On overlapping domain decomposition methods for high-contrast multiscale problems

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1 Summary

We review some important ideas in the design and analysis of robust overlapping domain decomposition algorithms for high-contrast multiscale problems. In recent years, there have been many contributions to the application of different domain decomposition methodologies to solve high-contrast multiscale problems. We mention two- and multi-levels methods, additive and additive average methods, iterative substructuring and non-overlapping methods and many others. See [11]. Due to page limitation, we focus only on two-levels overlapping methods developed by some of the authors that use a coarse-grid for the construction of the second level. We also propose a domain decomposition method with better performance in terms of the number of iterations. The main novelty of our approaches is the construction of coarse spaces, which are computed using spectral information of local bilinear forms. We present several approaches to incorporate the spectral information into the coarse problem in order to obtain minimal (locally constructed) coarse space dimension. We show that using these coarse spaces, we can obtain a domain decomposition preconditioner with the condition number independent of contrast and small scales. To minimize further the number of iterations until convergence, we use this minimal dimensional coarse spaces in a construction combining them with large overlap local problems that take advantage of the possibility of localizing global fields orthogonal to the coarse space. We obtain a condition number close to 1 for the new method. We discuss possible drawbacks and further extensions.

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2 High-contrast problems. Introduction

The methods and algorithms, discussed in the paper, can be applied to various PDEs, even though we will focus on Darcy flow equations. Given $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$, and $g : \partial D \to \mathbb{R}$, find $u : D \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial u}{\partial x_j}\right) = f$$

with a suitable boundary condition, for instance $u = 0$ on ∂D. The coefficient $\kappa_{ij}(x) = \kappa(x)\delta_{ij}$ represents the permeability of the porous media D. We focus on two-levels overlapping domain decomposition and use local spectral information in constructing "minimal" dimensional coarse spaces (MDCS) within this setting. After some review on constructing MDCS and their use in overlapping domain decomposition preconditioners, we present an approach, which uses MDCS to minimize the condition number to a condition number closer to 1. This approach requires a large overlap (when comparted to coarse-grid size) and, thus, is more efficient for small size coarse grids. We present the numerical results and state our main theoretical result. We assume that there exists $\kappa_{\min}$ and $\kappa_{\max}$ with $0 < \kappa_{\min} \leq \kappa(x) \leq \kappa_{\max}$ for all $x \in D$. **The coefficient κ has a multiscale structure** (significant local variations of κ occur across D at different scales). We also assume that the coefficient κ is a high-contrast coefficient (the constrast is $\eta = \kappa_{\max}/\kappa_{\min}$). We assume that η is large compared to the coarse-grid size.

It is well known that performance of numerical methods for high-contrast multiscale problems depends on η and local variations of κ across D. For classical finite element methods, the condition to obtain good approximation results is that the finite element mesh has to be fine enough to resolve the variations of the coefficient κ. Under these conditions, finite element approximation leads to the solution of very large (sparse) ill-conditioned problems (with the condition number scaling with h^{-2} and η). Therefore, the performance of solvers depends on η and local variations of κ across D. This was observed in several works, e.g., [8, 10, 1][1].

Let $\mathcal{T}^h$ be a triangulation of the domain D, where h is the size of typical element. We consider only the case of discretization by the classical finite element method $V = P_1(\mathcal{T}^h)$ of piecewise (bi)linear functions. Other discretizations can also be considered. The application of the finite element discretization leads to the solution of a very large ill-conditioned system $Ax = b$, where A is roughly of size h^{-2} and the condition number of A scales with η and h^{-2}. In general, the main goal is to obtain an efficient good approximation of solution u. The two main solution strategies are:

1. Choose h sufficiently small and implement an iterative method. It is important to implement a preconditioner M^{-1} to solve $M^{-1}Au = M^{-1}b$. Then, it is important to have the condition number of $M^{-1}A$ to be small and bounded independently of physical parameters, e.g., η and the multiscale structure of κ.

[1] Due to the page limitation, only a few references are cited throughout.

2. Solve a smaller dimensional linear system ($\mathcal{T}^H$ with $H > h$) so that computations of solutions can be done efficiently[2]. This usually involves the construction of a downscaling operator R_0 (from the coarse-scale to fine-scale $v_0 \mapsto v$) and an upscaling operator (from fine-scale to coarse-scale, $v \mapsto v_0$) (or similar operators). Using these operators, the linear system $Au = b$ becomes a coarse linear system $A_0 u_0 = b_0$ so that $R_0 u_0$ or functionals of it can be computed. The main goal of this approach it to obtain a sub-grid capturing such that $||u - R_0 u_0||$ is small.

The rest of the paper will focus on the design of overlapping domain decomposition methods by constructing appropriate coarse spaces. First, we will review existing results, which construct minimal dimensional coarse spaces, such that the condition number of resulting preconditioner is independent of η. These coarse spaces use local spectral problems to extract the information, which cannot be localized. This information is related to high-conductivity channels, which connect coarse-grid boundaries and it is important for the performance of domain decomposition preconditioners and multiscale simulations. Next, using these and oversampling ideas, we present a "hybrid" domain decomposition approach with a condition number close to 1 by appropriately selecting the oversampling size (i.e., overlapping size). We state our main result, discuss some limitations and show a numerical example. We compare the results to some existing contrast-independent preconditioners.

3 Classical overlapping methods. Brief review

We start with a non-overlapping decomposition $\{D_i\}_{i=1}^{N_S}$ of the domain D and obtain an overlapping decomposition $\{D_i'\}_{i=1}^{N_S}$ by adding a layer of width δ around each non-overlapping subdomain. Let A_j be the Dirichlet matrix corresponding to the overlapping subdomain D_j'. The one level method solves $M_1^{-1}A = M_1^{-1}b$ with $M_1^{-1} = \sum_{j=1}^{N_S} R_j (A_j)^{-1} R_j^T$ and the operators R_j^T, $j = 1, \ldots, N_S$, being the restriction to overlapping subdomain D_j' operator and with the R_j being the extension by zero (outside D_j') operator. We have the bound $\mathrm{Cond}(M_1^{-1}A) \leq C\,(1 + 1/\delta H)$. For high-contrast multiscale problems, it is known that $C \asymp \eta$.

Next, we introduce a coarse space, that is, a subspace $V_0 \subset V$ of small dimension (when compared to the fine-grid finite element space V). We consider A_0 as the matrix form of the discretization of the equation related to subspace V_0. For simplicity of the presentation, let A_0 be the Galerkin projection of A on the subspace V_0. That is $A_0 = R_0 A R_0^T$, where R_0 is a downscaling operator that converts coarse-space coordinates into fine-grid space coordinates. The two-levels preconditioner uses the coarse space and it is defined by $M_2^{-1} = R_0 A_0^{-1} R_0^T + \sum_{j=1}^{N_S} R_j (A_j)^{-1} R_j^T = R_0 A_0^{-1} R_0^T + M_1^{-1}$. It is known that $\mathrm{Cond}(M^{-1}A) \preceq \eta\,(1 + H/\delta)$. The classical two-levels method is robust with respect to the number of subdomains but it is not robust with respect to η.

[2] The coarse mesh does not necessarily resolve all the variations of κ.

The condition number estimates use a Poincaré inequality and small overlap trick; [13]. Without the small overlap trick $\text{Cond}(M^{-1}A) \preceq \eta(1 + H^2/\delta^2)$.

There were several works addressing the performance of classical domain decomposition algorithms for high-contrast problems. Many of these works considered simplified multiscale structures[3], see e.g., [13] for some works by O. Widlund and his collaborators. We also mention the works by Sarkis and his collaborators, where they introduce the assumption of quasi-monotonicity [4]. Sarkis also introduced the idea of using "extra" or additional basis functions as well as techniques that construct the coarse spaces using the overlapping decomposition (and not related to a coarse mesh); [12]. Scheichl and Graham [10] and Hou and Aarnes [1], started a systematic study of the performance of classical overlaping domain decomposition methods for high-contrast problems. In their works, they used coarse spaces constructed using a coarse grid and special basis functions from the family of multiscale finite element methods. These authors designed two-levels domain decomposition methods that were robust (with respect to η) for special multiscale structures. None of the results available in the literature (before the method in papers [8, 9] was introduced) were robust for a coefficient not-aligned with the construction of the coarse space (i.e., not aligned either with the non-overlapping decomposion or the coarse mesh if any), i.e., the condition number of the resulting preconditioner is independent of η for general multiscale coefficients.

4 Stable decomposition and eigenvalue problem. Review

A main tool in obtaining condition number bounds is the construction of a stable decomposition of a global field. That is, if for all $v \in V = P_0^1(D, \mathcal{T}^h)$ there exists a decomposition $v = v_0 + \sum_{j=1}^{N_S} v_j$ with $v_0 \in V_0$ and $v_j \in V_j = P_0^1(D'_j, \mathcal{T}^h)$, $j = 1, \ldots, N$, and

$$\int_D \kappa |\nabla v_0|^2 + \sum_{j=1}^{N_S} \int_{D'_j} \kappa |\nabla v_j|^2 \leq C_0^2 \int_D \kappa |\nabla v|^2$$

for $C_0 > 0$. Then, $\text{cond}(M_2^{-1}A) \leq c(\mathcal{T}^h, \mathcal{T}^H) C_0^2$. Existence of a suitable coarse interpolation $I_0 : V \to V_0 = \text{span}\{\Phi\}$ implies the stable decomposition above. Usually such stable decomposition is constructed as follows.

For the coarse part of the stable decomposition, we introduce a partition of unity $\{\chi_i\}$ subordinated to the coarse mesh (supp $\chi_i \subset \omega_i$ where ω_i is the coarse-block neighborhood of the coarse-node x_i). We begin by restricting the global field v to ω_i. For each coarse node neighborhood ω_i, we identify local field that will contribute to the coarse space $I_0^{\omega_i} v$ so that the coarse space will be defined as $V_0 = \text{Span}\{\chi_i I_0^{\omega_i} v\}$. In classical methods $I_0^{\omega_i} v$ is the average of v in ω_i. Later we present some more general examples for $I_0^{\omega_i}$. We

[3] These works usually assume some alignment between the coefficient heterogeneities and the initial non-overlapping decomposition.

assemble a coarse field as $v_0 = I_0 v = \sum_{i=1}^{N_S} \chi_i (I_0^{\omega_i} v)$. Note that in each block $v - v_0 = \sum_{x_i \in K} \chi_i (v - I_0^{\omega_i} v)$.

For the local parts of the stable decomposition, we introduce a partition of unity $\{\xi_j\}$ subordinated to the non-overlapping decomposition (supp $\xi_j \subset D'_j$). The local part of the stable decomposition is defined by $v_j = \xi_j (v - v_0)$. For instance, to bound the energy of v_j, we have in each coarse-block K,

$$\int_K \kappa |\nabla v_j|^2 \preceq \int_K \kappa |\nabla \xi_j \left(\sum_{x_i \in K} \chi_i (v - I_0^{\omega_i} v) \right)|^2$$
$$\preceq \sum_{i \in K} \int_K \kappa (\xi_j \chi_i)^2 |\nabla (v - I_0^{\omega_i} v)|^2 + \sum_{x_i \in K} \int_K \kappa |\nabla (\xi_j \chi_i)|^2 |v - I_0^{\omega_i} v|^2.$$

Adding up over K, we obtain,

$$\int_{D'_j} \kappa |\nabla v_j|^2 \preceq \sum_{x_i \in \overline{D}'_j} \int_{D'_j} \kappa (\xi_j \chi_i)^2 |\nabla (v - I_0^{\omega_i} v)|^2$$
$$+ \sum_{x_i \in \overline{\omega}_j} \int_{D'_j} \kappa |\nabla (\xi_j \chi_i)|^2 |v - I_0^{\omega_i} v|^2$$

and we would like to bound the last term by $C \int_{D'_j} \kappa |\nabla v|^2$.

For simplicity of our presentation, we consider the case when the coarse elements coincide with the non-overlapping decomposition subdomains. That is, $D'_j = \omega_j$. In this case, we can replace ξ by χ and replace $\nabla(\chi^2)$ by $\nabla \chi$ so that we need to bound $\sum_{x_i \in \overline{\omega}_j} \int_{\omega_j} \kappa |\nabla \chi_i|^2 |v - I_0^{\omega_i} v|^2$. We refer to this design as **coarse-grid based**.

Remark 1 (General case and overlapping decomposition based design). Similar analysis holds in the case when there is no coarse-grid and the coarse space is spanned by a partition of unity $\{\xi_j\}$. We can replace χ by ξ and $\nabla(\xi^2)$ by $\nabla \xi$. In general these two partitions are not related (see Sec. 4.1).

We now review the three main arguments to complete the required bound: 1) A Poincaré inequality. 2) L^∞ estimates. 3) Eigenvalue problem.

1. A Poincaré inequality: Classical analysis uses a Poincaré inequality to obtain the required bound above. That is, the inequality $\frac{1}{H^2} \int_\omega (v - \bar{v})^2 \leq C \int_\omega |\nabla v|^2$ to obtain $\sum_{x_i \in \omega_j} \int_{\omega_j} \kappa |\nabla \chi_i|^2 |v - I_0^{\omega_i} v|^2 \preceq \frac{1}{H^2} \int_{\omega_i} \kappa |v - I_0^{\omega_i} v|^2 \preceq C \int_{\omega_i} \kappa |\nabla v|^2$. In this case, $I_0^{\omega_i} v$ is the average of v on the subdomain. For the case of high-contrast coefficients, C depends on η, in general. For quasi-monotone coefficient it can be obtained that C is independent of the contrast [4]. We also mention [8] for the case **locally connected high-contrast region**. In this case $I_0^{\omega_i} v$ is a weighted average. From the argument given in [8], it was clear that when the high-contrast regions break across the domain,

defining only one average was not enough to obtain contrast independent constant in the Poincaré inequality.

2. L^∞ **estimates:** Another idea is to use an L^∞ estimate of the form

$$\sum_{x_i \in K} \int_{\omega_i} \kappa |\nabla \chi_i|^2 |v - I_0^{\omega_i} v|^2 \preceq \sum_{x_i \in K} ||\kappa |\nabla \chi_i|^2||_\infty \int_{\omega_i} |v - I_0^{\omega_i} v|^2.$$

The idea in [10, 1] was then to construct a partition of unity such that $||\kappa|\nabla\chi_i|^2||_\infty$ is bounded independently of the contrast and then to use classical Poincaré inequality estimates. Instead of minimizing the L^∞, one can intuitively try to minimize $\int_K \kappa |\nabla \chi_i|^2$. This works well when the multiscale structure of the coefficient is confined within the coarse blocks. For instance, for a coefficient and coarse-grid as depicted in Figure 1 (left picture), we have that a two-level domain decomposition method can be proven to be robust with respect the value of the coefficient inside the inclusions. In fact, the coarse space spanned by classical multiscale basis functions with linear boundary conditions ($-\text{div}(\kappa \nabla \chi_i) = 0$ in K and linear on each edge of ∂K) is sufficient and the above proof works. Now consider the coefficient in Figure 1 (center picture). For such cases, the boundary condition of the basis functions is important. In these cases, basis functions can be constructed such that the above argument can be carried on. Here, we can use multiscale basis functions with oscillatory boundary condition in its construction[4].

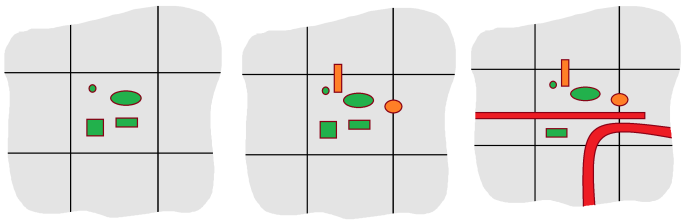

Fig. 1 Examples o multiscale coefficients with interior high-contrast inclusions (left), boundary inclusions (center) and long channels(right).

For the coefficient in Figure 1, right figure, the argument above using L^∞ cannot be carried out unless we can work with larger support basis functions (as large as to include the high-contrast channels of the coefficient). If the support of the coarse basis function does not include the high-contrast region, then $||\kappa|\nabla\chi_i|^2||_\infty$ increases with the contrast leading to non-robust two-level domain decomposition methods.

[4] We can include constructions of boundary conditions using $1D$ solution of the problem along the edges. Other choices include basis functions constructed using oversampling regions, energy minimizing partition of unity (global), constructions using limited global information (global), etc.

3. Eigenvalue problem. We can write $\sum_{x_j \in \omega_i} \int_{\omega_i} \kappa |\nabla \chi_j|^2 |v - I_0^{\omega_i} v|^2 \preceq \frac{1}{H^2} \int_{\omega_i} \kappa |(v - I_0^{\omega_i} v)|^2 \preceq C \int_{\omega_i} \kappa |\nabla v|^2$, where we need to justify the last inequality with constant independent of the contrast. The idea is then to consider the Rayleigh quotient, $\mathcal{Q}(v) := \frac{\int_{\omega_i} \kappa |\nabla v|^2}{\int_{\omega_i} \kappa |v|^2}$ with $v \in P^1(\omega_i)$. This quotient is related to an eigenvalue problem and we can define $I_0^{\omega_i} v$ to be the projection on low modes of this quotient on ω_i. The associated eigenproblem is given by $-\text{div}(k(x)\nabla \psi_\ell) = \lambda_\ell k(x) \psi_i$ in ω_i with homogeneous Neumann boundary condition for floating subdomains and a mixed homogeneous Neumann-Dirichlet condition for subdomains that touch the boundary. It turns out that the low part of the spectrum can be written as $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_L < \lambda_{L+1} \leq ...$ where $\lambda_1, ..., \lambda_L$ are small, asymptotically vanishing eigenvalues and λ_L can be bounded below independently of the contrast. After identifying the local field $I_0^{\omega_i} v$, we then define the coarse space as $V_0 = Span\{I^h \chi_i \psi_j^{\omega_i}\} = Span\{\Phi_i\}$.

Eigenvalue problem with a multiscale partition of unity. Instead of the argument presented earlier, we can include the gradient of the partition of unity in the bounds (somehow similar to the ideas of L^∞ bounds). We then need the following chain of inequalities, $\int_{\omega_i} \underbrace{\left(\sum_{x_j \in \omega_i} \kappa |\nabla \chi_j|^2 \right)}_{:= H^{-2} \widetilde{\kappa}} |v - I_0^{\omega_i} v|^2 = \frac{1}{H^2} \int_{\omega_i} \widetilde{\kappa} |v - I_0^{\omega_i} v)|^2 \preceq \int_{\omega_i} \kappa |\nabla v|^2$. Here we have to consider Rayleigh quotient $\mathcal{Q}_{ms}(v) := \frac{\int_{\omega_i} \kappa |\nabla v|^2}{\int_{\omega_i} \widetilde{\kappa} |v|^2}$, $v \in P^1(\omega_i)$ and define $I_0^{\omega_i} v$ as projection on low modes. Additional modes "complement" the initial space spanned by the partition of unity used so that the resulting coarse space leads to robust methods with minimal dimension coarse spaces; [9].

If we consider the two-level method with the (multiscale) spectral coarse space presented before, then

$$\text{cond}(M^{-1} A) \leq C(1 + (H/\delta)^2), \tag{1}$$

where C is independent of the contrast if enough eigenfunctions in each node neighborhood are selected for the construction of the coarse spaces. The constant C and the resulting coarse-space dimension depend on the partition of unity (initial coarse-grid representation) used.

4.1 Abstract problem eigenvalue problems

We consider an abstract variational problem, where the global bilinear form is obtained by assembling local bilinear forms. That is $a(u,v) = \sum_K a_K(R_K u, R_K v)$, where $a_K(u,v)$ is a bilinear form acting on functions with supports being the coarse block K. Define the subdomain bilinear form $a_{\omega_i}(u,v) = \sum_{K\subset\omega_i} a_K(u,v)$. We consider the abstract problem

$$a(u,v) = F(v) \quad \text{for all } v \in V.$$

We introduce $\{\chi_j\}$, a partition of unity subordinated to coarse-mesh blocks and $\{\xi_i\}$ a partition of unity subordinated to overlapping decomposition (not necessarily related in this subsection). We also define the "Mass" bilinear form (or energy of cut-off) m_{ω_i} and the Rayleigh quotient $\mathcal{Q}_{abs}$ by

$$m_{\omega_i}(v,v) := \sum_{j\in\omega_i} a(\xi_i\chi_j v, \xi_i\chi_j v) \quad \text{and} \quad \mathcal{Q}_{abs}(v) := \frac{a_{\omega_i}(v,v)}{m_{\omega_i}(v,v)}.$$

For the Darcy problem, we have $m_{\omega_i}(v,v) = \sum_{j\in\omega_i}\int_{\omega_i}\kappa|\nabla(\xi_i\chi_j v)|^2 \preceq \int_{\omega_i}\widetilde{\kappa}|v|^2$. The same analysis can be done by replacing the partition of unity functions by partition of degree of freedom (PDoF). Let $\{\boldsymbol{\chi}_j\}$ be PDoF subordianted to coarse mesh neighborhood and $\{\boldsymbol{\xi}_i\}$ be PDoF subordianted to overlapping decomposition. We define the cut-off bilinear form and quotient,

$$m_{\omega_i}(v,v) := \sum_{j\in\omega_i} a(\boldsymbol{\xi}_i\boldsymbol{\chi}_j v, \boldsymbol{\xi}_i\boldsymbol{\chi}_j v) \quad \text{and} \quad \mathcal{Q}_{abs2}(v) := \frac{a_{\omega_i}(v,v)}{m_{\omega_i}(v,v)}.$$

The previous construction alows applying the same design recursively and therefore to use the same ideas in a multilevel method. See [6, 7].

4.2 Generalized Multiscale Finite Element Method (GMsFEM) eigenvalue problem

We can consider the Rayleigh quotients presented before only in a suitable subspace that allows a good approximation of low modes. We call these subspace the snapshot spaces. Denote by W_i the snapshot space corresponding to subdomain ω_i, then we consider the Rayleigh quotient, $\mathcal{Q}_{gm}(v) := \dfrac{a_{\omega_i}(v,v)}{m_{\omega_i}(v,v)}$ with $v \in W_i$. The snapshot space can be obtained by dimension reduction techniques or similar computations. See [5, 2]. For example, we can consider the following simple example. In each subdomain ω_i, $i = 1,\ldots,N_S$:
(1) Generate forcing terms $f_1, f_2, \ldots, f_M$ randomly ($\int_{\omega_i} f_\ell = 0$);
(2) Compute the local solutions $-\mathrm{div}(\kappa\nabla u_\ell) = f_\ell$ with homogeneous Neumann boundary condition;

(3) Generate $W_i = \text{span}\{u_\ell\} \cup \{1\}$;
(4) Consider $\mathcal{Q}_{gm}$ with W_i in 3 and compute important modes.
In Table 1, we see the results of using the local eigenvalue problem versus using the GMsFEM eigenvalue problem.

η	MS	Full	8 rand.	15 rand
10^6	209	35	37	37
10^9	346	38	44	38

Table 1 PCG iterations for different values η. Here $H = 1/10$ with $h = 1/200$. We use the GMsFEM eigevalue problem with $W_i = V_i$ (full local fine-grid space), column 2; W_i spanned by 8 random samples, column 4, and W_i spanned by 15 samples, column 5.

5 Constrained coarse spaces, large overlaps, and DD

In this section, we introduce a hybrid overlapping domain decomposition preconditioner. We use the coarse spaces constructed in [3], which rely on minimal dimensional coarse spaces as discussed above. First, we construct local auxiliary basis functions. For each coarse-block $K \in \mathcal{T}^H$, we solve the eigenvalue problem with Rayleigh quotient $\mathcal{Q}_{ms}(v) := \frac{\int_K \kappa |\nabla v|^2}{\int_K \widehat{\kappa} |v|^2}$, where $\widehat{\kappa} = \kappa \sum_j |\nabla \chi_j|^2$. We assume $\lambda_1^K \leq \lambda_2^K \leq \dots$ and define the local auxiliary spaces,

$$V_{aux}(K) = \text{span}\{\phi_j^K | 1 \leq j \leq L_K\} \text{ and } V_{aux} = \oplus_K V_{aux}(K).$$

Next, define a projection operator π_K as the orthogonal projection on V_{aux} with respect to the inner product $\int_K \widehat{\kappa} uv$ and $\pi_D = \oplus_K \pi_K$. Let K^+ be obtained by adding l layers of coarse elements to the coarse-block K. The coarse-grid multiscale basis $\psi_{j,ms}^K \in V(K^+) = P_0^1(K^+)$ solve

$$\int_{K^+} \kappa \nabla \psi_{j,ms}^K \nabla v + \int_{K^+} \widehat{\kappa} \pi_D(\psi_{j,ms}^K) \pi_D(v) = \int_{K^+} \widehat{\kappa} \phi_j^K \pi_D(v), \ \forall v \in V(K^+).$$

The coarse-grid multiscale space is defined as $V_{ms} = \text{span}\{\psi_{j,ms}^{(i)}\}$. We remark that this space is used as the global coarse solver in our preconditioner. More precisely, we define the (coarse solution) operator $A_{0,ms}^{-1} : H^{-1}(\widehat{\kappa}, D) \mapsto V_{ms}$ by,

$$\int_D \kappa \nabla A_{0,ms}^{-1}(u) \nabla v = u(v) \quad \text{for all } v \in V_{ms}$$

where $H^{-1}(\widehat{\kappa}, D)$ is the space of bounded linear functionals on the weighted sobolev space, $H^1(\kappa, D)$. In our preconditioner, we also need local solution operators which are the operators $A_{i,ms}^{-1} : H^{-1}(\widehat{\kappa}, D) \mapsto V(\omega_i^+)$ defined by,

$$\int_{\omega_i^+} \kappa \nabla A_{i,ms}^{-1}(u_i) \nabla v + \int_{\omega_i^+} \widehat{\kappa} \pi(A_i^{-1}(u_i)) \pi_D(v) = u_i(\chi_i v) \quad \text{for all } v \in V(\omega_i^+),$$

where ω_i^+ is obtained by enlarging ω_i by k coarse-grid layers. Next, we can define the preconditioner[5] M by

$$M^{-1} = (I - A_{0,ms}^{-1}A)\Big(\sum_i A_{i,ms}^{-1}\Big)(I - AA_{0,ms}^{-1}) + A_{0,ms}^{-1}.$$

Note that this is a hybrid preconditioner as defined in [13]. We remark that the constructions of the global coarse space and local solution operators are motivated by [3], where a new multiscale space is developed and analyzed, and it is shown to have a good convergence property independent of the scales of the coefficient of the PDE. In addition, the size of the local problem is dictated by an exponential decay property.

Using some estimates in [3], we can establish the following condition number estimate for $\mathrm{cond}(M^{-1}A)$,

$$\mathrm{cond}(M^{-1}A) \leq \frac{1 + C(1+\Lambda^{-1})^{\frac{1}{2}} E^{\frac{1}{2}} \max\{\tilde{\kappa}^{\frac{1}{2}}\}}{1 - C(1+\Lambda^{-1})^{\frac{1}{2}} E^{\frac{1}{2}} \max\{\tilde{\kappa}^{\frac{1}{2}}\}} \tag{2}$$

where $E = 3(1+\Lambda^{-1})\Big(1+(2(1+\Lambda^{-\frac{1}{2}}))^{-1}\Big)^{1-k}$, C is a constant that depends on the fine and coarse grid only and $\Lambda = \min_K \lambda_{L_K+1}^K$. See [3] for the required estimates of the coarse space. The analysis of the local solvers of the hybrid method above will be presented elsewhere[6]. We see that the condition number is close to 1 if sufficient number of basis functions is selected (i.e., Λ is not close to zero)[7]. The overlap size usually involves several coarse-grid block sizes and thus, the method is effective when the coarse-grid sizes are small. We comment that taking the generous overlap $\delta = kH/2$ in (1), we get the bound $C(1 + 4/k^2)$ with C independent of the contrast. The estimate (2), on the other hand, gives a bound close to 1 if the oversampling is sufficiently large (e.g., the number of coarse-grid layers is related to $\log(\eta)$), which is due to the localization of global fields orthogonal to the coarse space.

Next, we present a numerical result and consider a problem with permeability κ shown in Fig. 2. The fine-grid mesh size h and the coarse-grid mesh size are considered as $h = 1/200$ and $H = 1/20$. In Table 2, we present the number of iterations for using varying numbers of oversampling layers k, values of the contrast η and $\kappa_{M^{-1}A} - 1$, which is the condition number of the preconditioned matrix minus one. We observe that when $k = 3$, the condition number $\kappa_{M^{-1}A}$ is almost one, which confirms (2). In practice, one can choose smaller local problems with a corresponding increase in the number of iterations. This balance can be determined by practical needs.

We would like to emphasize that the proposed method has advantages if the coarse mesh size is not very coarse. In this case, the oversampled coarse regions are still sufficiently small and the coarse-grid solves can be relatively

[5] Here we avoid restriction and extension operators for simplicity of notation.

[6] We mention that the analysis does not use a stable decomposition so, in principle, a new family of robust methods can be obtained.

[7] Having a robust condition number close to 1 is important, especially in applications where the elliptic equation needs to be solved many times.

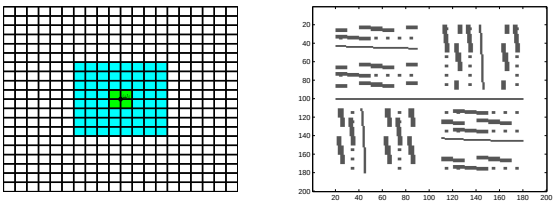

Fig. 2 Left: The coarse mesh used in the numerical experiments. We highlight a coarse neighborhood and the results of adding 3 coarse-block layers to it. Right: The permeability κ used in the experiments. The gray regions indicate high-permeability regions of order η while the white regions indicates a low (order 1) permeability.

# basis per ω	k	# iter	$\kappa_{M^{-1}A} - 1$
3	3	3	5.33e-04
3	4	2	2.57e-05
3	5	2	1.25e-06
3	6	1	5.50e-08

# basis per ω	η	# iter	$\kappa_{M^{-1}A} - 1$
3	1e+3	3	5.68e-04
3	1e+4	3	5.33e-04
3	1e+5	3	6.74e-04

Table 2 Condition number $\kappa_{M^{-1}A}$ and number of iterations until convergence for the PCG with $H = 1/20$, $h = 1/200$ and tol $= 1e-10$. Left: different number of oversampling layers k with $\eta = 1e+4$. Right: different values of the contrast η with $k = 3$.

expensive. Consequently, one wants to minimize the number of coarse-grid solves in addition to local solves. In general, the proposed approach can be used in a multi-level setup, in particular, at the finest levels, while at the coarsest level, we can use original spectral basis functions proposed in [8]. This is object of future research.

6 Conclusions

In this paper, we give an overview of domain decomposition preconditioners for high-contrast multiscale problems. In particular, we review the design of overlapping methods with an emphasis on the stable decomposition for the analysis of the method. We emphasize the use of minimal dimensional coarse spaces in order to construct optimal preconditioners with the condition number independent of physical scales (contrast and spatial scales). We discuss various approaches in this direction. Furthermore, using these spaces and oversampling ideas, we design a new preconditioner with a significant reduction in the number of iterations until convergence if oversampling regions are large enough (several coarse-grid blocks). We note that when using only minimal dimensional coarse spaces in additive Schwarz preconditioner with standard overlap size, we obtain around 19 iterations. in the new method, our main goal is to reduce even further the number of iterations due to large coarse problem sizes. We obtained around 3 iterations until convergence for the new approach. A main point of the new methodology is that after removing the channels we are able to localize the remaining multiscale information

via oversampling. Another interesting aspect of the new approach is that the bound can be obtained by estimating directly operator norms and do not require a stable decomposition.

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