Accelerated Convergence of the Pipelined Dynamic Iteration Method for RLC Circuits

Hélèna Shourick, Damien Tromeur-Dervout, and Laurent Chédot

1 Introduction

Since the pioneering work of Lelarasme et al. [9] that analyze in time domain large-scale problems arising from the modeling of integrated circuits, waveform relaxation methods (WR) [10] also known as dynamic iteration methods, a term first introduced by Miekkala and Nevanlinna [13, Eq. (2.2)], arouses more and more interest with the development of parallel computers. Let us recall the method of dynamic iteration as it was described by Miekkala[12] for ODE systems and adapted by Jiang & Wing for DAE systems [7]: Let $M \in \mathbb{C}^{n_1 \times n_1}$, $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_a}$, $C \in \mathbb{C}^{n_a \times n_1}$, $D \in \mathbb{C}^{n_a \times n_a}$ matrices and $f_1 \colon]0,T] \to \mathbb{C}^{n_1}$, $f_2 \colon]0,T] \to \mathbb{C}^{n_a}$ functions, $x_0 \in \mathbb{C}^{n_1}$ initial state value. We define the DAE system in its state-space form:

$$\begin{cases} M\dot{x}(t) + Ax(t) + By(t) = f_1(t),\ t \in [0,T], \\ Cx(t) + Dy(t) = f_2(t),\ t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x \colon]0,T] \to \mathbb{C}^{n_1}$ are the n_1 searched state solutions and $y \colon]0,T] \to \mathbb{C}^{n_a}$ are the n_a searched algebrical solutions.

Definition 1 (Dynamic Iteration for linear DAE) The Dynamic Iteration scheme for (1) considers the splitting of matrices M, A, B, C, D as $M = M_1 - M_2$, $A = A_1 - A_2$,

Hélèna Shourick
SuperGrid-Institute, 23 rue Cyprian, 69100 Villeurbanne, University of Lyon, UMR5208 U.Lyon1-CNRS, Institut Camille Jordan, e-mail: helena.shourick@supergrid-institute.com

Damien Tromeur-Dervout
University of Lyon, UMR5208 U.Lyon1-CNRS, Institut Camille Jordan,
e-mail: damien.tromeur-dervout@univ-lyon1.fr

Laurent Chédot
SuperGrid-Institute, 23 rue Cyprian, 69100 Villeurbanne,
e-mail: laurent.chedot@supergrid-institute.com

$B = B_1 - B_2$, $C = C_1 - C_2$, $D = D_1 - D_2$, where matrices M_1 and D_1 are assumed non-singular (which implies that the DAE system has index one)

$$\begin{cases} M_1\dot{x}^{(k)}(t) + A_1x^{(k)}(t) + B_1y^{(k)}(t) = M_2\dot{x}^{(k-1)}(t) + A_2x^{(k-1)}(t) + B_2y^{(k-1)}(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad +f_1(t),\ t \in [0,T] \\ C_1x^{(k)}(t) + D_1y^{(k)}(t) = C_2x^{(k-1)}(t) + D_2y^{(k-1)}(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad +f_2(t),\ t \in [0,T], \\ x^{(k)}(0) = x_0, \end{cases} \tag{2}$$

This fixed-point process must be contracting to converge. We propose to combine the DI method with the Restricted Additive Schwarz splitting and the Aitken's acceleration of the convergence technique to obtain a DI method less sensitive to the contracting property even with applying it on a pipeline of several time step or on a nonlinear problem. Related works on improvement of DI are that follow. Arnold & Gunther [1] proposed several techniques for preconditioning the fixed-point problem. Some waveform successive overrelaxation (SOR) techniques have been proposed by Janssen and Vandewalle [6] to accelerate the standard waveform method. Leimkuhler proposed to accelerate the WR by solving the defect equations with a larger timestep, or by using a recursive procedure based on a succession of increasing timesteps [8]. Lumdaisne & Wu proposed to accelerate the WR by Krylov subspace techniques (WGMRES) [11] to solve time-dependent problems. Gausling & al [5] analyzed the contraction and the rate of convergence of the co-simulation process for a test circuit subjected to uncertainties on the parameters of its components. In section 2, we consider the Restrictive Additive Schwarz [3] for the Eq. (1) with $M = I$ and we show that is a DI scheme with a specific splitting. Then we can apply the Aitken's acceleration of the convergence technique to obtain the true solution whether the DI is contracting or not (it must not stagnate). Section 3 considers advantages and drawbacks of the sequential (time step after time step) and the pipelined (several time steps at once) implementations of DI and their acceleration of convergence. Section 4 gives some numerical results of the pipelined DI accelerated by the Aitken's technique while section 5 concludes.

2 DI with RAS splitting

Let us consider Eq. (1) resulting from the modeling of an electrical network where we choose $M = I$, this choice corresponds to a change of variables on the voltage terms in the Kirchhoff's law. According to the RAS method notation of Cai & Sarkis [3] applied to the graph of the operator $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \epsilon A & B \\ C & D \end{pmatrix}$, where the ϵ is chosen in order to keep all data dependencies, we define the associated restriction operators R_i^p and $\tilde{R}_i^0$. Then, the k^{th} RAS iteration can be written as:

$$\begin{cases} \dot{x}_i^{(k)}(t) + A_i x_i^{(k)}(t) + B_i y_i^{(k)}(t) = b_i^b(t) - E_{i,d}^d x_{ie}^{(k-1)}(t) - E_{i,d}^a y_{ie}^{(k-1)}(t), \\ C_i x_i^{(k)}(t) + D_i y_i^{(k)}(t) = b_i^a(t) - E_{i,a}^d x_{ie}^{(k-1)}(t) - E_{i,a}^a y_{ie}^{(k-1)}(t), \\ x_i^{(k)}(0) = R_i^{p,d} x_0, \quad t \in [0,T]. \end{cases} \tag{3}$$

With $A_i = R_i^{p,d} A (R_i^{p,d})^T$, $B_i = R_i^{p,d} B (R_i^{p,a})^T$, $C_i = R_i^{p,a} C (R_i^{p,d})^T$, $D_i = R_i^{p,a} D (R_i^{p,a})^T$, $E_{i,d}^d = R_i^{p,d} A (R_{i,e}^{p,d})^T$, $E_{i,d}^a = R_i^{p,d} B (R_{i,e}^{p,a})^T$, $E_{i,a}^d = R_i^{p,a} C (R_{i,e}^{p,d})^T$ and $E_{i,a}^a = R_i^{p,a} D (R_{i,e}^{p,a})^T$. The operator $R_i^{p,d}$ (respectively $R_i^{p,a}$) is the restriction to the differential variables (respectively algebraical variables) of R_i^p. We also define $\tilde{R}_i^{0,d}$ and $\tilde{R}_i^{0,a}$ such that $\tilde{R}_i^0 = \begin{pmatrix} \tilde{R}_i^{0,d} & 0_{n_{i_1} \times n_2} \\ 0_{n_{i_2} \times n_1} & \tilde{R}_i^{0,a} \end{pmatrix}$ as we have chosen to separate the differential and algebraic parts.

By summing up the contribution of each RAS partition, we can show that the k^{th} RAS iteration for solving (1) (with $M = I$) is a DI as defined in (2) associated to the following splitting of the operators $A = A_1^d - A_2^d$, $B = B_1^d - B_2^d$, $C = C_1^a - C_2^a$, $D = D_1^a - D_2^a$:

$$\begin{cases} \dot{x}^{(k)}(t) + A_1^d x^{(k)}(t) + B_1^d y^{(k)}(t) = b^d(t) + A_2^d x^{(k-1)}(t) + B_2^d y^{(k-1)}(t), \\ C_1^a x^{(k)}(t) + D_1^a y^{(k)}(t) = b^a(t) + C_2^a x^{(k-1)}(t) + D_2^a y^{(k-1)}(t), \\ x^{(k)}(0) = x_0, \quad t \in [0,T]. \end{cases} \tag{4}$$

with

$$A_1^d = \sum_{i=0}^{N-1} \tilde{R}_i^{0,d} A_i R_i^{p,d},\ A_2^d = -\sum_{i=0}^{N-1} \tilde{R}_i^{0,d} E_{i,d}^d R_{ie}^{p,d},\ b^d(t) = \sum_{i=0}^{N-1} \tilde{R}_i^{0,d} R_i^{p,d} b^d(t),$$

$$B_1^d = \sum_{i=0}^{N-1} \tilde{R}_i^{0,d} B_i R_i^{p,a},\ B_2^d = -\sum_{i=0}^{N-1} \tilde{R}_i^{0,d} E_{i,d}^a R_{ie}^{p,a},$$

$$C_1^a = \sum_{i=0}^{N-1} \tilde{R}_i^{0,a} C_i R_i^{p,d},\ C_2^a = -\sum_{i=0}^{N-1} \tilde{R}_i^{0,a} E_{i,a}^d R_{ie}^{p,d},$$

$$D_1^a = \sum_{i=0}^{N-1} \tilde{R}_i^{0,a} D_i R_i^{p,a},\ D_2^a = -\sum_{i=0}^{N-1} \tilde{R}_i^{0,a} E_{i,a}^a R_{ie}^{p,a},\ b^a(t) = \sum_{i=0}^{N-1} \tilde{R}_i^{0,d} R_i^{p,a} b^a(t).$$

Thus, the RAS method applied to DAE system belongs to the DI methods with a specific splitting of the operators. Then we can reduce this specific DI method to an interface problem and we can accelerate its convergence to the true solution with the Aitken's acceleration of the convergence technique as in [4]. Denoting $W_{i,e}^{p,d}$ and $W_{i,e}^{p,a}$ the differential and algebraical components of $W_{i,e}^p$, we define $\Gamma^d = \{W_{0,e}^{p,d}, \ldots, W_{N-1,e}^{p,d}\}$, $\Gamma^a = \{W_{0,e}^{p,a}, \ldots, W_{N-1,e}^{p,a}\}$ and $\Gamma = \{\Gamma^d, \Gamma^a\}$ and R_Γ the restriction to the global interface $R_\Gamma = \begin{pmatrix} R_\Gamma^d & 0 \\ 0 & R_\Gamma^a \end{pmatrix}$ with $R_\Gamma^d = (R_{0,ie}^{p,d}, \ldots, R_{N-1,ie}^{p,d})^T$, $R_\Gamma^a = (R_{0,ie}^{p,a}, \ldots, R_{N-1,ie}^{p,a})^T$ and finally $z^{(k)} = (x^{(k)T}, y^{(k)T})^T$ and $z_\Gamma^{(k)} = R_\Gamma z^{(k)}$.

The DI with RAS splitting defined by Eq. (4) applied to a linear DAE system with D_1^a invertible has an error operator $P_{t,\Gamma}, t \in]0,T]$ for the problem interface that does not depend on the iteration number, such that the restriction of the iteration to the global interface satisfies: $z_\Gamma^{(k)} = P_{t,\Gamma} z_\Gamma^{(k-1)} + c$. Therefore, the convergence of the DI to the true solution $z^{(\infty)}$ can be performed using the Aitken's technique for accelerating the convergence, if 1 does not belong to the spectrum of $P_{t,\Gamma}$, as follows:

$$z_\Gamma^{(\infty)} = (I - P_{t,\Gamma})^{-1}(z_\Gamma^{(1)} + P_{t,\Gamma} z_\Gamma^{(0)}) \tag{5}$$

Numerically, the time derivative in Eq. (4) must be discretized using backward Euler scheme with a regular time step Δt for example. We write the DI with RAS splitting on the discretized system as:

$$\begin{cases} \tilde{A}_1^d x^{n+1,(k+1)} + \tilde{B}_1^d y^{n+1,(k+1)} = \tilde{b}^{n+1,d} + \tilde{A}_2^d x^{n+1,(k)} + \tilde{B}_2^d y^{n+1,(k)}, \\ C_1^a x^{n+1,(k+1)} + D_1^a y^{n+1,(k+1)} = b^{n+1,a} + C_2^a x^{n+1,(k)} + D_2^a y^{n+1,(k)}, \\ x^{0,(k+1)} = x_0. \end{cases} \tag{6}$$

with $\tilde{A}_1^d = I_{n,1}^d + \Delta t A_1^d, \tilde{B}_1^d = \Delta t B_1^d, \tilde{A}_2^d = \Delta t A_2^d, \tilde{B}_2^d = \Delta t B_2^d, \tilde{b}^{n+1,d} = x^{n,*} + \Delta t b^{n+1,d}$ where $x^{n,*} = x^{n,(k+1)}$ or $x^{n,*} = x^{n,(\infty)}$ leading to the sequential DI or pipelined DI strategies. Locally, it is written, with $x_i^{0,(k+1)} = R_i^{p,d} x_0$:

$$\underbrace{\begin{pmatrix} x_i^{n+1,(k+1)} \\ y_i^{n+1,(k+1)} \end{pmatrix}}_{z_i^{n+1,(k+1)}} = \underbrace{\begin{pmatrix} \tilde{A}_i & \tilde{B}_i \\ C_i & D_i \end{pmatrix}^{-1}}_{\tilde{\mathbb{A}}_i^{-1}} \left(\underbrace{\begin{pmatrix} \tilde{b}_{i,d}^{n+1} \\ b_{i,a}^{n+1} \end{pmatrix}}_{\tilde{b}_i^{n+1}} - \underbrace{\begin{pmatrix} \tilde{E}_{i,d}^d & \tilde{E}_{i,d}^a \\ E_{i,a}^d & E_{i,a}^a \end{pmatrix}}_{\tilde{\mathbb{E}}_i} \underbrace{\begin{pmatrix} x_{i,e}^{n+1,(k)} \\ y_{i,e}^{n+1,(k)} \end{pmatrix}}_{z_{i,e}^{n+1,(k)}} \right) \tag{7}$$

The choice for the term $x^{n,*}$ has an impact on the implementation and the Aitken's acceleration of convergence technique as described in the next section.

3 Pipelined time stepping strategy for DI

Let's consider the DI applied over a time interval $[t^0, t^F]$ with a constant time step Δt satisfying $t^F - t^0 = \Xi \Delta t$ with $\Xi \in \mathbb{N}^*$.

The sequential DI strategy consists in iterating the DI method until convergence on one time step before applying it to the next time step (Algorithm 1, $x^{n,*} = x^{n,(\infty)}$ in (6)). In the pipelined DI strategy, each DI iteration is performed over several time steps, these iterations are repeated until convergence (Algorithm 2, $x^{n,*} = x^{n,(k)}$ in (6)). The two algorithms differ by the choice of $x^{n,*} = x^{n,(k)}$ but also by the inversion of the order of loops 1 and 2.

In the following, we adapt the Aitken's acceleration of the convergence technique to accelerate the pipelined DI with RAS splitting.

Algorithm 1 Sequential DI strategy

1: **for** $n = 0 \dots \Xi - 1$ **do**
2: **for** $k = 1 \dots$ until convergence **do**
3: Solve $z_i^{n+1,(k+1)} = \tilde{\mathbb{A}}_i^{-1}\tilde{b}_i^{n+1} - \tilde{\mathbb{B}}_i z_{i,e}^{n+1,(k)}$.
4: **end for**
5: **end for**

Algorithm 2 Pipelined DI strategy

1: **for** $k = 1 \dots$ until convergence **do**
2: **for** $n = 0 \dots \Xi - 1$ **do**
3: Solve $z_i^{n+1,(k+1)} = \tilde{\mathbb{A}}_i^{-1}\tilde{b}_i^{n+1} - \tilde{\mathbb{B}}_i z_{i,e}^{n+1,(k)}$.
4: **end for**
5: **end for**

Definition 2 We note $Z_i^{(k)} \in \mathbb{C}^{\Xi n}$ the $(k)^{th}$ DI iteration corresponding to the concatenation over the Ξ time steps of the i^{th} partition W_i^p of the $(k+1)^{th}$ pipelined DI iteration: $Z_i^{(k)} = ((z_i^{1,(k)})^T, \dots, (z_i^{\Xi,(k)})^T)^T$, and the dependencies as $Z_{i,e}^{(k)} = ((z_{i,e}^{1,(k)})^T, \dots, (z_{i,e}^{\Xi,(k)})^T)^T$.

We define the operator $\mathbb{I}_{d,i}$ such that: $\mathbb{I}_{d,i} z_i^{n,(k)} = \begin{pmatrix} \Delta t\, x_i^{n,(k)} \\ 0_{n_{i,a}} \end{pmatrix}$. We also define $Z_\Gamma^{(k)} \in \mathbb{C}^{\Xi n_\Gamma}$ denote the k^{th} pipelined DI iterations of the global interface values of the Ξ time steps: $Z_\Gamma^{(k)} = ((z_\Gamma^{1,(k)})^T, \dots, (z_\Gamma^{\Xi,(k)})^T)^T$ and let $\mathbb{I}_d$ be the operator that follows: $\mathbb{I}_d = (\mathbb{I}_{d,1}^T, \dots, \mathbb{I}_{d,\Xi}^T)^T$.

Proposition 1 *The k^{th} pipelined DI iteration applied on to Ξ time steps Δt is written locally on the partition W_i^p :*

$$\begin{pmatrix} \tilde{\mathbb{A}}_i & & & \\ -\mathbb{I}_{d,i}\ \tilde{\mathbb{A}}_i & & & \\ & \ddots & \ddots & \\ & & -\mathbb{I}_{d,i}\ \tilde{\mathbb{A}}_i \end{pmatrix} Z_i^{(k)} = \begin{pmatrix} b_i^1 + \mathbb{I}_{d,i} z_i^0 \\ b_i^2 \\ \vdots \\ b_i^\Xi \end{pmatrix} - \begin{pmatrix} \tilde{\mathbb{E}}_i & & \\ & \tilde{\mathbb{E}}_i & \\ & & \ddots \\ & & \tilde{\mathbb{E}}_i \end{pmatrix} Z_{i,e}^{(k-1)} \tag{8}$$

Equation (8) has the same form (and the same properties) than the sequential DI. We can then apply the same methodology as in the sequential case. That is to say: as the convergence is purely linear one can thus build an operator of error $\mathbb{P}_\Gamma$ and use the Aitken acceleration of the convergence method. $\mathbb{P}_\Gamma$ can be computed algebraically or numerically. It will be of size $\Xi n_\Gamma \times \Xi n_\Gamma$ and we will need $\Xi n_\Gamma + 1$ RAS iterations to calculate it numerically. Nevertheless, we can take advantage of the structure of $\mathbb{P}_\Gamma$ for linear DAE with regular time stepping as the $\mathbb{P}_\Gamma$ operator and the P_Γ operator are linked as shown below.

By noting $M_{n,RAS}^{-1}$ the RAS operator (defined as in [3]) and $P_{n,\Gamma}$ the error operator associated to the n^{th} time step. Then, similarly to the sequential DI, we can restrict the pipelined DI iteration to the global interface of (5) of all the Ξ time steps:

Proposition 2 *The k^{th} iteration of the pipelined DI can be written on the global interface of the Ξ time steps:*

$$Z_\Gamma^{(k)} = \begin{pmatrix} I & & & \\ M_2^{-1} & I & & \\ & \ddots & \ddots & \\ & & M_\Xi^{-1} & I \end{pmatrix}^{-1} \begin{pmatrix} P_{1,\Gamma} & & & \\ & P_{2,\Gamma} & & \\ & & \ddots & \\ & & & P_{\Xi,\Gamma} \end{pmatrix} Z_\Gamma^{(k-1)} + \begin{pmatrix} R_\Gamma M_{1,RAS}^{-1} \mathbb{I}_d z^0 + c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad (9)$$

where $M_i^{-1} = R_\Gamma M_{i,RAS}^{-1} \mathbb{I}_d R_\Gamma^T$, $i = 2 \ldots \Xi$

The error operator can be calculated in two ways algebraically or numerically. We recall that the global interface Γ is defined as the concatenation of $W_{i,e}^p$, that is $\Gamma = \left\{ W_{0,e}^p, \ldots, W_{N-1,e}^p \right\}$ of size $n_\Gamma = \sum_{i=0}^{N-1} n_{i,e}$. It is pointed out that to numerically calculate the error operator, it is necessary to perform one more iteration than the size of the vector to be accelerated.

In the sequential DI strategy, we can apply the Aitken's technique for accelerating convergence, after $n_\Gamma + 1$ DI iterations for the first regular time step, in order to numerically build the P_Γ operator. Then, if we use the same time step size for the following time steps, and if there is no non-linearity and no change in the topology, we can perform the Aitken's convergence acceleration technique after one DI iteration. However, it is necessary to recalculate the error operator P_Γ at each change of topology or at each change of time step. Indeed the matrices $\mathbb{A}$ and $\mathbb{E}$ which have an impact in the P_Γ are modified by the changes in topology and changes in time step size (because of the discretization).

Moreover, in the pipelined DI strategy, the interface to be accelerated is the concatenation of the interfaces over the entire period of interest ($\Xi \Delta t$ here). This interface is of size $\Xi \times n_\Gamma$, it will therefore take $\Xi \times n_\Gamma + 1$ DI iterations in order to compute the error operator $\mathbb{P}_\Gamma$, whether or not there is a change in topology or time step. On the other hand, changes in the size of the time steps must be planned before launching the simulation.

Table 1 summarizes the number of iterations needed to simulate the problem on the period $\Xi \Delta t$, depending on the strategy chosen.

Table 1 Number of iterations needed to performed the simulation using the DI method accelerated with the Aitken acceleration technique

DI Strategy	Sequential	First time step sequential, computation of the error operator then pipelined	Pipelined
no non-linearity& fixed and equidistant time steps	$\Xi + n_\Gamma$	$2(\Xi - 1) + n_\Gamma + 1$	$\Xi \times n_\Gamma + 1$
fixed variable time step distribution with j changes in time step length	$\Xi - j + j \times (n_\Gamma + 1)$	$2(\Xi - j) + j \times (n_\Gamma + 1)$	$\Xi \times n_\Gamma + 1$
non-fixed variable time step distribution with j changes in time step length	$\Xi - j + j \times (n_\Gamma + 1)$	Strategy non valid	Strategy non valid
presence of non-linear components	$\Xi \times (n_\Gamma + 1)$	Strategy non valid	$\Xi \times n_\Gamma + 1$
j non-linearity events	$\Xi - j + j \times (n_\Gamma + 1)$	$2(\Xi - j) + j \times (n_\Gamma + 1)$	$\Xi \times n_\Gamma + 1$

4 Numerical results

The numerical results are performed on the test circuit of Pade and Tischendorf [14], with the sequential strategy we know from the spectrum of the error operator that the convergence of the method depends on the value of the components but especially on the size of the time step Δt. The size of the problem is 9 and the global interface Γ size is 3. We refer to [2] for parallel results approximating P_Γ by using singular values decomposition of the RAS interface solution iterates and the Aitken's acceleration for solving large-scale elliptic Darcy flows 3D problems. Figure 1 (left) shows the evolution of the spectral radius of $\mathbb{P}_\Gamma$ computed numerically with respect to the number of regular time steps in the pipeline. It shows that the convergence of the pipelined DI with RAS splitting deteriorates with increasing number of pipelined time steps. This result is corroborated by Figure 1 (middle) which shows the error of the pipelined DI with RAS splitting between two consecutive iterations, with respect to the RAS iterations, for each time step in the pipeline. Although the first few time steps in the pipeline the DI converge, this is not the case for the last few time steps. This shows the limitation of the pipeline size. Nevertheless, with Aitken's convergence acceleration, we can still accelerate the DI for all time steps. We should note that we also have a limitation on the pipeline size due to numerical problems in the numerical computation of the error operator if the DI diverges too strongly for some time steps. Nevertheless, in the pipelined strategy, the numerically calculated error operator can take into account some changes in the behavior of the electrical components, which allows us to apply Aitken's convergence acceleration technique even in the presence of nonlinear components, as shown in the figure (right) with a nonlinear resistor in the test circuit [14].

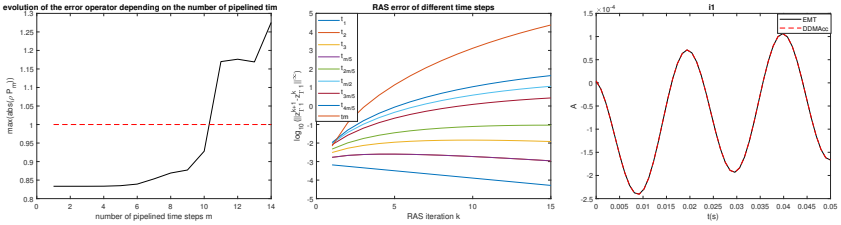

Fig. 1 (left) Evolution of the spectral radius of the error operator depending on the number of pipelined regular time steps of size $\Delta t = 1.1\ 10^{-3}$ (left, $\Xi = 14$), (middle) DI with the RAS splitting convergence behavior ($\log_{10}(||z^{(2k)} - z_{\text{ref}}||_\infty)$) on each of the pipelined time steps with respect to the iterations, (right) Comparison between the DI with the RAS splitting with the Aitken's technique for accelerating convergence and the DAE monolithic reference with a non-linear component, with $\Delta t = 1, 1.10^{-4}$, $\Xi = 100$.

5 Conclusion

Starting from the RAS method applied to a state-space DAE system, we show that this method is a dynamic iteration method with a specific operator splitting.

Then, the DI with RAS splitting inherits the property of reducing the size of the error operator to the size of a global interface problem. It also inherits its pure linear convergence/divergence when applied to a linear problem. We are then able to accelerate the convergence by Aitken acceleration by working on the global interface, thus, we get rid of the contracting constraint of the error operator. Writing RAS as a DI with RAS splitting also makes us consider the implementation of the pipelined strategy performing iteration over several time steps. We have shown the link between the error operators of the sequential DI and pipelined DI strategies, which allows us to apply Aitken's convergence acceleration technique on the pipelined DI. The optimal use cases for these strategies were also discussed. Numerical results show that pipelined DI with RAS splitting successfully applies Aitken acceleration to both contracting and non-contracting DI. It also opens up the Aitken acceleration of DI convergence for nonlinear problems (using a different linearization of the nonlinear problem on the time steps in the pipeline) and for pipelined steps with different sizes.

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