INTERNODES for heterogeneous couplings

Paola Gervasio and Alfio Quarteroni

Abstract The INTERNODES (INTERpolation for NOnconforming DEcompositionS) method is an interpolation based approach to solve partial differential equations on non-conforming discretizations. In this paper we apply the INTERNODES method to different problems such as the Fluid Structure Interaction problem and the Stokes-Darcy coupled problem that models the filtration of fluids in porous media. Our results highlight the flexibility of the method as well as its optimal rate of convergence.

1 Introduction

The INTERNODES (INTERpolation for NOnconforming DEcompositionS) method is an interpolation based approach to solve partial differential equations on non-conforming discretizations [3, 9]. It is an alternative to projection-based methods like mortar [1], or other interpolation-based method like GFEM/XFEM [10]. Differently than in mortar methods, no cross-mass matrix involving basis functions living on different grids of the interface are required by INTERNODES to build the intergrid operators. Instead, two separate interface mass matrices (separately on either interface) are used. The substantial difference between GFEM/XFEM methods and INTERNODES consists in the fact that the former ones use a partition of unity to enrich the finite element space, while the latter does not add any shape function to those of the local finite element subspaces.

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In this paper we apply the INTERNODES method to different problems such as the Fluid Structure Interaction problem and the Stokes-Darcy coupled problem that models the filtration of fluids in porous media. Our results highlight the flexibility of the method as well as its optimal rate of convergence. Before addressing the two specific problems above, we introduce an abstract formulation for heterogeneous problems. This will also be useful to state the definition of the interface matching operators that will stand at the base of the INTERNODES method.

Let $\Omega \subset \mathbb{R}^d$, with $d = 2,3$, be an open domain with Lipschitz boundary $\partial\Omega$, Ω_1 and Ω_2 be two non-overlapping subdomains with Lipschitz boundary such that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ be their common interface.

Given a function f defined in Ω, we look for u_1 in Ω_1 and u_2 in Ω_2 such that

$$
\begin{aligned}
&L_k(u_k) = f && \text{in } \Omega_k, \quad k = 1,2, && (1)\\
&\Phi_2(u_2) = \Phi_1(u_1) && \text{on } \Gamma \quad \text{(Dirichlet-like condition)}, && (2)\\
&\Psi_1(u_1) + \Psi_2(u_2) = 0 && \text{on } \Gamma \quad \text{(Neumann-like condition)}, && (3)\\
&\text{boundary conditions} && \text{on } \partial\Omega, && (4)
\end{aligned}
$$

where L_1 and L_2 are two differential operators (that may also coincide) while, for $k = 1,2$, Φ_k and Ψ_k are suitable boundary operators restricted to the interface Γ, that depend upon the nature of the differential operators L_1 and L_2. More specifically, Neumann conditions refer here to natural conditions that are enforced *weakly*, whereas Dirichlet conditions identify those *essential* conditions that are enforced directly in the solution subspaces, via the suitable choice of trial functions (see, e.g., [13]). Typically for second order differential operators there is one Dirichlet-like condition and one Neumann-like condition, however more general situations are admissible.

Problem (1)–(4) provides an abstract setting for several kinds of differential problems; here we present two instances of (1)–(4) which the INTERNODES method is applied to.

2 Fluid Structure Interaction problem

When modeling the coupling between fluids and solids, the viscous incompressible Navier-Stokes equations are typically written in ALE (Arbitrarian Lagrangian Eulerian) coordinates in the fluid domain, whereas an elasticity model (either linear or nonlinear, depending on the type of structure) is solved in a reference frame; a third field, the so-called geometry problem, allows to determine the displacement of the fluid domain which defines, in turn, the ALE map, see, e.g., [14, 8, 5].

Let $\widehat{\Omega}_s$ and $\widehat{\Omega}_f$ be two non-overlapping reference configurations for the structure and fluid domains, respectively, and $\widehat{\Gamma} = \partial\widehat{\Omega}_s \cap \partial\widehat{\Omega}_f$ be the fluid-structure reference interface. We assume that the boundaries $\partial\widehat{\Omega}_k$, for $k = s, f$ are Lipschitz continuous and that $(\partial\widehat{\Omega}_k \setminus \widehat{\Gamma})$ is the union of two nonoverlapping subsets $\partial\widehat{\Omega}_k^N$ and $\partial\widehat{\Omega}_k^D$ on which Neumann and Dirichlet boundary conditions will be

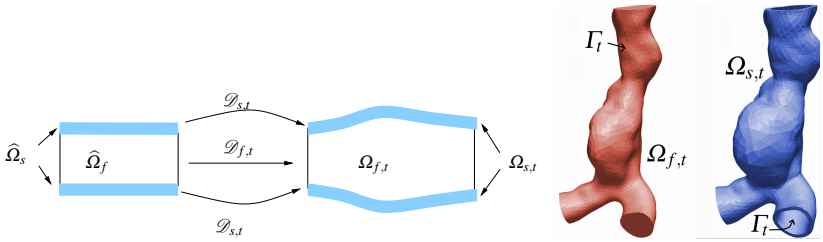

Fig. 1 At left: the ALE frame of reference. At right: the computational domains for the FSI problem: the fluid domain $\Omega_{f,t}$ and the structure domain $\Omega_{s,t}$. $\Gamma_t = \partial\Omega_{f,t} \cap \partial\Omega_{s,t}$

imposed, respectively. Then, for any $t \in (0,T)$ let $\Omega_{s,t}$ and $\Omega_{f,t}$ be the computational structure and fluid domains, respectively, such that $\Omega_{s,0} = \widehat{\Omega}_s$, $\Omega_{f,0} = \widehat{\Omega}_f$ and $\overline{\Omega_t} = \overline{\Omega_{s,t}} \cup \overline{\Omega_{f,t}}$. The current configurations $\Omega_{s,t}$ and $\Omega_{f,t}$ are defined as $\Omega_{k,t} = \{\mathbf{x} = \mathscr{D}_{k,t}(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}} + \widehat{\mathbf{d}}_k(\widehat{\mathbf{x}},t),\ \forall \widehat{\mathbf{x}} \in \widehat{\Omega}_k\}$, with $k = s, f$, where $\widehat{\mathbf{d}}_s$ and $\widehat{\mathbf{d}}_f$ are the displacements induced by the dynamics (see Fig. 1).

We introduce the following entities:

- the outward unit normal vectors $\mathbf{n}_k$ to $\partial\Omega_{k,t}$ (current configuration) and $\widehat{\mathbf{n}}_k$ to $\partial\widehat{\Omega}_k$ (reference configuration),
- the *Arbitrary-Lagrangian-Eulerian (ALE) velocity* $\mathbf{w} = \frac{\partial \widehat{\mathbf{d}}_f}{\partial t}|_{\widehat{\mathbf{x}}}$,
- the deformation gradient tensor for both structure ($k = s$) and fluid ($k = f$) $\mathbf{F}_k = \frac{\partial \mathbf{x}}{\partial \widehat{\mathbf{x}}} = \mathbf{I} + \frac{\partial \widehat{\mathbf{d}}_k}{\partial \widehat{\mathbf{x}}}$ for any $\widehat{\mathbf{x}} \in \widehat{\Omega}_k$,
- the fluid velocity $\mathbf{u}_f$ and the fluid pressure p_f, the dynamic viscosity of the fluid μ, the fluid density ρ_f,
- the Cauchy stress tensor for the fluid $\boldsymbol{\sigma}_f = \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = -p_f\mathbf{I} + \mu(\nabla\mathbf{u}_f + (\nabla\mathbf{u}_f)^T)$, and $\widehat{\boldsymbol{\sigma}}_f$ such that $\widehat{\boldsymbol{\sigma}}_f\widehat{\mathbf{n}}_f = det(\mathbf{F}_f)\mathbf{F}_f^{-T}\boldsymbol{\sigma}_f\mathbf{n}_f \circ \mathscr{D}_{f,t}$,
- the Cauchy stress tensor $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(\widehat{\mathbf{d}}_s)$ and the first Piola-Kirchhoff tensor $\widehat{\boldsymbol{\sigma}}_s = \widehat{\boldsymbol{\sigma}}_s(\widehat{\mathbf{d}}_s) = det(\mathbf{F}_s)\boldsymbol{\sigma}_s(\widehat{\mathbf{d}}_s)\mathbf{F}_s^{-T}$ for the structure, the structure density ρ_s.

Then, for any $t \in (0,T)$ the structure and fluid displacements ($\widehat{\mathbf{d}}_s$ and $\widehat{\mathbf{d}}_f$) and the fluid velocity and pressures ($\mathbf{u}_f$ and p_f) are the solution of the FSI system:

structure problem (in reference configuration)

$$\rho_s \frac{\partial^2 \widehat{\mathbf{d}}_s}{\partial t^2} - \nabla \cdot \widehat{\boldsymbol{\sigma}}_s = \mathbf{0} \qquad \text{in } \widehat{\Omega}_s, \qquad (5)$$

fluid problem (in current configuration)

$$\rho_f \frac{\partial \mathbf{u}_f}{\partial t}\Big|_{\widehat{\mathbf{x}}} + \rho_f((\mathbf{u}_f - \mathbf{w}) \cdot \nabla)\mathbf{u}_f - \nabla \cdot \boldsymbol{\sigma}_f = \mathbf{0}, \qquad \text{in } \Omega_{f,t}, \qquad (6)$$

$$\nabla \cdot \mathbf{u}_f = 0 \qquad \text{in } \Omega_{f,t}, \qquad (7)$$

geometry problem (in reference configuration)

$$-\Delta \widehat{\mathbf{d}}_f = \mathbf{0} \qquad \text{in } \widehat{\Omega}_f, \qquad (8)$$

interface conditions (at interface in reference configuration)

$$\widehat{\boldsymbol{\sigma}}_s\widehat{\mathbf{n}}_s+\widehat{\boldsymbol{\sigma}}_f\widehat{\mathbf{n}}_f=\mathbf{0} \qquad \text{(dynamic)} \quad \text{on } \widehat{\Gamma}, \qquad (9)$$

$$\mathbf{u}_f\circ\mathscr{D}_{f,t}=\frac{\partial\widehat{\mathbf{d}}_s}{\partial t} \qquad \text{(kinematic)} \quad \text{on } \widehat{\Gamma}, \qquad (10)$$

$$\widehat{\mathbf{d}}_f=\widehat{\mathbf{d}}_s \qquad \text{(adherence)} \quad \text{on } \widehat{\Gamma}, \qquad (11)$$

completed with: the Dirichlet boundary conditions $\mathbf{u}_f=\mathbf{g}_f^D$ on $\Gamma_{f,t}^D$ and $\widehat{\mathbf{d}}_f=\mathbf{g}_g^D$ on $\widehat{\Gamma}_f^0\subset\partial\widehat{\Omega}_f$, $\widehat{\mathbf{d}}_s=\mathbf{g}_s^D$ on $\widehat{\Gamma}_s^D$, the Neumann conditions $\boldsymbol{\sigma}_f\mathbf{n}_{f,t}=\mathbf{g}_f^N$ on $\Gamma_{f,t}^N$, $\widehat{\boldsymbol{\sigma}}_s\widehat{\mathbf{n}}_s=\mathbf{g}_s^N$ on $\widehat{\Gamma}_s^N$, and the initial conditions $\mathbf{u}_f=\mathbf{u}_0$ in $\Omega_{f,0}$, $\widehat{\mathbf{d}}_s=\widehat{\mathbf{d}}_0$, $\frac{\partial\widehat{\mathbf{d}}_s}{\partial t}=\widehat{\mathbf{d}}_1$ in $\Omega_{s,0}$.

System (5)–(11) can be recast in the form (1)–(4) by associating the structure problem with $L_1(u_1)$ (now representing nonlinear operators, the choices of u_1 and u_2 are obvious), the fluid problem and the geometric problem with $L_2(u_2)$, both the adherence and the kinematic interface conditions are interpreted as Φ-like conditions (they involve the traces of the unknowns functions on $\widehat{\Gamma}$), whereas the dynamic interface condition is interpreted as a Ψ-like condition (as it involves normal stresses on $\widehat{\Gamma}$).

3 Fluids filtration in porous media (Stokes-Darcy coupling)

Flow processes in a free-fluid region adjacent to a porous medium occur in many relevant applications. Under the (realistic) assumption that the Reynolds number in the porous domain is small, the Navier-Stokes equations could be therein up-scaled to a macroscopic level and replaced by the Darcy law.

Consider the case of a tangential flow of a fluid over a porous bed. This situation is known in literature also as *near parallel flows* [12], i.e. flows for which the pressure gradient is not normal to the interface and the Darcy velocity inside the porous domain is much smaller than the velocity in the fluid domain. The most widely used approach to couple the free fluid regime with the porous-medium one consists of:
- the introduction of an artificial sharp interface Γ between the Stokes (or fluid) domain Ω_s and the Darcy (or porous) domain Ω_d;
- the imposition of the mass conservation, the balance of normal forces and the Beavers-Joseph-Saffman (BJS) experimental law on Γ ([7]), see Fig. 2.

To write down the associated mathematical model, we introduce the following entities:

- the outward unit normal vectors $\mathbf{n}_k$ to $\partial\Omega_k$,
- the dynamic viscosity μ, the density ρ, the velocity $\mathbf{u}_s$ and the pressure p_s of the fluid in Ω_s,
- the Cauchy stress tensor for the fluid $\boldsymbol{\sigma}_s=\boldsymbol{\sigma}_s(\mathbf{u}_s,p_s)=-p_s\mathbf{I}+\mu(\nabla\mathbf{u}_s+(\nabla\mathbf{u}_s)^T)$,
- the Darcy velocity $\mathbf{u}_d$ and the intrinsic average pressure p_d in the porous domain, the intrinsic permeability $\boldsymbol{\kappa}=\boldsymbol{\kappa}(\mathbf{x})$ (for any $\mathbf{x}\in\Omega_d$) of the porous media,
- two given body forces $\mathbf{f}_s$ and $\mathbf{f}_d$,
- the normal unit vector $\mathbf{n}_\Gamma$ to Γ directed from Ω_s to Ω_d (then $\mathbf{n}_\Gamma=\mathbf{n}_s=-\mathbf{n}_d$ on Γ) and an orthonormal system of tangent vectors $\boldsymbol{\tau}_j$, with $j=1,\ldots,d-1$ on Γ.

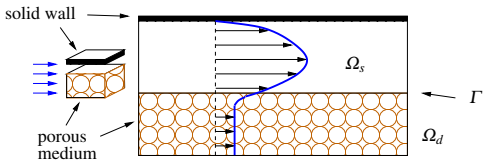

Fig. 2 A typical setting of the Stokes-Darcy coupled problem for a fluid over a porous bed

The coupled problem that we consider reads:

Stokes problem (fluid domain)

$$-\nabla\cdot\boldsymbol{\sigma}_s=\mathbf{f}_s,\quad \nabla\cdot\mathbf{u}_s=0 \qquad \text{in } \Omega_s, \quad (12)$$

Darcy problem (porous domain)

$$\mathbf{u}_d=-\frac{\boldsymbol{\kappa}}{\mu}(\nabla p_d-\mathbf{f}_d),\quad \nabla\cdot\mathbf{u}_d=0 \qquad \text{in } \Omega_d, \quad (13)$$

interface conditions (sharp interface)

$$\mathbf{u}_s\cdot\mathbf{n}_s+\mathbf{u}_d\cdot\mathbf{n}_d=0 \qquad \text{(mass conservation)} \qquad \text{on } \Gamma, \quad (14)$$

$$(\boldsymbol{\sigma}_s\mathbf{n}_s)\cdot\mathbf{n}_s+p_d=0 \qquad \text{(balance of normal forces)} \qquad \text{on } \Gamma, \quad (15)$$

$$(\boldsymbol{\sigma}_s\mathbf{n}_s)\cdot\boldsymbol{\tau}_j+\frac{\alpha\mu}{\sqrt{\boldsymbol{\tau}_j{}^T\boldsymbol{\kappa}\boldsymbol{\tau}_j}}\mathbf{u}_s\cdot\boldsymbol{\tau}_j=0,\ j=1,\ldots,d-1, \text{(BJS condition)} \qquad \text{on } \Gamma, \quad (16)$$

where α is a suitable parameter depending on the porous media. Indeed, the BJS condition is not a coupling condition, as it only involves quantities from one side.

The system (12)–(16) is completed with suitable boundary conditions that read (as usual, D stands for Dirichlet and N for Neumann): $\mathbf{u}_s=\mathbf{g}_s^D$ on $\partial\Omega_s^D$, $\sigma_s\mathbf{n}_s=\mathbf{0}$ on $\partial\Omega_s^N$, $p_d=0$ on $\partial\Omega_d^D$, $\mathbf{u}_d\cdot\mathbf{n}_d=g_d^N$ on $\partial\Omega_d^N$, where we assume that $\partial\Omega_k^N$ and $\partial\Omega_k^D$ are non-intersecting subsets of $\partial\Omega_k\setminus\Gamma$ such that $\overline{\partial\Omega_k^N\cup\partial\Omega_k^D}=\overline{\partial\Omega_k\setminus\Gamma}$.

The coupled system (12)–(16) can be recast in the form (1)–(4) by associating the Stokes problem with $L_2(u_2)$ and the Darcy problem with $L_1(u_1)$. When considering the weak (variational) formulation of the coupled problem (12)–(16), the interface coupling conditions (14) and (15) can be treated in different ways depending on the specific variational form used. In the form used in Sect. 6, the balance of normal forces (15) plays the role of a Φ-like condition (2), while the mass conservation condition (14) will be treated as a Ψ-like condition (3). In specific circumstances, however, for instance when the interface Γ is parallel to one of the cartesian coordinates, condition (14) can be easily enforced as a Dirichlet condition (thus under the form (2)) on the space of trial functions and condition (15) as a Neumann (natural) condition, e.g., like (3).

4 Intergrid operators for non-conforming discretization

We consider two a-priori *independent families of triangulations* $\mathscr{T}_{1,h_1}$ in Ω_1 and $\mathscr{T}_{2,h_2}$ in Ω_2, respectively. The meshes in Ω_1 and in Ω_2 can be non-conforming on Γ and characterized by different mesh-sizes h_1 and h_2. Moreover, different poly-

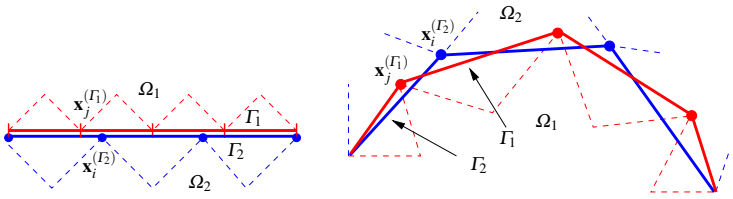

Fig. 3 Γ_1 and Γ_2 induced by the triangulations $\mathscr{T}_{1,h_1}$ and $\mathscr{T}_{2,h_2}$, when $d=2$

nomial degrees $\mathtt{p}_1$ and $\mathtt{p}_2$ can be used to define the finite element spaces. Inside each subdomain Ω_k we assume that the triangulations $\mathscr{T}_{k,h_k}$ are affine, regular and quasi-uniform ([15, Ch.3]).

Then, for $k=1,2$, let $X_{k,h_k}=\{v\in C^0(\overline{\Omega_k}):\ v_{|T}\in\mathbb{P}_{\mathtt{p}_k},\forall T\in\mathscr{T}_{k,h_k}\}$ be the usual Lagrangian finite element spaces associated with $\mathscr{T}_{k,h_h}$, while $Y_{k,h_k}=\{\lambda=v|_\Gamma,\ v\in X_{k,h_k}\}$ are the spaces of traces on Γ of functions in X_{k,h_k}, whose dimension is n_k.

We denote by Γ_1 and Γ_2 the internal boundaries of Ω_1 and Ω_2, respectively, induced by the triangulations $\mathscr{T}_{1,h_1}$ and $\mathscr{T}_{2,h_2}$. If Γ is a straight segment, then $\Gamma_1=\Gamma_2=\Gamma$, otherwise Γ_1 and Γ_2 may not coincide (see Fig. 3).

For $k=1,2$, let $\{\mathbf{x}_1^{(\Gamma_k)},\ldots,\mathbf{x}_{n_k}^{(\Gamma_k)}\}\in\overline{\Gamma}_k$ be the nodes induced by the mesh $\mathscr{T}_{k,h_k}$.

We introduce two independent operators that exchange information between the two independent grids on the interface Γ: $\Pi_{12}:Y_{2,h_2}\to Y_{1,h_1}$ and $\Pi_{21}:Y_{1,h_1}\to Y_{2,h_2}$.

If $\Gamma_1=\Gamma_2$, Π_{12} and Π_{21} are the classical Lagrange interpolation operators defined by the relations:

$$(\Pi_{12}\mu_{2,h_2})(\mathbf{x}_i^{(\Gamma_1)})=\mu_{2,h_2}(\mathbf{x}_i^{(\Gamma_1)}),\quad i=1,\ldots,n_1,\qquad\forall\mu_{2,h_2}\in Y_{2,h_2},\tag{17}$$

$$(\Pi_{21}\mu_{1,h_1})(\mathbf{x}_i^{(\Gamma_2)})=\mu_{1,h_1}(\mathbf{x}_i^{(\Gamma_2)}),\quad i=1,\ldots,n_2,\qquad\forall\mu_{1,h_1}\in Y_{1,h_1}.\tag{18}$$

If, instead, Γ_1 and Γ_2 are geometrically non-conforming, we define Π_{12} and Π_{21} as the Rescaled Localized Radial Basis Function (RL-RBF) interpolation operators introduced in formula (3.1) of [4]. In both cases, the (rectangular) matrices associated with Π_{12} and Π_{21} are, respectively, $R_{12}\in\mathbb{R}^{n_1\times n_2}$ and $R_{21}\in\mathbb{R}^{n_2\times n_1}$ and they are defined by:

$$\begin{aligned}(R_{12})_{ij}&=(\Pi_{12}\mu_j^{(2)})(\mathbf{x}_i^{(\Gamma_1)})\quad i=1,\ldots,n_1,\ j=1,\ldots,n_2,\\(R_{21})_{ij}&=(\Pi_{21}\mu_j^{(1)})(\mathbf{x}_i^{(\Gamma_2)})\quad i=1,\ldots,n_2,\ j=1,\ldots,n_1,\end{aligned}\tag{19}$$

where $\{\mu_i^{(k)}\}$ are the Lagrange basis functions of Y_{k,h_k}, for $k=1,2$ and $i=1,\ldots,n_k$.

In the special conforming case for which $\Gamma_1=\Gamma_2$, $h_1=h_2$ and $\mathtt{p}_1=\mathtt{p}_2$, the interpolation operators Π_{12} and Π_{21} are the identity operator and $R_{12}=R_{21}=I$ (the identity matrix of size $n_1=n_2$). Finally, let M_{Γ_k} such that

$$(M_{\Gamma_k})_{ij}=(\mu_j^{(k)},\mu_i^{(k)})_{L^2(\Gamma_k)},\qquad k=1,2,\tag{20}$$

be the *interface mass matrices*. To assemble both the interface mass matrices and the interpolation matrices, for both the Lagrange and the RL-RBF approaches, the only information that are needed are the coordinates of the interface nodes.

Let $\ell,k=1,2$. If $\boldsymbol{\mu}^{(k)} \in [Y_{k,h_k}]^d$ with $d=2,3$; by writing $\Pi_{\ell k}\boldsymbol{\mu}^{(k)}$ we mean that the interpolation operator $\Pi_{\ell k}$ is applied to each component of the vector-value function $\boldsymbol{\mu}^{(k)}$. Finally, $\mathbf{M}_{\Gamma_k} = diag(M_{\Gamma_k},\ldots,M_{\Gamma_k})$ and $\mathbf{R}_{\ell k} = diag(R_{\ell k},\ldots,R_{\ell k})$ are block diagonal matrices with d blocks.

5 INTERNODES applied to the FSI system

We define the functional spaces:

$$\begin{array}{l}
\mathbf{V}_{f,t} = [H^1(\Omega_{f,t})]^d, \quad Q_{f,t} = L^2(\Omega_{f,t}), \quad \mathbf{V}^D_{f,t} = \{\mathbf{v}\in\mathbf{V}_{f,t}:\ \mathbf{v}=\mathbf{0} \text{ on } \partial\Omega^D_{f,t}\},\\
\mathbf{V}^0_{f,t} = \{\mathbf{v}\in\mathbf{V}_{f,t}:\ \mathbf{v}=\mathbf{0} \text{ on } \partial\Omega^D_{f,t}\cup\Gamma_t\}, \quad \mathbf{V}_s = [H^1(\widehat{\Omega}_s)]^d,\\
\mathbf{V}^D_s = \{\mathbf{v}\in\mathbf{V}_s:\ \mathbf{v}=\mathbf{0} \text{ on } \partial\widehat{\Omega}^D_s\}, \quad \mathbf{V}^0_s = \{\mathbf{v}\in\mathbf{V}_s:\ \mathbf{v}=\mathbf{0} \text{ on } \partial\widehat{\Omega}^D_s\cup\widehat{\Gamma}\},\\
\mathbf{V}_g = [H^1(\widehat{\Omega}_f)]^d, \quad \mathbf{V}^D_g = \{\mathbf{v}\in\mathbf{V}_g:\ \mathbf{v}=\mathbf{0} \text{ on } \partial\widehat{\Omega}^0_f\}, \quad \widehat{\boldsymbol{\Lambda}} = [H^{1/2}_{00}(\widehat{\Gamma})]^d,
\end{array} \tag{21}$$

and the lifting operators $\mathscr{R}_s : \widehat{\boldsymbol{\Lambda}} \to \widehat{\mathbf{V}}^D_s$ s.t. $(\mathscr{R}_s\widehat{\boldsymbol{\lambda}})|_{\widehat{\Gamma}} = \widehat{\boldsymbol{\lambda}}$, $\mathscr{R}_{f,t} : \widehat{\boldsymbol{\Lambda}} \to \mathbf{V}^D_{f,t}$ s.t. $(\mathscr{R}_{f,t}\widehat{\boldsymbol{\lambda}})|_{\Gamma_t} = \widehat{\boldsymbol{\lambda}}\circ\mathscr{D}^{-1}_{f,t}$.

Let us discretize the time derivatives by standard finite difference schemes (e.g. a backward differentiation formula to approximate the first order derivative and the Newmark method to approximate the second one). The weak semi-discrete (continuous in space) counterpart of the FSI system (5)–(11) reads: for any time level t^n, with $n\geq 1$, find $\mathbf{u}^n_f \in \mathbf{V}_{f,t^n}$, $p^n_f \in Q_{f,t^n}$, $\widehat{\mathbf{d}}^n_f \in \mathbf{V}_g$ and $\widehat{\mathbf{d}}^n_s \in \mathbf{V}_s$ satisfying the Dirichlet boundary conditions $\mathbf{u}^n_f = \mathbf{g}^D_f(t^n)$ on Γ^D_{f,t^n} and $\widehat{\mathbf{d}}^n_f = \mathbf{g}^D_g(t^n)$ on $\widehat{\Gamma}^0_f \subset \partial\widehat{\Omega}_f$, $\widehat{\mathbf{d}}^n_s = \mathbf{g}^D_s(t^n)$ on $\widehat{\Gamma}^D_s$ and the initial conditions $\mathbf{u}^0_f = \mathbf{u}_0$ in $\Omega_{f,0}$, $\widehat{\mathbf{d}}^0_s = \widehat{\mathbf{d}}_0$, and $\frac{\partial\widehat{\mathbf{d}}_s}{\partial t}|_{t=0} = \widehat{\mathbf{d}}_1$ in $\Omega_{s,0}$, such that:

$$\mathscr{A}_s(\widehat{\mathbf{d}}^n_s,\widehat{\mathbf{v}}_s) = \mathscr{F}^n_s(\widehat{\mathbf{v}}_s) \qquad \forall\widehat{\mathbf{v}}_s\in\mathbf{V}^0_s, \tag{22}$$

$$\mathscr{A}_f(\mathbf{u}^n_f,\widehat{\mathbf{d}}^n_f;\mathbf{v}_f) + \mathscr{B}_f(\mathbf{v}_f,p^n_f) = \mathscr{F}^n_f(\mathbf{v}_f) \qquad \forall\mathbf{v}_f\in\mathbf{V}^0_{f,t^n}, \tag{23}$$

$$\mathscr{B}_f(\mathbf{u}^n_f,q) = 0 \qquad \forall q\in Q_{f,t^n}, \tag{24}$$

$$\mathscr{G}(\widehat{\mathbf{d}}^n_f,\widehat{\mathbf{v}}_g) = 0 \qquad \forall\widehat{\mathbf{v}}_g\in\mathbf{V}^D_g, \tag{25}$$

$$\begin{array}{ll}
\mathscr{A}_s(\widehat{\mathbf{d}}^n_s,\mathscr{R}_s\widehat{\boldsymbol{\mu}}) + \mathscr{A}_f(\mathbf{u}^n_f,\widehat{\mathbf{d}}^n_f;\mathscr{R}_f\widehat{\boldsymbol{\mu}}) + \mathscr{B}_f(\mathscr{R}_f\widehat{\boldsymbol{\mu}},p^n_f) & \\
\qquad = \mathscr{F}^n_s(\mathscr{R}_s\widehat{\boldsymbol{\mu}}) + \mathscr{F}^n_f(\mathscr{R}_f\widehat{\boldsymbol{\mu}}) & \forall\widehat{\boldsymbol{\mu}}\in\widehat{\boldsymbol{\Lambda}},
\end{array} \tag{26}$$

$$\mathbf{u}^n_f\circ\mathscr{D}_{f,t^n} = a_3\widehat{\mathbf{d}}^n_s + \widehat{\mathbf{b}}^{n-1}_3, \qquad \widehat{\mathbf{d}}^n_f = \widehat{\mathbf{d}}^n_s \qquad \text{on } \widehat{\Gamma}, \tag{27}$$

where

$$\mathscr{A}_s(\widehat{\mathbf{d}}_s,\widehat{\mathbf{v}}_s)=\int_{\widehat{\Omega}_s}(\rho_s a_1\widehat{\mathbf{d}}_s\cdot\widehat{\mathbf{v}}_s+\widehat{\boldsymbol{\sigma}}_s:\nabla_{\hat{\mathbf{x}}}\widehat{\mathbf{v}}_s)\,d\widehat{\Omega},$$
$$\mathscr{F}_s^n(\widehat{\mathbf{v}}_s)=\int_{\partial\widehat{\Omega}_s^N}\mathbf{g}_{s,N}^n\cdot\widehat{\mathbf{v}}_s d\widehat{\gamma}+\int_{\widehat{\Omega}_s}\mathbf{b}_1^{n-1}\,d\widehat{\Omega},\quad \mathscr{G}(\widehat{\mathbf{d}}_f,\widehat{\mathbf{v}}_g)=\int_{\widehat{\Omega}_f}\nabla_{\hat{\mathbf{x}}}\widehat{\mathbf{d}}_f:\nabla_{\hat{\mathbf{x}}}\widehat{\mathbf{v}}_g\,d\Omega,$$
$$\mathscr{A}_f(\mathbf{u}_f,\widehat{\mathbf{d}}_f;\mathbf{v}_f)=\int_{\Omega_{f,t}}\rho_f(a_2\mathbf{u}_f+((\mathbf{u}_f-\mathbf{w})\cdot\nabla)\mathbf{u}_f)\cdot\mathbf{v}_f\,d\Omega$$
$$+\int_{\Omega_{f,t}}\mu(\nabla\mathbf{u}_f+(\nabla\mathbf{u}_f)^T):\nabla\mathbf{v}_f)\,d\Omega,$$
$$\mathscr{B}_f(\mathbf{u}_f,q_f)=\int_{\Omega_{f,t}}(\nabla\cdot\mathbf{u}_f)q\,d\Omega,\quad \mathscr{F}_f^n(\mathbf{v}_f)=\int_{\partial\Omega_{f,t}^N}\mathbf{g}_{f,N}^n\cdot\mathbf{v}_f d\gamma+\int_{\Omega_{f,t}}\mathbf{b}_2^{n-1}\cdot\mathbf{v}_f\,d\Omega,$$

with a_1,a_2,a_3 suitable real values and $\mathbf{b}_1^{n-1},\mathbf{b}_2^{n-1}$, and $\mathbf{b}_3^{n-1}$ (depending on the solution at the previous time levels) suitable vector functions arising from the finite difference discretization of the time derivatives.

Equation (26) is the weak counterpart of the dynamic interface condition (9).

We consider now independent finite element space discretizations (as described in Sect. 4) in $\widehat{\Omega}_f$ and $\widehat{\Omega}_s$ (a suitable inf-sup stable couple of finite elements will be considered in the fluid domain) that may induce two different discrete interfaces $\widehat{\Gamma}_f=\mathscr{T}_{f,h_f}\cap\widehat{\Gamma}$ and $\widehat{\Gamma}_s=\mathscr{T}_{s,h_s}\cap\widehat{\Gamma}$ in the case that $\widehat{\Gamma}$ is curved as in Fig. 3, right. Then we use the subindices h_k, for $k=s,f$, to characterize the subspaces of the functional spaces (21) as well as the discrete counterpart of each variable appearing in the system (22)–(27). From now on, in $\widehat{\mathbf{d}}_{s,h_s}^n$, $\mathbf{u}_{f,h_f}^n$, $\widehat{\mathbf{d}}_{f,h_f}^n$, and p_{f,h_f}^n, the super-index n will be omitted for sake of notations.

In order to apply the INTERNODES method to the discrete counterpart of (22)–(27), we define the scalar quantities:

$$\begin{aligned}
r_{s,i}=&\ \mathscr{A}_s(\widehat{\mathbf{d}}_{s,h_s},\mathscr{R}_s\widehat{\boldsymbol{\mu}}_i^{(s)})-\mathscr{F}_s^n(\mathscr{R}_s\widehat{\boldsymbol{\mu}}_i^{(s)}), && i=1,\dots,d\cdot n_s,\\
r_{f,i}=&\ \mathscr{A}_f(\mathbf{u}_{f,h_f},\widehat{\mathbf{d}}_{f,h_f};\mathscr{R}_f\widehat{\boldsymbol{\mu}}_i^{(f)})+\mathscr{B}_f(\mathscr{R}_f\widehat{\boldsymbol{\mu}}_i^{(f)},p_{f,h_f})\\
&-\mathscr{F}_f^n(\mathscr{R}_f\widehat{\boldsymbol{\mu}}_i^{(f)}), && i=1,\dots,d\cdot n_f
\end{aligned}\tag{28}$$

(where $\{\widehat{\boldsymbol{\mu}}_i^{(k)}\}_{i=1}^{d\cdot n_k}$ are the Lagrange basis functions of $[Y_{k,h_k}]^d$) and

$$z_{k,j}=\sum_{i=1}^{d\cdot n_k}(\mathbf{M}_{\widehat{\Gamma}_k}^{-1})_{ji}r_{k,i},\qquad k=s,f,\ j=1,\dots,d\cdot n_k,\tag{29}$$

and the functions $\mathbf{r}_{k,h_k}=\sum_{j=1}^{d\cdot n_k}z_{k,j}\widehat{\boldsymbol{\mu}}_j^{(k)}$,which are the so called *discrete residuals* and are the discrete counterpart of $\widehat{\boldsymbol{\sigma}}_k\widehat{\mathbf{n}}_k$.

The INTERNODES method applied to system (9)–(11) at any t^n reads:

$$\mathscr{A}_s(\widehat{\mathbf{d}}_{s,h_s},\widehat{\mathbf{v}}_{s,h_s})=\mathscr{F}_s^n(\widehat{\mathbf{v}}_{s,h_s})\qquad\forall\widehat{\mathbf{v}}_{s,h_s}\in\mathbf{V}_{s,h_s}^0,\tag{30}$$
$$\mathscr{A}_f(\mathbf{u}_{f,h_f},\widehat{\mathbf{d}}_{f,h_f};\mathbf{v}_{f,h_f})+\mathscr{B}_f(\mathbf{v}_{f,h_f},p_{f,h_f})=\mathscr{F}_f^n(\mathbf{v}_{f,h_f})\qquad\forall\mathbf{v}_{f,h_f}\in\mathbf{V}_{f,h_f,t^n}^0,\tag{31}$$
$$\mathscr{B}_f(\mathbf{u}_{f,h_f},q_{f,h_f})=0\qquad\forall q_{f,h_f}\in Q_{f,h_f,t^n},\tag{32}$$
$$\mathscr{G}(\widehat{\mathbf{d}}_{f,h_f},\widehat{\mathbf{v}}_{g,h_g})=0\qquad\forall\widehat{\mathbf{v}}_{g,h_g}\in\mathbf{V}_{g,h_g}^D,\tag{33}$$

$$\mathbf{r}_{s,h_s}+\Pi_{sf}\mathbf{r}_{f,h_f}=\mathbf{0}\qquad\text{(dynamic)}\qquad\text{on }\widehat{\Gamma}_s,\tag{34}$$
$$\mathbf{u}_{f,h_f}\circ\mathscr{D}_{f,t^n}=\Pi_{fs}(a_3\widehat{\mathbf{d}}_{s,h_s}+\widehat{\mathbf{b}}_3^{n-1})\qquad\text{(kynematic)}\qquad\text{on }\widehat{\Gamma}_f,\tag{35}$$

$$\widehat{\mathbf{d}}_{f,h_f} = \Pi_{fs}\widehat{\mathbf{d}}_{s,h_s} \qquad \text{(adherence)} \quad \text{on } \widehat{\Gamma}_f. \tag{36}$$

The conditions (34)–(36) are the INTERNODES counterpart of the interface condition (9)–(11), obtained by applying the intergrid operators Π_{12} and Π_{21} defined in Sect. 4. More precisely, if we make the associations $s \leftrightarrow 1$ and $f \leftrightarrow 2$, the operator $\Pi_{fs}(=\Pi_{21})$ is used to interpolate on $\widehat{\Gamma}_f$ each component of the discrete traces $\widehat{\mathbf{d}}_{s,h_s}$ and (the discretization of) $\frac{\partial \widehat{\mathbf{d}}_{s,h_s}}{\partial t}|_{t^n}$ that are known on $\widehat{\Gamma}_s$, while $\Pi_{sf}(=\Pi_{12})$ is used to interpolate on $\widehat{\Gamma}_s$ each component of the discrete counterpart of the normal stress $\widehat{\boldsymbol{\sigma}}_f\widehat{\mathbf{n}}_f$ that is known on $\widehat{\Gamma}_f$.

By construction, $\mathbf{r}_{k,h_k} \in \mathbf{Y}_k = [Y_{k,h_k}]^d$, for $k = s, f$, and then $\mathbf{r}_{f,h_f}$ has the sufficient regularity to be interpolated.

Remark 1. The scalar values (28), typically computed as algebraic residuals at the interface of the finite element system, *are not* the coefficients of the function $\mathbf{r}_{k,h_k}$ w.r.t. the Lagrange expansion $\{\widehat{\boldsymbol{\mu}}_j^{(k)}\}$, rather the coefficients of $\mathbf{r}_{k,h_k}$ w.r.t. the canonical basis $\{\widehat{\boldsymbol{\psi}}_i^{(k)}\}_{i=1}^{d\cdot n_k}$ of $\mathbf{Y}'_{k,h_k}$. The latter is the dual to $\{\widehat{\boldsymbol{\mu}}_j^{(k)}\}$, that is it satisfies the relations $(\widehat{\boldsymbol{\psi}}_i^{(k)}, \widehat{\boldsymbol{\mu}}_j^{(k)})_{L^2(\widehat{\Gamma}_k)} = \delta_{ij}$, for $i, j = 1, d\cdot\ldots, n_k$, with δ_{ij} the Kronecker delta. It can proved (see [2]) that $\widehat{\boldsymbol{\psi}}_i^{(k)} = \sum_{j=1}^{d\cdot n_k}(\mathbf{M}_{\Gamma_k}^{-1})_{ji}\widehat{\boldsymbol{\mu}}_j^{(k)}$, i.e., the interface mass matrix $\mathbf{M}_{\Gamma_k}$ and its inverse play the role of transfer matrices from the Lagrange basis to the dual one and viceversa, respectively.

Denoting by r_f, r_s, u_f, d_s, d_f, b_3^{n-1}, and d_f the arrays whose entries are the Lagrangian degrees of freedom of $\mathbf{r}_{f,h_f}$, $\mathbf{r}_{s,h_s}$, $\mathbf{u}_{f,h_f}$, $\widehat{\mathbf{d}}_{s,h_s}$, $\widehat{\mathbf{d}}_{f,h_f}$, and $\mathbf{b}_3^{n-1}$, respectively, the algebraic form of the INTERNODES conditions (34)–(36) reads:

$$\mathbf{M}_{\Gamma_s}^{-1}\mathsf{r}_s + \mathbf{R}_{sf}\mathbf{M}_{\Gamma_f}^{-1}\mathsf{r}_f = 0, \tag{37}$$
$$\mathsf{u}_f = \mathbf{R}_{fs}(a_3\mathsf{d}_s + \mathsf{b}_3^{n-1}), \tag{38}$$
$$\mathsf{d}_f = \mathbf{R}_{fs}\mathsf{d}_s. \tag{39}$$

Notice that (37) can be equivalently written as $\mathsf{r}_s + \mathbf{M}_{\Gamma_s}\mathbf{R}_{sf}\mathbf{M}_{\Gamma_f}^{-1}\mathsf{r}_f = 0$.

The INTERNODES method has been successfully applied to the FSI system in [8, 5].

6 INTERNODES applied to the Stokes-Darcy system

We define the functional spaces:

$$\mathbf{V}_s = [H^1(\Omega_s)]^d, \quad \mathbf{V}_s^D = \{\mathbf{v} \in \mathbf{V}_s :\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_s^D\}, \tag{40}$$
$$\mathbf{V}_d = \{\mathbf{v} \in [L^2(\Omega_d)]^d :\ \nabla\cdot\mathbf{v} \in L^2(\Omega_d)\}, \quad \mathbf{V}_d^N = \{\mathbf{v} \in \mathbf{V}_d :\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_d^N\},$$
$$Q_s = L^2(\Omega_s), \quad Q_d = L^2(\Omega_d), \quad \Lambda = H_{00}^{1/2}(\Gamma).$$

Then we consider the following weak form of the Stokes-Darcy coupled problem (12)–(16) ([11]): find $\mathbf{u}_s \in \mathbf{V}_s$, $p_s \in Q_s$, $\mathbf{u}_d \in \mathbf{V}_d$, $p_d \in Q_d$, and $\lambda \in \Lambda$ with $\mathbf{u}_s = \mathbf{g}_s^D$ on $\partial\Omega_s^D$, $\mathbf{u}_d \cdot \mathbf{n}_d = g_d^N$ on $\partial\Omega_d^N$ such that:

$$
\begin{aligned}
&2\mu \int_{\Omega_s} D(\mathbf{u}_s) : D(\mathbf{v}_s)\, d\Omega - \int_{\Omega_s} p_s \nabla \cdot \mathbf{v}_s\, d\Omega + \int_{\Gamma} \lambda \mathbf{v}_s \cdot \mathbf{n}_s\, d\Gamma \qquad (41)\\
&\quad + \sum_{j=1}^{d-1} \int_{\Gamma} \alpha_j (\mathbf{u}_s \cdot \boldsymbol{\tau}_j)(\mathbf{v}_s \cdot \boldsymbol{\tau}_j)\, d\Gamma = \int_{\Omega_s} \mathbf{f}_s \cdot \mathbf{v}_s\, d\Omega \qquad \forall \mathbf{v}_s \in \mathbf{V}_s^D,\\
&\int_{\Omega_s} q_s \nabla \cdot \mathbf{u}_s\, d\Omega = 0 \qquad \forall q_s \in Q_s,\\
&\mu \int_{\Omega_d} (\boldsymbol{\kappa}^{-1} \mathbf{u}_d) \cdot \mathbf{v}_d\, d\Omega - \int_{\Omega_d} p_d \nabla \cdot \mathbf{v}_d\, d\Omega + \int_{\Gamma} \lambda \mathbf{v}_d \cdot \mathbf{n}_d\, d\Gamma \qquad (42)\\
&\quad = \int_{\Omega_d} \mathbf{f}_d \cdot \mathbf{v}_d\, d\Omega \qquad \forall \mathbf{v}_d \in \mathbf{V}_d^N,\\
&\int_{\Omega_d} q_d \nabla \cdot \mathbf{u}_d\, d\Omega = 0 \qquad \forall q_d \in Q_d,\\
&\int_{\Gamma} \mathbf{u}_s \cdot \mathbf{n}_s \eta + \int_{\Gamma} \mathbf{u}_d \cdot \mathbf{n}_d \eta = 0 \qquad \forall \eta \in \Lambda, \qquad (43)
\end{aligned}
$$

where $D(\mathbf{v}) = (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)/2$, while $\alpha_j = \alpha\mu / \sqrt{\boldsymbol{\tau}_j^T \boldsymbol{\kappa} \boldsymbol{\tau}_j}$.

The Lagrange multiplier $\lambda \in \Lambda$ is in fact $\lambda = p_d = -(\boldsymbol{\sigma}_s \mathbf{n}_s) \cdot \mathbf{n}_s$ on Γ.

We discretize both Stokes problem (12) and Darcy problem (13) by inf-sup stable (or stabilized) couples of finite elements (see, e.g., [6]). Independent finite element space discretizations (as described in Sect. 4) are considered in Ω_s and Ω_d that may induce two different discrete interface $\Gamma_s = \mathscr{T}_{s,h_s} \cap \Gamma$ and $\Gamma_d = \mathscr{T}_{d,h_d} \cap \Gamma$ in the case that Γ is curved as in Fig. 3, right. Then we use the subindices h_k, for $k = s, d$, to characterize the subspaces of the functional spaces (40) as well as the discrete counterpart of each variable appearing in the system (41)–(43). For $k = s, d$, $\Lambda_{k,h_k} = \Lambda \cap Y_{k,h_k}$.

In order to apply the INTERNODES method to the discrete counterpart of (41)–(43), we define the scalar quantities:

$$
r_{k,i} = \int_{\Gamma} (\mathbf{u}_{k,h_k} \cdot \mathbf{n}_k) \mu_i^{(k)}, \qquad i = 1, \ldots, n_k,\ k = s, d, \qquad (44)
$$

(where $\{\mu_i^{(k)}\}_{i=1}^{n_k}$ are the Lagrange basis functions of Y_{k,h_k}) and

$$
z_{k,j} = \sum_{i=1}^{n_k} (M_{\Gamma_k}^{-1})_{ji} r_{k,i}, \qquad j = 1, \ldots, n_k,\ k = s, d, \qquad (45)
$$

and the discrete functions (belonging to Y_{k,h_k})

$$
w_{k,h_k} = \sum_{j=1}^{n_k} z_{k,j} \mu_j^{(k)}. \qquad (46)
$$

The INTERNODES form of problem (41)–(43) reads: find $\mathbf{u}_{s,h_s} \in \mathbf{V}_{s,h_s}$, $p_{s,h_s} \in Q_{s,h_s}$, $\mathbf{u}_{d,h_d} \in \mathbf{V}_{d,h_d}$, $p_{d,h_d} \in Q_{d,h_d}$, $\lambda_{s,h_s} \in \Lambda_{s,h_s}$ and $\lambda_{d,h_d} \in \Lambda_{d,h_d}$ (satisfying the given boundary conditions) such that:

$$2\mu \int_{\Omega_s} D(\mathbf{u}_{s,h_s}) : D(\mathbf{v}_{s,h_s})\, d\Omega - \int_{\Omega_s} p_{s,h_s} \nabla \cdot \mathbf{v}_{s,h_s}\, d\Omega + \int_{\Gamma} \lambda_{s,h_s} \mathbf{v}_{s,h_s} \cdot \mathbf{n}_s\, d\Gamma \tag{47}$$
$$+ \sum_{j=1}^{d-1} \int_{\Gamma} \alpha_j (\mathbf{u}_{s,h_s} \cdot \boldsymbol{\tau}_j)(\mathbf{v}_{s,h_s} \cdot \boldsymbol{\tau}_j)\, d\Gamma = \int_{\Omega_s} \mathbf{f}_s \cdot \mathbf{v}_{s,h_s}\, d\Omega \qquad \forall \mathbf{v}_{s,h_s} \in \mathbf{V}^D_{s,h_s},$$
$$\int_{\Omega_s} q_{s,h_s} \nabla \cdot \mathbf{u}_{s,h_s}\, d\Omega = 0 \qquad \forall q_{s,h_s} \in Q_{s,h_s},$$
$$\mu \int_{\Omega_d} (\boldsymbol{\kappa}^{-1} \mathbf{u}_{d,h_d}) \cdot \mathbf{v}_{d,h_d}\, d\Omega - \int_{\Omega_d} p_{d,h_d} \nabla \cdot \mathbf{v}_{d,h_d}\, d\Omega \tag{48}$$
$$+ \int_{\Gamma} \lambda_{d,h_d} \mathbf{v}_{d,h_d} \cdot \mathbf{n}_d\, d\Gamma = \int_{\Omega_d} \mathbf{f}_d \cdot \mathbf{v}_{d,h_d}\, d\Omega \qquad \forall \mathbf{v}_{d,h_d} \in \mathbf{V}^N_{d,h_d},$$
$$\int_{\Omega_d} q_{d,h_d} \nabla \cdot \mathbf{u}_{d,h_d}\, d\Omega = 0 \qquad \forall q_{d,h_d} \in Q_{d,h_d},$$
$$\Pi_{ds} w_{s,h_s} + w_{d,h_d} = 0 \qquad \text{on } \Gamma_d, \tag{49}$$
$$\lambda_{s,h_s} = \Pi_{sd} \lambda_{d,h_d} \qquad \text{on } \Gamma_s. \tag{50}$$

The conditions (49)–(50) are the INTERNODES counterpart of the interface condition (14)–(15), obtained by applying the intergrid operators Π_{12} and Π_{21} defined in Sect. 4. More precisely, if we make the associations $d \leftrightarrow 1$ and $s \leftrightarrow 2$, the operator $\Pi_{sd} (= \Pi_{21})$ is used to interpolate on Γ_s the discrete trace of p_{d,h_d} that is known on Γ_d, while $\Pi_{ds} (= \Pi_{12})$ is used to interpolate on Γ_d the weak counterpart of $\mathbf{u}_{s,h_s} \cdot \mathbf{n}_s$ that is known on Γ_s.

Denoting by $\mathtt{w}_s$, $\mathtt{w}_d$, $\mathtt{t}_s$, and $\mathtt{t}_d$, the arrays whose entries are the Lagrangian degrees of freedom of w_{s,h_s}, w_{d,h_d}, λ_{s,h_s}, and λ_{d,h_d} respectively, the algebraic form of the INTERNODES conditions (49)–(50) reads:

$$R_{ds} M_{\Gamma_s}^{-1} \mathtt{w}_s + M_{\Gamma_d}^{-1} \mathtt{w}_d = 0, \qquad \mathtt{t}_s = R_{sd} \mathtt{t}_d. \tag{51}$$

We test the accuracy of INTERNODES by solving problem (12)–(16) with: $\Omega_s = (0,1) \times (1,2)$, $\Omega_d = (0,1) \times (0,1)$, $\mu = 1$, $\kappa = 10^{-2}$, $\boldsymbol{\kappa} = \kappa I$, boundary data and $\mathbf{f}_s = \mathbf{f}_d$ are such that the exact solution is $\mathbf{u}_s = \kappa[-\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) - y + 1, \cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) - 1 + x]$, $p_s = 1 - x$, $\mathbf{u}_d = \kappa[\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) + y, \cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) - 1 + x]$, $p_d = \frac{2}{\pi}\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y) - y(x-1)$. The approximation in each subdomain is performed with stabilized hp-fem on quadrilaterals ([6]). The errors $e_s = \|\mathbf{u}_s - \mathbf{u}_{s,h_s}\|_{H^1(\Omega_s)} + \|p_s - p_{s,h_s}\|_{L^2(\Omega_s)}$ and $e_d = \|\mathbf{u}_d - \mathbf{u}_{d,h_d}\|_{L^2(\Omega_d)} + \|p_d - p_{d,h_d}\|_{H^1(\Omega_d)}$ are shown in Figure 4, versus either the mesh sizes h_s, h_d and the polynomial degrees $\mathtt{p}_s$ and $\mathtt{p}_d$, they decay exponentially w.r.t. the polynomial degrees (Fig. 4, at left) and with order $q = \mathtt{p}_s = \mathtt{p}_d$ w.r.t. the mesh sizes (Fig. 4, at center and at right).

In Fig. 5 we show the INTERNODES solution computed for the *cross-flow membrane filtration* test case with non-flat interface Γ. The setting of the problem is given in Sect. 5.3 of [6]. We have considered either a cubic spline interface (Fig. 5 at the left) and a piece-wise interface (Fig. 5 at the right). Quadrilaterals hp-fem are used for the discretization in either Ω_s and Ω_d. The solution at left is obtained with

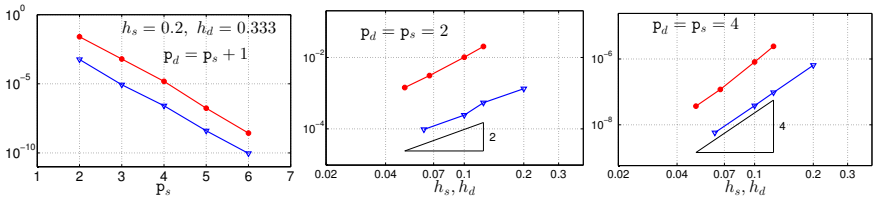

Fig. 4 Errors e_s (red) and e_d (blue) for the Stoked-Darcy problem (12)–(16) solved on non-conforming meshes by the INTERNODES method

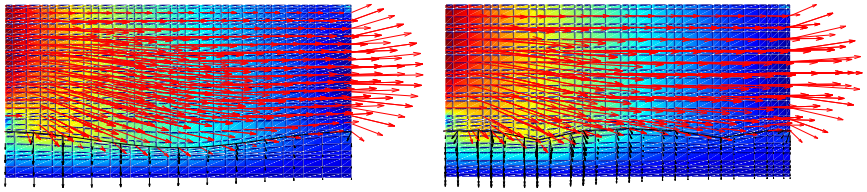

Fig. 5 INTERNODES solution of the Stokes-Darcy coupling. The velocity field $\mathbf{u}_s$ is red in Ω_s and black in Ω_d, the underground colored scalar field the hydrodynamic pressure. Γ is curved at left and piece-wise linear at right

$h_s = 3/8$, $h_d = 1/2$, and $\mathrm{p}_s = \mathrm{p}_d = 4$, that at right with $h_s = h_d = 3/8$, $\mathrm{p}_s = 4$ and $\mathrm{p}_d = 3$. RL-RBF interpolation is used to build the intergrid operators (17) when Γ is curved, and Lagrange interpolation when Γ is piece-wise linear.

Numerical results show that INTERNODES keeps the optimal accuracy of the local discretizations and that it is a versatile method to deal with non-conforming interfaces.

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