Two-level preconditioners for the Helmholtz equation

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1 Introduction

Solving the Helmholtz equation $-\Delta u - k^2 u = f$ is a challenging task because of its indefinite nature and its highly oscillatory solution when the wavenumber k is high. Although there have been different attempts to solve it efficiently, we believe that there is no established and robust preconditioner, whose behavior is independent of k, for general decompositions into subdomains. In Conen et al. [2014] a two-level preconditioner was introduced, where the coarse correction involves local eigenproblems of Dirichlet-to-Neumann (DtN) maps. This method proved to be very robust with respect to heterogeneous coefficients compared to the reference preconditioner based on plane waves, and its construction is completely automatic without the need for parameter tuning. Another method was developed in Graham et al. [2017b,a], where two-level domain decomposition approximations of the Helmholtz equation with absorption $-\Delta u - (k^2 + \mathrm{i}\varepsilon)u = f$ were used as preconditioners for the pure Helmholtz equation without absorption; there the coarse correction is based on a coarse mesh with diameter constrained by k. Our purpose is to compare numerically the performance of the latter with the two-level method based on DtN maps, both in two and three dimensions.

2 Definition of the problem

Consider the interior Helmholtz problem of the following form: let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polyhedral, bounded domain; find $u \colon \Omega \to \mathbb{C}$ such that

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$$-\Delta u - (k^2 + \mathtt{i}\varepsilon)u = f \qquad \text{in } \Omega, \tag{1a}$$

$$\frac{\partial u}{\partial n} - \mathtt{i}\eta u = 0 \qquad \text{on } \Gamma = \partial\Omega. \tag{1b}$$

The wavenumber k is given by $k(\mathbf{x}) = \omega/c(\mathbf{x})$, where ω is the angular frequency and c is the speed of propagation that might depend on $\mathbf{x} \in \Omega$; we take $\eta = \text{sign}(\varepsilon)k$ if $\varepsilon \neq 0$, $\eta = k$ if $\varepsilon = 0$,[R2] as Robin boundary condition parameter. We are interested in solving the problem in the case $\varepsilon = 0$, using ε as a parameter when building the preconditioner. The variational formulation of Problem (1) is: find $u \in V = H^1(\Omega)$ such that $a_\varepsilon(u, v) = F(v)$, $\forall v \in V$, where $a_\varepsilon(.,.) \colon V \times V \to \mathbb{C}$ and $F \colon V \to \mathbb{C}$ are defined by

$$a_\varepsilon(u, v) = \int_\Omega \left(\nabla u \cdot \overline{\nabla v} - (k^2 + \mathtt{i}\varepsilon)u\overline{v}\right) - \int_\Gamma \mathtt{i}\eta u\overline{v}, \quad F(v) = \int_\Omega f\overline{v}.$$

Note that if $\varepsilon \neq 0$ and $\eta = \text{sign}(\varepsilon)k$, a_ε is coercive (see §2 in Graham et al. [2017b]). We consider a discretization of the variational problem using piecewise linear finite elements on a uniform simplicial mesh $\mathcal{T}_h$ of Ω. Denoting by $V_h \subset V$ the corresponding finite element space and by $\{\phi_k\}_{k=1}^n$ its basis functions, $n := \dim(V_h)$, the discretized problem reads: find $u_h \in V_h$ such that $a_\varepsilon(u_h, v_h) = F(v_h)$, $\forall v_h \in V_h$, that is, in matrix form,

$$A_\varepsilon \mathbf{u} = \mathbf{f}, \tag{2}$$

where the coefficients of the matrix $A_\varepsilon \in \mathbb{C}^{n\times n}$ and the right-hand side $\mathbf{f} \in \mathbb{C}^n$ are given by $(A_\varepsilon)_{k,l} = a(\phi_l, \phi_k)$ and $(\mathbf{f})_k = F(\phi_k)$. The matrix A_ε is complex, symmetric (but not Hermitian), and indefinite if $\varepsilon = 0$.

3 Two-level domain decomposition preconditioners

In the following we will define the domain decomposition preconditioners for the linear system $A_0\mathbf{u} = \mathbf{f}$ resulting from the discretization of the Helmholtz problem without absorption ($\varepsilon = 0$). These are two-level Optimized Restricted Additive Schwarz (ORAS) algorithms, where "optimized" refers to the use of Robin boundary conditions at the interface between subdomains. In the terminology of Graham et al. [2017b], the prefix O is replaced with Imp, which stands for impedance (i.e. Robin) boundary conditions.

First of all, consider a decomposition of the domain Ω into a set of overlapping subdomains $\{\Omega_j\}_{j=1}^{N_{\text{sub}}}$, with each subdomain consisting of a union of elements of the mesh $\mathcal{T}_h$. Let $V_h(\Omega_j) = \left\{v|_{\Omega_j} : v \in V_h\right\}$, $1 \leq j \leq N_{\text{sub}}$, denote the space of functions in V_h restricted to the subdomain Ω_j. Let n_j be the dimension of $V_h(\Omega_j)$, $1 \leq j \leq N_{\text{sub}}$. ~~Let $n_j := \#\,\text{dof}\,(\Omega_j)$, $1 \leq j \leq N_{\text{sub}}$, where $\text{dof}(D) := \left\{k : \text{supp}\,(\phi_k) \subset \overline{D}\right\}$ represents the degrees of freedom (dofs) of $V_h(\Omega_j)$.~~[R1] For $1 \leq j \leq N_{\text{sub}}$, we define a restriction operator $\mathcal{R}_j \colon V_h \to$

$V_h(\Omega_j)$ by injection, i.e. for $u \in V_h$ we set $(\mathcal{R}_j u)(\mathbf{x}_i) = u(\mathbf{x}_i)$ for all $\mathbf{x}_i \in \Omega_j$. We denote by R_j the corresponding Boolean matrix in $\mathbb{R}^{n_j \times n}$ that maps coefficient vectors of functions in V_h to coefficient vectors of functions in $V_h(\Omega_j)$. Let $D_j \in \mathbb{R}^{n_j \times n_j}$ be a diagonal matrix corresponding to a partition of unity in the sense that $\sum_{i=1}^{N_{\text{sub}}} \tilde{R}_i^T R_i = I$, where $\tilde{R}_j := D_j R_j$. Then the *one-level ORAS* preconditioner (which is also the one-level ImpRAS ~~ImpHRAS~~[R2] of Graham et al. [2017b]) reads

$$M_{1,\varepsilon}^{-1} := \sum_{j=1}^{N_{\text{sub}}} \tilde{R}_j^T A_{j,\varepsilon}^{-1} R_j. \tag{3}$$

We define the matrices $A_{j,\varepsilon}$ in (3) to be the matrices stemming from the discretization of the following local Robin problems with absorption

$$\begin{aligned} -\Delta u_j - (k^2 + \mathtt{i}\varepsilon) u_j &= f && \text{in } \Omega_j, \\ \frac{\partial u_j}{\partial n_j} - \mathtt{i}\eta u_j &= 0 && \text{on } \partial\Omega_j. \end{aligned}$$

In order to achieve weak dependence on the wavenumber k and number of subdomains, we add a coarse component to (3). The *two-level* preconditioner can be written in a generic way as follows

$$M_{2,\varepsilon}^{-1} = Q M_{1,\varepsilon}^{-1} P + Z E^{-1} Z^*, \tag{4}$$

where $*$ denotes the conjugate transpose, $M_{1,\varepsilon}^{-1}$ is the one-level preconditioner (3), Z is a rectangular matrix with full column rank, $E = Z^* A_\varepsilon Z$ is the so-called coarse grid matrix, $\Xi = Z E^{-1} Z^*$ is the so-called coarse grid correction matrix. If $P = Q = I$ this is an *additive* two-level preconditioner (which would be called two-level ImpRAS in Graham et al. [2017b]). If $P = I - A_\varepsilon \Xi$ and $Q = I - \Xi A_\varepsilon$, this is a *hybrid* two-level preconditioner (ImpHRAS in Graham et al. [2017b]), also called the *Balancing* Neumann Neumann (BNN) preconditioner. Preconditioner (4) is characterized by the choice of Z, whose columns span the *coarse space* (CS). We will consider the following two cases:

The grid coarse space The most natural coarse space would be one based on a coarser mesh, we subsequently call it "grid coarse space". Let us consider $\mathcal{T}_{H_{\text{coarse}}}$ a simplicial mesh of Ω with mesh diameter H_{coarse} and $V_{H_{\text{coarse}}} \subset V$ the corresponding finite element space. Let $\mathcal{I}_0 \colon V_{H_{\text{coarse}}} \to V_h$ be the nodal interpolation operator and define Z as the corresponding matrix. Then ~~Let $\mathcal{R}_0 \colon V_h \to V_{H_{\text{coarse}}}$ be the nodal interpolation operator and R_0 the corresponding matrix. Define $Z = R_0^T$, then~~[R2] in this case $E = Z^* A_\varepsilon Z$ is really the stiffness matrix of the problem (with absorption) discretized on the coarse mesh. Related preconditioners without absorption are used in Kimn and Sarkis [2007].

The DtN coarse space This coarse space (see Conen et al. [2014]) is based on local Dirichlet-to-Neumann (DtN) eigenproblems on the subdomain interfaces. For a subdomain Ω_i, first of all consider $a^{(i)}\colon H^1(\Omega_i)\times H^1(\Omega_i)\to\mathbb{R}$

$$a^{(i)}(v,w)=\int_{\Omega_i}\left(\nabla v\cdot\overline{\nabla w}-(k^2+\mathtt{i}\varepsilon)v\overline{w}\right)-\int_{\partial\Omega_i\cap\partial\Omega}\mathtt{i}\eta u\overline{v}.$$

Let $(A^{(i)})_{kl}=a^{(i)}(\phi_k,\phi_l)$, and let I and Γ_i be the sets of indices corresponding, resp., to the interior and boundary dofs on Ω_i, with n_I and n_{Γ_i} their cardinalities. With the usual block notation, the subscripts I and Γ_i for the matrices A and $A^{(i)}$ denote the entries of these matrices associated with the respective dofs. Let $M_{\Gamma_i}=\left(\int_{\Gamma_i}\phi_k\phi_l\right)_{k,l\in\Gamma_i}$ be the mass matrix on the interface $\Gamma_i=\partial\Omega_i\setminus\partial\Omega$ of subdomain Ω_i. We need to solve the following eigenproblem: find $(\mathbf{u},\lambda)\in\mathbb{C}^{n_{\Gamma_i}}\times\mathbb{C}$, s.t.

$$(A^{(i)}_{\Gamma_i\Gamma_i}-A_{\Gamma_iI}A_{II}^{-1}A_{I\Gamma_i})\mathbf{u}=\lambda M_{\Gamma_i}\mathbf{u}. \tag{5}$$

Now, the matrix Z of the DtN coarse space is a rectangular, block-diagonal matrix with blocks W_i, associated with the subdomain Ω_i, $1\le i\le N_{\text{sub}}$, given by Algorithm 3.1. If m_i is the number of eigenvectors selected by the automatic criterion in Line 2 of Algorithm 3.1, the block W_i has dimensions $n_i\times m_i$, and the matrix Z has dimensions $n\times\sum_{j=1}^{N_{\text{sub}}}m_i$. Due to the overlap in the decomposition, the blocks may share some rows inside the matrix Z.

Algorithm 3.1 Construction of the block W_i of the DtN CS matrix Z

1: Solve the discrete DtN eigenproblem (5) on subdomain Ω_i for the eigenpairs $(\lambda_j,\mathbf{g}_i^j)$.
2: Choose the m_i eigenvectors $\mathbf{g}_i^j\in\mathbb{C}^{n_{\Gamma_i}}$ such that $\Re(\lambda_j)<k$, $1\le j\le m_i$.
3: **for** $j=1$ to m_i **do**
4: Compute the discrete Helmholtz extension $\mathbf{u}_i^j\in\mathbb{C}^{n_i}$ to Ω_i of $\mathbf{g}_i^j$ as $\mathbf{u}_i^j=[-A_{II}^{-1}A_{I\Gamma_i}\mathbf{g}_i^j,\quad\mathbf{g}_i^j]^T$.
5: **end for**
6: Define the matrix $W_i\in\mathbb{C}^{n_i\times m_i}$ as $W_i=\left(D_i\mathbf{u}_i^1,\ldots,D_i\mathbf{u}_i^{m_i}\right)$.

4 Numerical experiments

We solve (2) with $\varepsilon=0$ on the unit square/cube, with a uniform simplicial mesh of diameter $h\sim k^{-3/2}$, which is believed to remove the pollution effect. The right-hand side is given by $f=-\exp(-100((x-0.5)^2+(y-0.5)^2))$ for $d=2$, $f=-\exp(-400((x-0.5)^2+(y-0.5)^2+(z-0.5)^2))$ for $d=3$.

We use GMRES with right preconditioning (with a tolerance $\tau=10^{-6}$), starting with a *random initial guess*, which ensures, unlike a zero initial

guess, that all frequencies are present in the error;[R1] the stopping criterion is based on the relative residual. We consider a regular decomposition into subdomains (squares/cubes), the overlap for each subdomain is of size $\mathcal{O}(2h)$ in all directions and the two-level preconditioner (4) is used in the hybrid way. All the computations are done in the open source language FreeFem++ (`http://www.freefem.org/ff++/`). The $3d$ code is parallelized and run on the TGCC Curie supercomputer. We assign each subdomain to one processor. So in our experiments the number of processors increases if the number of subdomains increases. To apply the preconditioner, the local problems in each subdomain (with matrices $A_{j,\varepsilon}$ in (3)) and the coarse space problem (with matrix E in (4)) are solved with a direct solver.

As in Graham et al. [2017b,a], in the experiments we take the subdomain diameter H_{sub} and the coarse mesh diameter H_{coarse} constrained by k: $H_{\text{sub}} \sim k^{-\alpha}$ and $H_{\text{coarse}} \sim k^{-\alpha'}$, for some choices of $0 < \alpha, \alpha' <= 1$ detailed in the following; if not differently specified, we take $\alpha = \alpha'$, which is the setting of all numerical experiments in Graham et al. [2017b]. Note that H_{coarse} does not appear as a parameter in the DtN coarse space. We denote by n_{CS} the size of the coarse space. For the grid coarse space $n_{\text{CS}} = (1/H_{\text{coarse}} + 1)^d$, the number of dofs for the nodal linear finite elements in the unit square/cube. For the DtN coarse space $n_{\text{CS}} = \sum_{j=1}^{N_{\text{sub}}} m_i$, the total number of computed eigenvectors for all the subdomains. While we solve the pure Helmholtz problem without absorption, both the one-level preconditioner (3) and the two-level preconditioner (4) are built from problems which can have non zero absorption given by $\varepsilon_{\text{prec}} = k^\beta$. In the experiments we put $\beta = 1$ or $\beta = 2$.

In the following tables we compare the one-level preconditioner, the two-level preconditioners with the grid coarse space and with the DtN coarse space in terms of number of iterations of GMRES and size of the coarse space (n_{CS}), for different values of the wavenumber k and of the parameters α, β. We also report the number of subdomains N_{sub}, which is controlled by k and α as mentioned above. Since $h \sim k^{-3/2}$, the dimension n of the linear system matrix is of order $k^{3d/2}$; for 3d experiments we report n explicitly. Tables 1, 2 concern the $2d$ problem, Table 3 the $3d$ problem.

In Table 1, we let the DtN coarse space size be determined by the automatic choice criterion in Line 2 of Algorithm 3.1 (studied in Conen et al. [2014]) and the grid coarse space size by $H_{\text{coarse}} \sim k^{-\alpha}$. We see that the DtN coarse space is much larger than the grid coarse space and gives fewer iterations. The preconditioners with absorption $\varepsilon_{\text{prec}} = k^2$ perform much worse than those with absorption $\varepsilon_{\text{prec}} = k$ independently of n_{CS}. For $\varepsilon_{\text{prec}} = k$, when $\alpha = 1$ the number of iterations grows as $k^{0.9}$, respectively $k^{1.1}$, for the grid coarse space, respectively DtN coarse space (excluding the first two values for k small where the asymptotic behaviour is not reached yet) ~~the number of iterations grows mildly with the wavenumber k for both coarse spaces (but at the cost of an increasing coarse space size),~~[R2] while the one-level preconditioner performs poorly (for $k = 80$ it needs more than 500 iterations to converge)[R2]. When $\alpha < 1$, i.e. for coarser coarse meshes, the

		$\beta = 1$					$\beta = 2$				
		$\alpha = 0.6$					$\alpha = 0.6$				
k	N_{sub}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}
10	9	22	19	16	11	39	28	27	16	23	40
20	36	48	46	49	26	204	67	56	49	40	220
40	81	78	98	100	37	531	121	114	100	72	578
60	121	109	114	144	43	1037	169	165	144	109	920
80	169	139	138	196	93	1588	223	216	196	126	1824
		$\alpha = 0.8$					$\alpha = 0.8$				
k	N_{sub}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}
10	36	35	19	49	10	122	39	27	49	28	86
20	100	71	35	121	13	394	83	51	121	41	362
40	361	158	88	400	22	1440	182	95	400	71	1370
60	676	230	187	729	39	2700	268	150	729	103	2698
80	1089	304	331	1156	68	4352	355	214	1156	138	4350
		$\alpha = 1$					$\alpha = 1$				
k	N_{sub}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}
10	100	65	26	121	11	324	57	30	121	23	324
20	400	122	26	441	14	1120	130	49	441	42	1120
40	1600	286	33	1681	20	4640	296	80	1681	72	4640
60	3600	445	45	3721	29	10560	455	112	3721	101	10560
80	6400	>500	62	6561	44	18880	>500	149	6561	134	18880

Table 1: ($d = 2$) Number of iterations (and coarse space size n_{CS}) for the one-level preconditioner and the two-level preconditioners with the grid coarse space/DtN coarse space, with $H_{\text{sub}} = H_{\text{coarse}} \sim k^{-\alpha}$, $\varepsilon_{\text{prec}} = k^{\beta}$.

growth with k is higher, and for $\alpha = 0.6$ the two-level preconditioner is not much better than the one-level preconditioner because the coarse grid problem is too coarse; for $\alpha = 0.8$ with the DtN coarse space the growth with k degrades less than with the grid coarse space.

We have seen in Table 1 that the DtN coarse space gives fewer iterations than the grid coarse space, but their sizes differed significantly. Therefore, in Table 2 we compare the two methods forcing n_{CS} to be similar. On the left, we force the DtN coarse space to have a smaller size, similar to the one of the grid coarse space, by taking just $m_i = 2$ eigenvectors for each subdomain. On the right, we do the opposite, we force the grid coarse space to have the size of the DtN coarse space obtained in Table 1, by prescribing a smaller coarse mesh diameter H_{coarse}, while keeping the same number of subdomains as in Table 1 with $H_{\text{sub}} \sim k^{-\alpha}$. We can observe that for smaller coarse space sizes (left) the grid coarse space gives fewer iterations than the DtN coarse space, while for larger coarse space sizes (right) the result is reversed.

We have seen that the coarse mesh obtained with $H_{\text{coarse}} \sim k^{-\alpha'}$, $\alpha' = \alpha$ can be too coarse if $\alpha = 0.6$. At the same time, for $\alpha = 1$ the number of subdomains gets quite large since $H_{\text{sub}} \sim k^{-\alpha}$, especially in $3d$; this is not desirable because in our parallel implementation we assign each subdomain to one processor, so communication among them would prevail and each

		n_{CS} forced by grid CS				n_{CS} forced by DtN CS			
		$\alpha = 0.6$				$\alpha = 0.6$			
k	N_{sub}	grid CS	n_{CS}	DtN CS	n_{CS}	grid CS	n_{CS}	DtN CS	n_{CS}
10	9	19	16	18	18	17	36	11	39
20	36	46	49	44	72	24	196	26	204
40	81	98	100	85	162	50	529	37	531
60	121	114	144	109	242	104	841	43	1037
80	169	138	196	140	338	173	1521	93	1588
		$\alpha = 0.8$				$\alpha = 0.8$			
k	N_{sub}	grid CS	n_{CS}	DtN CS	n_{CS}	grid CS	n_{CS}	DtN CS	n_{CS}
10	36	19	49	26	72	15	121	10	122
20	100	35	121	61	200	20	361	13	394
40	361	88	400	139	722	35	1369	22	1440
60	676	187	729	191	1352	52	2601	39	2700
80	1089	331	1156	250	2178	78	4225	68	4352
		$\alpha = 1$				$\alpha = 1$			
k	N_{sub}	grid CS	n_{CS}	DtN CS	n_{CS}	grid CS	n_{CS}	DtN CS	n_{CS}
10	100	26	121	52	200	17	324	11	324
20	400	26	441	43	800	23	1089	14	1120
40	1600	33	1681	157	3200	22	4624	20	4640
60	3600	45	3721	338	7200	26	10404	29	10560
80	6400	62	6561	>500	12800	30	18769	44	18880

Table 2: ($d = 2$) Number of iterations (and coarse space size n_{CS}) for the two-level preconditioners with the grid coarse space/DtN coarse space *forcing similar* n_{CS}, with $H_{sub} \sim k^{-\alpha}$, $\varepsilon_{prec} = k$.

processor would not be fully exploited since the subdomains would become very small. Therefore, to improve convergence with the grid coarse space while maintaining a reasonable number of subdomains, we consider separate H_{coarse} and H_{sub}, taking $\alpha' \neq \alpha$. For load balancing (meant as local problems having the same size as the grid coarse space problem), in $3d$ we choose $\alpha' = 3/2 - \alpha$. The DtN coarse space is still built by keeping the eigenvectors verifying the automatic choice criterion; note that in $3d$ the number of selected eigenvectors is larger than in $2d$, but we only keep a maximum of 20 eigenvectors in each subdomain. ~~The DtN coarse space size is still determined by the automatic choice criterion (among 20 computed local eigenvectors) in each subdomain.~~[R2] In Table 3 we report the results of this experiment. As expected, for the grid coarse space the best iteration counts are obtained for $\alpha = 0.5$ because then $\alpha' = 1$ gives the coarse mesh with the smallest diameter among the experimented ones: the number of iterations grows slowly, with $\mathcal{O}(k^{0.61}) \cong \mathcal{O}(n^{0.13})$. With higher α the iteration counts get worse quickly, and $\alpha = 0.8$ is not usable. For the DtN coarse space, the larger coarse space size is obtained by taking α bigger (recall that α' is not a parameter in the DtN case): for $\alpha = 0.8$ the number of iterations grows slowly, with $\mathcal{O}(k^{0.2}) \cong \mathcal{O}(n^{0.04})$, but this value may be optimistic, there is a decrease in iteration number between $k = 20$ and 30. We believe that for the other

			$\alpha = 0.5$, $\alpha' = 1$				
k	n	N_{sub}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}
10	39304	27	25	12	1331	14	316
20	704969	64	39	17	9261	31	1240
30	5000211	125	55	21	29791	54	2482
40	16194277	216	74	29	68921	80	4318
			$\alpha = 0.6$, $\alpha' = 0.9$				
k	n	N_{sub}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}
10	39304	27	25	15	512	14	316
20	912673	216	61	24	3375	41	2946
30	4826809	343	73	34	10648	65	6226
40	16194277	729	98	48	21952	108	13653
			$\alpha = 0.7$, $\alpha' = 0.8$				
k	n	N_{sub}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}
10	46656	125	34	19	343	11	896
20	912673	512	73	35	1331	18	4567
30	5929741	1000	103	57	4096	65	12756
40	17779581	2197	139	89	8000	116	30603
			$\alpha = 0.8$, $\alpha' = 0.7$				
k	n	N_{sub}	1-level	grid CS	n_{CS}	DtN CS	n_{CS}
10	50653	216	39	23	216	19	1354
20	1030301	1000	46	86	729	23	7323
30	5929741	3375	137	116	1331	21	26645
40	28372625	6859	189	200	2744	27	54418

Table 3: ($d = 3$) Number of iterations (and coarse space size n_{CS}) for the one-level preconditioner and the two-level preconditioners with the grid coarse space/DtN coarse space, with $H_{\text{sub}} \sim k^{-\alpha}$, $H_{\text{coarse}} \sim k^{-\alpha'}$, $\varepsilon_{\text{prec}} = k$.

values of α, where the iteration counts are not much better or worse than with the one-level preconditioner, we did not compute enough eigenvectors in each subdomain to build the DtN coarse space.

5 Conclusion

We tested numerically two different coarse space definitions for two-level domain decomposition preconditioners for the pure Helmholtz equation (discretized with piecewise linear finite elements), both in $2d$ and $3d$, reaching more than 15 million degrees of freedom in the resulting linear systems. The preconditioners built with absorption $\varepsilon_{\text{prec}} = k^2$ appear to perform much worse than those with absorption $\varepsilon_{\text{prec}} = k$. We have seen that in most cases for smaller coarse space sizes the grid coarse space gives fewer iterations than the DtN coarse space, while for larger coarse space sizes the grid coarse space gives generally more iterations than the DtN coarse space. The best iteration counts for the grid coarse space are obtained by separating the coarse

mesh diameter $H_{\text{coarse}} \sim k^{-\alpha'}$ from the subdomain diameter $H_{\text{sub}} \sim k^{-\alpha}$, taking $\alpha' > \alpha$. Both for the grid coarse space ~~the coarse grid space~~[R2] and the DtN coarse space, for appropriate choices of the method parameters we have obtained iteration counts which grow quite slowly with the wavenumber k. Further experiments to compare the two coarse spaces ~~the two definitions of coarse space~~[R2] should be carried out in the heterogenous case.

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References

L. Conen, V. Dolean, R. Krause, and F. Nataf. A coarse space for heterogeneous Helmholtz problems based on the Dirichlet-to-Neumann operator. *J. Comput. Appl. Math.*, 271:83–99, 2014.

I. G. Graham, E. A. Spence, and E. Vainikko. *Recent Results on Domain Decomposition Preconditioning for the High-Frequency Helmholtz Equation Using Absorption*, pages 3–26. Geosystems Mathematics. Springer, 2017a.

I. G. Graham, E. A. Spence, and E. Vainikko. Domain decomposition preconditioning for high-frequency Helmholtz problems with absorption. *Math. Comp.*, 86(307):2089–2127, 2017b.

J.-H. Kimn and M. Sarkis. Restricted overlapping balancing domain decomposition methods and restricted coarse problems for the Helmholtz problem. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1507–1514, 2007.