we first discretize the Lagrangian in time with time-shape-functions $\Phi^{(s)}(t)$ and $\Theta(t)$, that fulfill partition of unity, we can approximate displacements, velocities and Lagrange-multipliers as

$$\mathbf{q}^{(s)}(t) \approx \sum_{m=0}^{N_m^{(s)}}\Phi_m^{(s)}(t)\hat{\mathbf{q}}_m^{(s)}, \quad \dot{\mathbf{q}}^{(s)}(t) \approx \sum_{m=0}^{N_m^{(s)}}\frac{d\Phi_m^{(s)}(t)}{dt}\hat{\mathbf{q}}_m^{(s)}, \quad \boldsymbol{\lambda}(t) \approx \sum_{j=0}^{N_j}\Theta_j\hat{\boldsymbol{\lambda}}_j.$$

Throughout this paper, we assume linear time-shape-functions. This results in the discrete Lagrangian

$$\mathcal{L}_d(\hat{\mathbf{q}}_0^{(s)}, \ldots, \hat{\mathbf{q}}_{N_m}^{(s)}, \hat{\boldsymbol{\lambda}}_0, \ldots, \hat{\boldsymbol{\lambda}}_{N_j}, t) = \sum_{s=1}^{N_s}\left(\mathcal{T}(\hat{\mathbf{q}}^{(s)}, t) - \mathcal{V}(\hat{\mathbf{q}}^{(s)}, t)\right) + \mathcal{G}(t) \tag{2}$$

which is then integrated with a numerical quadrature rule, such as the generalized midpoint-rule, to the discrete action integral

$$\mathcal{S}_d = \sum_{k=0}^{N_k}\Delta t_k\,\mathcal{L}_d(t_{k+\alpha}), \tag{3}$$

where we have N_k common integration-segments and $\mathcal{L}_d$ is evaluated at a generalized mid-point of these segments $t_{k+\alpha}$. This discrete action integral has to remain

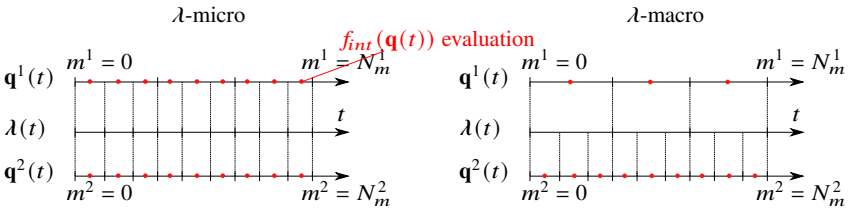
stationary $\sum_{s=1}^{N_s}\sum_{m=0}^{N_m^{(s)}}\frac{\partial \mathcal{S}_d}{\partial \hat{\mathbf{q}}_m^{(s)}}\delta\hat{\mathbf{q}}_m^{(s)}+\sum_{j=0}^{N_j}\frac{\partial \mathcal{S}_d}{\partial \hat{\boldsymbol{\lambda}}_j}\delta\hat{\boldsymbol{\lambda}}_j=\mathbf{0}$ for arbitrary variations of time-nodal quantities, while the endpoints $\delta\hat{\mathbf{q}}_0^{(s)}$ and $\delta\hat{\mathbf{q}}_{N_m}^{(s)}$ remain fixed. This way, we also obtain a local variational integration scheme, such as the non-dissipative Newmark-β method and a variational coupling condition. A variational method comes with some beneficial properties by design, such as symplecticity, conservation of momentum and energy-oscillations remain bounded [6]. We could now solve this problem with a Newton-Raphson scheme and solve the Lagrange-multipliers at each Newton-iteration with a FETI-solver. However, in general, all equations have to be solved at once and a more memory-efficient time-stepping can only be applied on the subdomain-level [9]. The constraint equation for Lagrange-multiplier j

$$\frac{\partial \mathcal{S}_d}{\partial \hat{\boldsymbol{\lambda}}_j}=\sum_{k=0}^{N_k}\Delta t_k\Theta_j(t_{k+\alpha})\sum_{s=1}^{N_s}\mathbf{B}^{(s)}\sum_{m=0}^{N_m^{(s)}}\Phi_m^{(s)}(t_{k+\alpha})\hat{\mathbf{q}}_m^{(s)}=\mathbf{0},$$

is a constraint for several time-nodal displacements $\hat{\mathbf{q}}_m^{(s)}$. Hence, In the following sections 3.1 and 3.2, we introduce special cases and some modifications to this variational method, to still enable time-stepping.

3.1 Downsampling of Lagrange-multipliers

The quadrature (3) suggests the evaluation of the discrete Lagrangian is performed at each time-step $t_{k+\alpha}$ regardless of each subdomain's time-discretization and therefore the evaluation of the nonlinear potential energy derivative $\frac{\partial \mathcal{V}}{\partial \mathbf{q}}$. Hence, in terms of computational efficiency, one could as well choose a micro-discretization in all subdomains. In the following, we consider a subcycled time-discretization on all subdomains and Lagrange-multipliers. If the macro-discretization is chosen for the Lagrange-multiplier, one can just evaluate at the local time-step's midpoint, as depicted in Fig. 2b to properly integrate the Lagrangian. This high number of evaluations is especially needed if the time-discretization of the Lagrange-multiplier is chosen as a micro-discretization, as shown in Fig. 2a. For such cases, we introduce a downsampling of the Lagrange-multiplier by inserting an additional local Lagrange-multiplier field $\bar{\boldsymbol{\lambda}}^{(s)}$, as depicted in Fig. 3. With an artificial



(a) Evaluation-points for micro-discretization of the Lagrange-multiplier.

(b) Evaluation-points for macro-discretization of the Lagrange-multiplier.

Fig. 2 Subcycling time-discretization of two subdomains and evaluation-points for quadrature.

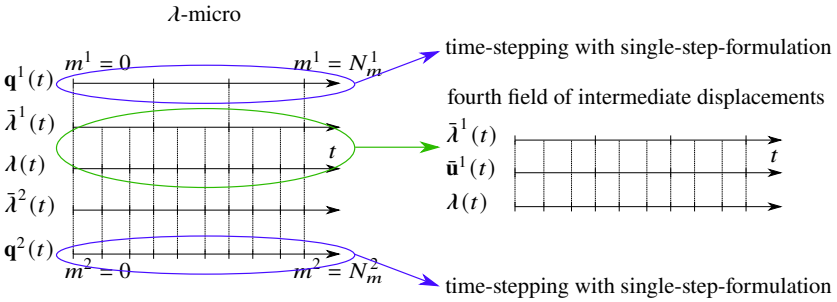


Fig. 3 Additional local Lagrange-multiplier-field and artificial displacement-field for local downsampling.

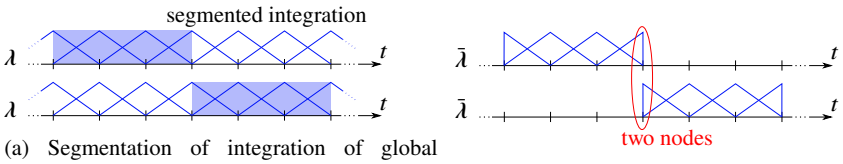
displacement-field $\bar{\mathbf{u}}^{(s)}$, connecting both Lagrange-multiplier fields, we can reformulate the constraint-energy $\mathcal{G} = \left(\sum_{s=1}^{N_s} \mathbf{B}^{(s)} \sum_{m=0}^{N_m^{(s)}} \Phi_m^{(s)}(t)\bar{\mathbf{u}}_m^{(s)}\right)^T \sum_{j=0}^{N_j} \Theta_j(t)\hat{\boldsymbol{\lambda}}_j + \sum_{s=1}^{N_s}\left(\left(\sum_{m=0}^{N_m^{(s)}} \Phi_m^{(s)}(t)\hat{\mathbf{q}}_m^{(s)} - \sum_{m=0}^{N_m^{(s)}} \Phi_m^{(s)}(t)\bar{\mathbf{u}}_m^{(s)}\right)^T \sum_{m=0}^{N_m^{(s)}} \Phi_m^{(s)}(t)\bar{\boldsymbol{\lambda}}_m^{(s)}\right)$ and apply the generalized midpoint-rule and variational calculus to obtain constraints $\frac{\partial \mathcal{G}}{\partial \hat{\boldsymbol{\lambda}}_j} = \mathbf{0}$ and $\frac{\partial \mathcal{G}}{\partial \bar{\boldsymbol{\lambda}}_m^{(s)}} = \mathbf{0}$, that are fulfilled in a weak sense. With the variation for $\bar{\mathbf{u}}_m^{(s)}$ follows the downsampling-equation

$$\sum_{k=0}^{N_k} \Delta t_k \left(\mathbf{B}^{(s)^T} \Phi_m^{(s)}(t_{k+\alpha}) \sum_{j=0}^{N_j} \Theta_j(t_{k+\alpha})\hat{\boldsymbol{\lambda}}_j - \Phi_m^{(s)}(t_{k+\alpha}) \sum_{m=0}^{N_m^{(s)}} \Phi_m^{(s)}(t_{k+\alpha})\bar{\boldsymbol{\lambda}}_m^{(s)} \right) = \mathbf{0}$$

and from $\sum_{m=0}^{N_m^{(s)}} \Delta t_m \Phi_m^{(s)}(t_{m+\alpha}) \sum_{m=0}^{N_m^{(s)}} \Phi_m^{(s)}(t_{m+\alpha})\bar{\boldsymbol{\lambda}}_m^{(s)}$ follows together with the kinetic and potential part of the discrete Lagrangian the local equation of motion and the time-stepping-scheme. All these equations can now be solved by a Newton-Raphson scheme. Due to $\hat{\boldsymbol{\lambda}}$ at the macro-discretization influencing both sides, the left and the right, the interface-problem still has to be solved all at once.

3.2 Reduce to time-stepping

To enable at least a time-stepping on the interface-problem from one macro-time-step to the next and only solve the subcycled Lagrange-multipliers between two macro-time-steps at once, we have to reduce the global integration and introduce some errors that way. The integration of the previously introduced equations is no longer performed from 0 to N_k, but only from one macro-time-step to the next one, which is visualized in Fig. 4a. While the global Lagrange-multiplier-field itself stays continuous, this requires the local Lagrange-multiplier to become discontinuous at the macro-time-steps, as can be seen in Fig. 4b. Finally, we have to apply some numerical dissipation or formulate the constraints for velocities, instead of displacements.



(a) Segmentation of integration of global Lagrange-multipliers.

(b) Discontinuous local Lagrange-multipliers.

Fig. 4 Segmentation of integration of Lagrange-multipliers according to macro-discretization.

Otherwise, high-frequency instabilities might prevent the solver from converging as pointed out by Farhat e.a. [2]. Hence, we replace the nodal displacements in the constraints with nodal velocities. Of course, with these modifications, our framework is no longer a variational method, but as shown in section 4, some beneficial properties of variational methods are still preserved, which is why we call it a variational-based framework instead.

4 Numerical experiments

In this section, we compare the accuracy of the BGC-macro method with our variational-based multirate method. To this end, we apply both methods to a nonlinear split Duffing-oscillator, as proposed by Prakash e.a. [8] and depicted in Fig. 5 and solve the interface-problem with a GMRes-method. The velocities from the BGC-macro in Fig. 6a exhibit the well-known spurious oscillations [8, 9], leading to rather large incompatibilities in the displacements. These spurious oscillations are reduced by the micro-discretization of the variational based method in Fig. 6b, which improves the compatibility of displacements. The solution from the BGC-macro method shows slightly less phase-error, as the displacement-curve is closer to the fine-solution, compared to the variational-based method, but the solution still remains in the margin between the fine and the coarse singlerate Newmark solution. The energy-behavior of the variational-based method in Fig. 7b is also still better compared to the BGC-macro method in Fig. 7a, despite the modifications made. The total energy's oscillations remain bounded, while we can observe a slight decline in the BGC-macro's total energy. Also the amplitude of the interface-energy's oscillations is smaller for the variational-based method. However, all this comes at the cost of a larger interface-problem.

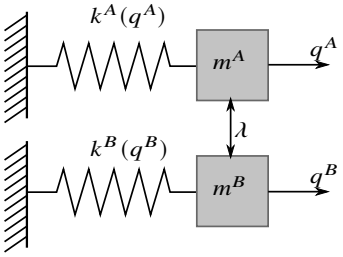
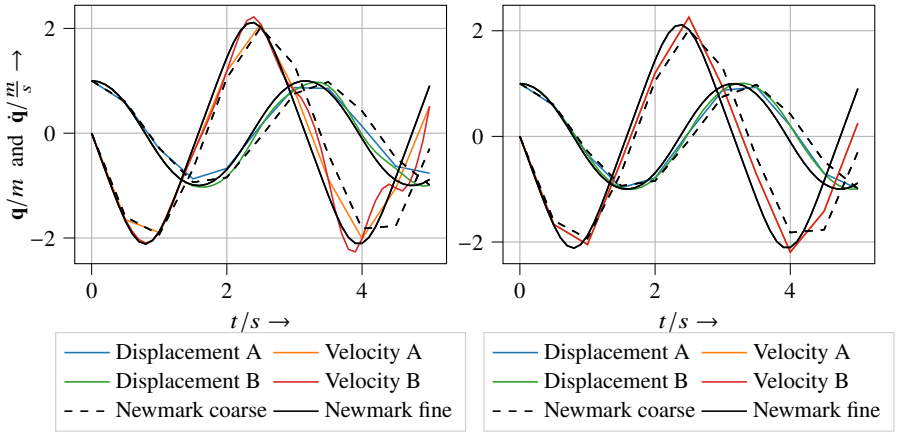


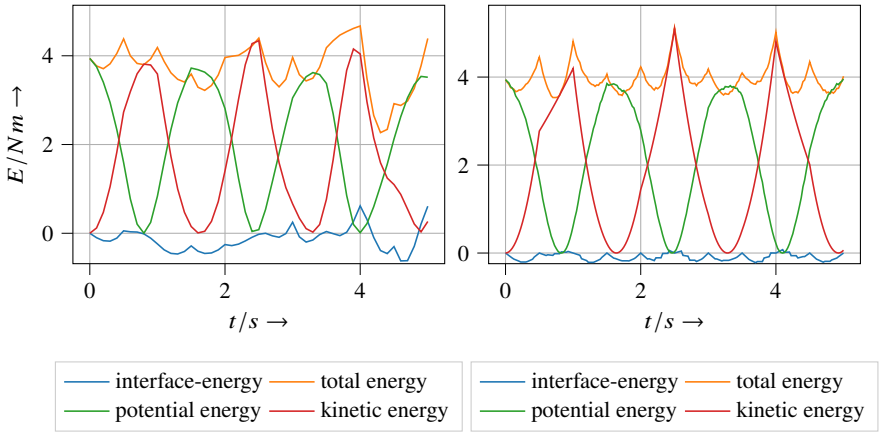
Fig. 5 Split Duffing-oscillator with stiffnesses $k^{(1)}(q^{(1)}) = 1\frac{N}{m} \cdot q^{(1)} + 1\frac{N}{m} \cdot q^{(1)^3}$, $k^{(2)}(q^{(2)}) = 10\frac{N}{m} \cdot q^{(2)} - 5\frac{N}{m} \cdot q^{(2)^3}$, masses $m^{(1)} = 1kg$, $m^{(2)} = 1kg$, time-step-sizes $\Delta t^{(1)} = 0.5s$, $\Delta t^{(2)} = 0.1s$.



(a) BGC-macro.

(b) Variational-based multirate integrator.

Fig. 6 Displacements and Velocities of the split Duffing-oscillator.



(a) BGC-macro.

(b) Variational-based multirate integrator.

Fig. 7 Energies of the split Duffing-oscillator.

5 Conclusions

The derived variational-based multirate method and its interface-problem is solved by a FETI-solver. The method enables a macro-time-stepping and still exhibits a better accuracy than the BGC-macro method. This comes at the cost of a larger interface-problem. A suitable preconditioner remains to be constructed.

Acknowledgements We thank the Deutsche Forschungsgemeinschaft (DFG) for the funding of project 357361040, in which context this work has been done.

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